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On shrinkage Estimation Procedure Combining Direct and Randomized Responses in Unrelated Question Model

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SUMMARY

In this paper, we consider the estimation problem of population proportion, say, θ_A bearing a stigmatizing characteristic in a community. We take into account the unrelated question model with the assumption that the population proportion of unrelated innocuous attribute is not known. We modify that model on combining the direct and randomized responses and an unbiased estimator of θ_A based on randomized responses obtained from persons chosen by simple random sampling with replacement is obtained for this modified model. In this work, a new attempt has been made to construct shrinkage estimator on this modified model based on an adequate prior value θ_{A0} of θ_A. The efficiency properties of the new shrinkage estimator over the usual modified estimator are discussed here theoretically along with some numerical illustrations. In addition the unbiased estimator of the mean squared error is also derived.

Key words : Direct-cum-Randomized responses, Bias, Mean squared error, Improved efficiency, Prior value.

1. INTRODUCTION

Warner (1965) introduced a device of eliciting randomized responses (RR) instead of direct responses (DR) from persons chosen by simple random sampling with replacement (SRSWR) for estimating unbiasedly the proportion θ_A of people bearing a stigmatizing characteristic, say A like habitual alcoholism, durnken driving, drug addiction, induced abortions etc. in a community. In his technique, each respondent is provided with a randomization device by which he/she chooses one of two questions 'Do you belong to A?' or 'Do you belong to A^c?' with respective probabilities p and $1-p$ and then is asked to give a truthful 'yes' or 'no' answer to the question chosen unnoticed by the interviewer. As the interviewer does not see the question chosen, the randomization device with p close to 0.5, protects the privacy of respondent and so he/she may be willing to cooperate by responding truthfully.

Subsequently, the unrelated question model was suggested in Horvitz *et al*. (1967), Greenberg *et al*. (1969) where instead of the question related to A or A^c, the respondent is asked to respond 'yes' or 'no' to whether he/she belongs to A or to another group, say, B which is unrelated in the sense of statistical association to A. The group B and its complement B^c should both be innocuous, e.g. the question could be 'Does your birthday fall in the month of January?' Since this is unrelated to the sensitive attribute A, one can expect that the respondent will be more confident about privacy protection. However, in this approach there is an added difficulty as the true proportion θ_B in group B is also unknown. To remove this difficulty, the respondent is asked to give two randomized resposes from two randomization devices (one from each device) made with questions relating to A and B with probabilities p_1 and $1-p_1$ for 1st device and p_2 and

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$1 - p_2$ for 2nd device, ($p_1 \neq p_2$), respectively. Many researchers like Moor (1971), Folsom *et al*. (1973), Lanke (1975), Abul-Ela *et al*. (1967), Mangat *et al*. (1992), Singh *et al*. (1994) and Singh (1994) studied this model in detail. Dowling and Shatchman (1975) compared the Warner's (1965) model with the unrelated question model. Extensive reviews in this regard are available in Chaudhuri and Mukerjee (1985).

Mangat and Singh (1990) gave a method of combining the direct and randomized responses to estimate the proportion of people bearing a stigmatizing attribute in a community and compared the efficiency of their estimator with the estimator given by Warner (1965). In their method a sampled respondent chosen by SRSWR scheme is asked to respond directly his/her true value ('yes' or 'no') regarding the possessing of stigmatizing characteristic with a fixed probability, say, T and to answer ('yes' or 'no') by Warner's (1965) randomized response device with probability $(1 - T)$, unnoticed by the interviewer. We modify the unrelated question model along the line of Mangat and Singh (1990) on assuming that the population proportion of innocuous attribute is not known and an unbiased estimator of population proportion of bearing stigmatizing character is found out. We call this unbiased estimator as the usual estimator in the present context. The variance and an unbiased estimator of that variance of the estimator are also found out.

In usual practice, the experimenter often possesses some prior knowledge about an adequate guess, say, θ_{A0} of the value of θ_A based on some past experience or from some authentical source. Thompson (1968) suggested to construct a shrinkage estimator on using such prior guess. Singh *et al*. (2007) constructed a shrinkage estimator based on such prior guess in unrelated question model developed by Horvitz *et al*. (1967) on assuming that the population proportion of innocuous attribute is known. They also compared the efficiencies of their suggested shrinkage estimators with the estimator given by Horvitz *et al*. (1967).

In our present work, motivated by Thompson (1968) and Singh *et al*. (2007), we make an attempt to construct the new shrinkage estimator in unrelated question model on assuming that the population proportion of innocuous attribute is not known. We show here how the shrinkage estimator performs better in comparison to the usual estimator after combining the direct and randomized response technique in view of the mean squared error. An unbiased estimator of the mean squared error of the shrinkage estimator is also suggested. Finally some numerical computations are done to display the ranges of the required parameters to ensure the gain in effeciencies by our suggested estimator.

2. DIRECT-CUM-RANDOMIZED RESPONSE TECHNIQUE IN UNRELATED QUESTION MODEL ASSUMING θ_B IS UNKNOWN

2.1 Basic Notations and Definition of Unrelated Question Model

Suppose in a finite survey population $U = (1, \ldots, i, \ldots, N)$ a person labelled i has the value y_i on a sensitive variable y. Let

$y_i = 1$ if i bears the stigmatizing feature A.
$= 0$ if he/she bears the complementary feature A^c.

Our problem is to estimate $\theta_A = \dfrac{\sum_{i=1}^{N} y_i}{N}$, the proportion in U bearing A.

An unrelated question RR model studied by Horvitz *et al*. (1967), Greenberg *et al*. (1969) needs another variable x defined on U related to an innocuous human character, say, B not correlated with y in the following way.

$x_i = 1$ if i bears the unrelated innocuous feature B.
$= 0$ if he/she bears the complementary feature B^c.

To apply this RR technique, one box is to be prepared with a number of cards marked as 'Do you belong to A?' with probability $p_1 (0 < p_1 < 1)$ and the rest are marked as 'Do you belong to B?' And a second box is also to be prepared with a number of cards marked as 'Do you belong to A?' with probability $p_2 (0 < p_2 < 1, p_2 \neq p_1)$ and the rest are marked as 'Do you belong to B? These two boxes are offered to a sampled person i. Following Chaudhuri (2001), we note as follows. Using the above two boxes independently, two responses, realizing I_i and J_i are generated leading to the RR's.

$I_i = 1$ if card type matches his/her true feature from 1st box.
$= 0$, else.

J_i = 1 if card type matches his/her true feature from the second box.
= 0, else.

So, generically writing E_R and V_R to denote expectation and variance operator with respect to the personal 'response-making', we have

$E_R(I_i) = p_1 y_i + (1 - p_1)x_i$
$E_R(J_i) = p_2 y_i + (1 - p_2)x_i$

$$r_i = \frac{(1-p_2)\,I_i - (1-p_1)\,j_i}{p_1 - p_2}, \text{ taking } p_1 \neq p_2,$$

with $E_R(r_i) = y_i$ and on simplification

$$V_R(r_i) = \frac{[(1-p_1)(1-p_2)\{p_1(1-p_2) + p_2(1-p_1)\}]}{(p_1 - p_2)^2}\,(y_i - x_i)^2 \qquad (2.1)$$

2.2 Modified Definition of Unrelated Question Model

To modify this procedure along the line of Mangat and Singh (1990), our suggestion is that a sampled person i be requested first to draw one card from a box, say, BOXT, containing the cards marked 'True' with probability T and the rest marked as 'RR'. Then if 'RR'-marked card appears, he is requested to produce two independent responses I_i and J_i as above or else give out the true response y_i if 'True'-marked card appears. We write generically

$z_i = y_i$ with probability T using BOXT
$= I_i$ with probability $(1 - T)$ using the 1st box as in Subsection 2.1

$z'_i = y_i$ with probability T using BOXT
$= J_i$ with probability $(1 - T)$ using the 2nd box as in Subsection 2.1

Then

$E_R(z_i) = Ty_i + (1 - T)\,[p_1 y_i + (1 - p_1)x_i]$ and
$E_R(z_i') = Ty_i + (1 - T)\,[p_2 y_i + (1 - p_2)x_i]$

yielding

$$r'_i = \frac{(1-p_2)\,z_i - (1-p_1)\,z'_i}{(p_1 - p_2)} \text{ with } p_1 \neq p_2$$

$E_R(r'_i) = y_i$

and on suppressing detailed algebra,

$$V_R(r'_i) = \phi(y_i - x_i)^2, \text{ say} \qquad (2.2)$$

where

$$\phi = \frac{[(1-T)(1-p_1)(1-p_2)[(1-p_2)(T + p_1 - p_1 T) + (1-p_1)(T + p_2 - p_2 T)]]}{(p_1 - p_2)^2}$$

Based on this modification, an unbiased estimator of θ_A is

$$\hat{\theta}_A = e = \frac{1}{n}\sum_{k=1}^{n} r'_k \qquad (2.3)$$

since on writing E_P and V_P to denote the expectation and variance operator with respect to the sampling of respondents

$$E_P E_R(e) = E_P\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) = \bar{Y} = \theta_A$$

and

$$V(e) = V_P E_R(e) + E_P V_R(e)$$

$$= V_P\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) + E_P\left[\frac{\phi}{n^2}\sum_{k=1}^{n}(y_k + x_k - 2y_k x_k)\right]$$

$$= \frac{\theta_A(1-\theta_A)}{n} + \frac{\phi}{n}(\theta_A + \theta_B) - \frac{2\phi}{n} E_P\left(\frac{1}{n}\sum_{k=1}^{n} y_k x_k\right)$$

$$= \frac{\theta_A(1-\theta_A)}{n} + \frac{\phi}{n}(\theta_A + \theta_B) - \frac{2\phi}{n}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i X_i\right)$$

$$= \frac{\theta_A(1-\theta_A)}{n} + \frac{\phi}{n}(\theta_A + \theta_B) - \frac{2\phi}{n}\theta_A\theta_B$$

$$= \frac{\theta_A(1-\theta_A)}{n} + \frac{\phi}{n}(\theta_A + \theta_B - 2\theta_A\theta_B) \qquad (2.4)$$

It is easy to prove that irrespective of how a sample of respondents is drawn, the estimator based on unrelated question model by above modification perfoms better than the usual estimator based on usual unrelated question model without above modification if one chooses T satisfying

$$T \geq 1 - \frac{p_1(1-p_2)+p_2(1-p_1)}{2(1-p_1)(1-p_2)} \tag{2.5}$$

2.3 An Unbiased Variance Estimation for Modified Definition

An unbiased variance estimator of $\hat{\theta}_A = e$ may be taken as

$$v(e) = \frac{1}{n(n-1)}\sum_{k=1}^{n}(r_k' - \bar{r}_n')^2$$

where $$\bar{r}_n' = \frac{1}{n}\sum_{k=1}^{n} r_k' \tag{2.6}$$

because on writing

$$s_n^2 = \frac{1}{n-1}\sum_{k=1}^{n}(y_k - \bar{y}_n)^2$$

with $$\bar{y}_n = \frac{1}{n}\sum_{k=1}^{n} y_k$$

$$E_R\left[\frac{1}{n(n-1)}\sum_{k=1}^{n}(r_k' - \bar{r}_n')^2\right]$$

$$= \frac{1}{n(n-1)}\left[\sum_{k=1}^{n}\{V_R(r_k') + E_R^2(r_k')\} - n\{V_R(\bar{r}_n') + E_R^2(\bar{r}_n')\}\right]$$

$$= \frac{1}{n(n-1)}\left[\phi\sum_{k=1}^{n}(y_k - x_k)^2 + \sum_{k=1}^{n} y_k^2 - n\frac{\phi}{n^2}\sum_{k=1}^{n}(y_k - x_k)^2 - n\bar{y}_n^2\right]$$

$$= \frac{\phi}{n^2}\sum_{k=1}^{n}(y_k - x_k)^2 + \frac{s_n^2}{n}$$

Then $$E_p\left[\frac{\phi}{n^2}\sum_{k=1}^{n}(y_k - x_k)^2 + \frac{s_n^2}{n}\right]$$

$$= \frac{\phi}{nN}\sum_{i=1}^{N}(Y_i - X_i)^2 + \frac{1}{nN}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$$

$$= \frac{\phi}{n}(\theta_A + \theta_B - 2\theta_A\theta_B) + \frac{\theta_A(1-\theta_A)}{n} = V(e)$$

3. SHRINKAGE ESTIMATOR ON MODIFIED UNRELATED QUESTION MODEL ASSUMING θ_B IS UNKNOWN

3.1 Estimator (i) and its Mean Squared Error

In usual practice, we may often have some adequate guess, say, θ_{A0} of the value of θ_A based on past knowledge. Let θ_{A0} denotes that prior estimate of θ_A. Motivated by Thompson (1968), we may define a belief parameter, say, δ with $(0 < \delta < 1)$ defined as a value of δ close to 0 implying a strong belief on θ_{A0} and a value of δ close to 1 implying a strong belief on sample estimate $\hat{\theta}_A = e$. Then our suggested shrinkage estimator on using such prior guess and belief parameter is

$$e_{S1} = \delta e + (1-\delta)\theta_{A0} \tag{3.1}$$

According to this set-up, bias of e_{S1} is given by

$$\begin{aligned} B(e_{S1}) &= E(e_{S1}) - \theta_A \\ &= E[\delta e + (1-\delta)\theta_{A0}] - \theta_A \\ &= \delta\theta_A + (1-\delta)\theta_{A0} - \theta_A \\ &= (1-\delta)(\theta_{A0} - \theta_A) \end{aligned} \tag{3.2}$$

The mean squared error of e_{S1} is given by

$$\begin{aligned} MSE(e_{S1}) &= V(e_{S1}) + B^2(e_{S1}) \\ &= V[\delta e + (1-\delta)\theta_{A0}] + (1-\delta)^2(\theta_{A0} - \theta_A)^2 \\ &= V[\delta e] + (1-\delta)^2(\theta_{A0} - \theta_A)^2 \\ &= \delta^2 V[e] + (1-\delta)^2(\theta_{A0} - \theta_A)^2 \\ &= \delta^2\left[\frac{\phi}{n}(\theta_A + \theta_B - 2\theta_A\theta_B) + \frac{\theta_A(1-\theta_A)}{n}\right] + (1-\delta)^2(\theta_{A0} - \theta_A)^2 \\ &= \delta^2\left[\frac{\phi}{n}(\theta_A + \theta_B - 2\theta_A\theta_B) + \frac{\theta_A(1-\theta_A)}{n}\right] + (1-\delta)^2(\theta_{A0}^2 + \theta_A^2 - 2\theta_{A0}\theta_A) \end{aligned} \tag{3.3}$$

3.2 An Unbiased Estimator for Mean Squared Error of Estimator (i)

An unbiased estimator for mean squared error (3.3) of above shrinkage estimator (3.1) may be taken as

$$mse(e_{S1}) = \delta^2 v(e) + (1-\theta)^2(\theta_{A0}^2 + e^2 - v(e) - 2\theta_{A0}e)$$

because of the following reasons.

(i) e and $v(e)$ are respectively the unbiased estimators of θ_A and $V(e)$.

(ii) $E_P E_R[e^2 - v(e)] = E_P[V_R(e) + E_R^2(e)] - V(e)$

$$= E_p\left[V_e(e) + \bar{y}_n^2\right] - V(e)$$

$$= E_P(V_R(e)] - V_P(\bar{y}_n) + E_P^2(\bar{y}_n) - V(e)$$

$$= E_P\,[V_R(e)\} + V_P E_R(e) + E_P^2(\bar{y}_n) - V(e)$$

$$= V(e) + \bar{Y}^2 - V(e) = \bar{Y}^2 = \theta_A^2$$

Hence after simplification the above expression reduces to

$$mse\,(e_{S1}) = (1 - \delta^2)\,(\theta_{A0} - e)^2 - (1 - 2\delta)\,v(e) \tag{3.4}$$

for which $v(e)$ is to be substituted from (2.6).

3.3 Efficiency Comparison between e_{S1} and e

The new estimator e_{S1} in (3.1) will be more efficient than the usual estimator e in (2.3) if

$$MSE\,(e_{S1}) \le V\,(e)$$

On suppressing considerable algebra this condition reduces to

$e_{S1} \succ e$ if

$$\delta \ge \frac{n(\theta_{A0} - \theta_A)^2 - [\theta_A(1-\theta_A) + \phi(\theta_A + \theta_B - 2\theta_A\theta_B)]}{n(\theta_{A0} - \theta_A)^2 + [\theta_A(1-\theta_A) + \phi(\theta_A + \theta_B - 2\theta_A\theta_B)]} \tag{3.5}$$

This lower bound of δ for different values of ϕ (i.e. for different values of p_1, p_2, T) and θ_A, θ_B and θ_{A0} are presenented in Tables 1a and 1b. The bounds yielding negative values are presented as zero.

3.4 Optimum Value of δ to Minimize MSE(e_{S1})

The optimum value of δ which minimizes $MSE(e_{S1})$ can be obtained by solving

$$\frac{\partial}{\partial\delta}\,[MSE(es_1)] = 0$$

And this gives rise to the following solution which also after substitution to the $MSE(e_{S1})$ yields a positive quantity.

$$\delta_{\text{optimum}} = \frac{n(\theta_{A0} - \theta_A)^2}{\left[\theta_A(1-\theta_A) + \phi(\theta_A + \theta_B - 2\theta_A\theta_B) + n(\theta_{A0} - \theta_A)^2\right]} \tag{3.6}$$

But, we note that this optimum value of δ depends on two unknown quantities, θ_A and θ_B. So, to use this optimum value in practical survey situation, we may take into consideration the guessed value θ_{A0} of θ_A and also one more guessed value of θ_B. Suppose, θ_{B0} denotes an initial estimate or prior guessed value of θ_B. In order to obtain a usable value of δ_{optimum} utilizing the prior guesses of θ_A and θ_B, we may replace θ_A by $\alpha_A\theta_{A0}$ and θ_B by $\alpha_B\theta_{B0}$ in (3.6), where α_A and α_B are positive constants ($0 < \alpha_A$, $\alpha_B < 1$). We shall also examine the efficacy of this new estimator on changes of these two constants. The new value of δ thus obtained as $\hat{\delta}_{opt}$, say, is

$$\hat{\delta}_{opt} = \frac{n(\theta_{A0} - \alpha_A\theta_{A0})^2}{[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0}) + n(\theta_{A0} - \alpha_A\theta_{A0})^2]} \tag{3.7}$$

3.5 Estimator (ii) and its Mean Squared Error

Substitution of above $\hat{\delta}_{opt}$ from (3.7) in place of δ in (3.1) yields a second shrinkage estimator as

$$e_{S2} = \theta_{A0} + \frac{(e - \theta_{A0})n\theta_{A0}^2(1-\alpha_A)^2}{[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0}) + n(\theta_{A0} - \alpha_A\theta_{A0})^2]}. \tag{3.8}$$

The bias of e_{S2} is given by

$$B(e_{S2}) = (\theta_{A0} + \theta_A)\left[\frac{[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0})]}{[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0}) + n(\theta_{A0} - \alpha_A\theta_{A0})^2]}\right] \tag{3.9}$$

The mean squared error of this estimator (ii) is given by

$$MSE(e_{S2}) = \frac{[n^2\theta_{A0}^4(1-\alpha_A)^4 V(e) + (\theta_{A0}-\theta_A)^2 [\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0})]^2]}{[[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0}) + n(\theta_{A0}-\alpha_A\theta_{A0})^2]^2]}$$

This after substitution of the expression for $V(e)$ from (2.4) reduces to

$$MSE(e_{S2}) = \frac{[n\theta_{A0}^4(1-\alpha_A)^4\{\theta_A(1-\theta_A) + \phi(\theta_A+\theta_B-2\theta_A\theta_B)\}]}{[[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0}) + n(\theta_{A0}-\alpha_A\theta_{A0})^2]^2]}$$

$$+ \frac{[(\theta_{A0}-\theta_A)^2[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0})]^2]}{[[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0}) + n(\theta_{A0}-\alpha_A\theta_{A0})^2]^2]} \quad (3.10)$$

3.6 An Unbiased Estimator for Mean Squared Error of Estimator (ii)

An unbiased estimator for mean squared error (3.10) of above shrinkage estimator (3.8) may be taken as

$$mse\ (e_{S2}) = \frac{[n^2\theta_{A0}^4(1-\alpha_A)^4 v(e) + (\theta_{A0}^2 - 2\theta_{A0}e + e^2 - v(e))[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0})]^2]}{[[\alpha_A\theta_{A0}(1-\alpha_A\theta_{A0}) + \phi(\alpha_A\theta_{A0} + \alpha_B\theta_{B0} - 2\alpha_A\theta_{A0}\alpha_B\theta_{B0}) + n(\theta_{A0}-\alpha_A\theta_{A0})^2]^2]}$$

This after simplification reduces to

$$mse\ (e_{S2}) = (1-\hat{\delta}_{opt}^2)(\theta_{A0}-e)^2 - (1-2\hat{\delta}_{opt})\, v(e) \quad (3.11)$$

for which $v(e)$ is to be substituted from (2.6).

3.7 Efficiency Comparison between e_{S2} and e

The new estimator e_{S2} in (3.8) will be more efficient than the usual estimator e in (2.3) if

$$MSE\ (e_{S2}) \le V\ (e)$$

Let $k = \theta_A - \theta_A^2 + \phi\theta_A + \phi\theta_B - 2\phi\theta_A\theta_B$. Then on suppressing considerable algebra the required condition for e_{S2} to be more efficient than e is that

$$\alpha_A^2\ P_A - \alpha_A Q_A + R_A \ge 0 \quad (3.12)$$

where

$$P_A = \theta_{A0}^2\,[n(\theta_{A0}-\theta_A)^2 + (2n-1)k]$$

$$Q_A = \theta_{A0}[(1+\phi-2\phi\alpha_B\theta_{B0})\ \{n(\theta_{A0}-\theta_A)^2 - k\} + 4n\theta_{A0}k]$$

and

$$R_A = 2nk\theta_{A0}^2 - \alpha_B\theta_{B0}\phi\{n(\theta_{A0}-\theta_A)^2 - k\}$$

To get a real solution of this inequality the necessary condition $Q_A^2 - 4P_AR_A \ge 0$ gives us the admissible ranges of α_B. On suppressing considerable algebra, this condition reduces to

$$\alpha_B^2\,P_B + \alpha_B Q_B + R_B \ge 0 \quad (3.13)$$

where

$$P_B = 4\phi^2\theta_{B0}^2\ [n(\theta_{A0}-\theta_A)^2 - k]$$

$$Q_B = 4\theta_{B0}\phi\,[2nk - \phi\ \{n(\theta_{A0}-\theta_A)^2 - k\} - 4n\theta_{A0}k]$$

and

$$R_B = (1+\phi)^2\{n(\theta_{A0}-\theta_A)^2 - k\} + 8nk\theta_{A0}\ \{1+\phi-\theta_{A0}\}$$

For different values of ϕ (i.e. for different values of p_1, p_2, T), θ_A, θ_B, θ_{A0}, θ_{B0} and n, the ranges of α_B are displayed in Tables 2*a* and 2*b*. Then corresponding to a particular value of $\alpha_B = 0.40$ satisfying the most of the solution ranges of (3.13), the required solution in terms of upper bounds of α_A satisfying (3.12) for $e_{S2} \succ e$ are presented in Tables 3*a* and 3*b*.

4. NUMERICAL ILLUSTRATIONS

We compute the lower bound of δ as proved in (3.5) for efficiency comparison between e_{S1} and e and our illustrative findings are presented in Tables 1*a* and 1*b*. We also compute the bounds of α_A as proved in (3.12) for efficiency comparison between e_{S2} and e and our illustrative findings are presented in Tables 3*a* and 3*b*. But prior to that we first compute the ranges of α_B

Table 1a. Illustrative lower bound of δ for assured gain in efficiency of e_{S1} over e

	p_1	p_2	Range of T		T	p_1	p_2	Range of T		T	p_1	p_2	Range of T		T
	0.3	0.45	≥ 0.377		0.40	0.5	0.3	≥ 0.286		0.30	0.5	0.6	Any		0.2
	n					n					n				
θ_{A0}	10	25	50	75	100	10	25	50	75	100	10	25	50	75	100
$\theta_A = 0.20$ and $\theta_B = 0.15$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.086	0.226	0.0	0.0	0.0	0.0	0.0
$\theta_A/8$	0.0	0.0	0.0	0.0	0.100	0.0	0.0	0.037	0.236	0.366	0.0	0.0	0.0	0.0	0.036
$\theta_A/10$	0.0	0.0	0.0	0.0	0.128	0.0	0.0	0.065	0.262	0.390	0.0	0.0	0.0	0.0	0.064
$\theta_A = 0.30$ and $\theta_B = 0.15$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.118	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.100	0.237	0.0	0.0	0.175	0.363	0.481	0.0	0.0	0.0	0.033	0.176
$\theta_A/8$	0.0	0.0	0.0	0.247	0.376	0.0	0.0	0.320	0.489	0.590	0.0	0.0	0.0	0.185	0.320
$\theta_A/10$	0.0	0.0	0.077	0.273	0.400	0.0	0.013	0.345	0.510	0.608	0.0	0.0	0.013	0.213	0.345
$\theta_A = 0.40$ and $\theta_B = 0.15$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.036	0.0	0.0	0.0	0.174	0.310	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.095	0.290	0.415	0.0	0.032	0.362	0.524	0.620	0.0	0.0	0.031	0.229	0.361
$\theta_A/8$	0.0	0.0	0.244	0.424	0.534	0.0	0.184	0.488	0.626	0.706	0.0	0.0	0.183	0.369	0.487
$\theta_A/10$	0.0	0.0	0.271	0.447	0.554	0.0	0.211	0.509	0.643	0.720	0.0	0.0	0.210	0.394	0.508
$\theta_A = 0.20$ and $\theta_B = 0.25$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.002	0.144	0.0	0.0	0.0	0.0	0.0
$\theta_A/8$	0.0	0.0	0.0	0.0	0.012	0.0	0.0	0.0	0.155	0.291	0.0	0.0	0.0	0.0	0.0
$\theta_A/10$	0.0	0.0	0.0	0.0	0.040	0.0	0.0	0.0	0.182	0.317	0.0	0.0	0.0	0.0	0.0
$\theta_A = 0.30$ and $\theta_B = 0.25$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.072	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.048	0.190	0.0	0.0	0.130	0.321	0.444	0.0	0.0	0.0	0.0	0.127
$\theta_A/8$	0.0	0.0	0.0	0.200	0.333	0.0	0.0	0.277	0.452	0.559	0.0	0.0	0.0	0.137	0.275
$\theta_A/10$	0.0	0.0	0.028	0.227	0.358	0.0	0.0	0.303	0.474	0.578	0.0	0.0	0.0	0.165	0.300
$\theta_A = 0.40$ and $\theta_B = 0.25$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.015	0.0	0.0	0.0	0.155	0.291	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.074	0.270	0.397	0.0	0.012	0.344	0.509	0.608	0.0	0.0	0.009	0.209	0.342
$\theta_A/8$	0.0	0.0	0.224	0.406	0.519	0.0	0.165	0.472	0.614	0.696	0.0	0.0	0.162	0.351	0.470
$\theta_A/10$	0.0	0.0	0.251	0.429	0.539	0.0	0.192	0.494	0.631	0.710	0.0	0.0	0.189	0.375	0.492

Table 1b. Illustrative lower bound of δ for assured gain in efficiency of e_{S1} over e

	p_1	p_2	Range of T		T	p_1	p_2	Range of T		T	p_1	p_2	Range of T		T
	0.3	0.45	≥ 0.377		0.40	0.5	0.3	≥ 0.286		0.30	0.5	0.6	Any		0.2
	n					n					n				
θ_{A0}	10	25	50	75	100	10	25	50	75	100	10	25	50	75	100
$\theta_A = 0.20$ and $\theta_B = 0.55$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/8$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.108	0.0	0.0	0.0	0.0	0.0
$\theta_A/10$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.136	0.0	0.0	0.0	0.0	0.0
$\theta_A = 0.30$ and $\theta_B = 0.55$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.0	0.068	0.0	0.0	0.012	0.211	0.344	0.0	0.0	0.0	0.0	0.003
$\theta_A/8$	0.0	0.0	0.0	0.078	0.219	0.0	0.0	0.165	0.353	0.472	0.0	0.0	0.0	0.013	0.156
$\theta_A/10$	0.0	0.0	0.0	0.106	0.245	0.0	0.0	0.192	0.377	0.494	0.0	0.0	0.0	0.041	0.183
$\theta_A = 0.40$ and $\theta_B = 0.55$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.099	0.239	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.015	0.214	0.347	0.0	0.0	0.294	0.466	0.571	0.0	0.0	0.0	0.152	0.288
$\theta_A/8$	0.0	0.0	0.168	0.356	0.474	0.0	0.110	0.427	0.578	0.666	0.0	0.0	0.104	0.298	0.423
$\theta_A/10$	0.0	0.0	0.195	0.380	0.496	0.0	0.137	0.450	0.596	0.681	0.0	0.0	0.132	0.323	0.445
$\theta_A = 0.20$ and $\theta_B = 0.70$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/8$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.035	0.0	0.0	0.0	0.0	0.0
$\theta_A/10$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.063	0.0	0.0	0.0	0.0	0.0
$\theta_A = 0.30$ and $\theta_B = 0.70$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.0	0.016	0.0	0.0	0.0	0.163	0.299	0.0	0.0	0.0	0.0	0.0
$\theta_A/8$	0.0	0.0	0.0	0.026	0.169	0.0	0.0	0.115	0.308	0.432	0.0	0.0	0.0	0.0	0.104
$\theta_A/10$	0.0	0.0	0.0	0.055	0.196	0.0	0.0	0.143	0.334	0.455	0.0	0.0	0.0	0.0	0.132
$\theta_A = 0.40$ and $\theta_B = 0.70$															
θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.074	0.214	0.0	0.0	0.0	0.0	0.0
$\theta_A/4$	0.0	0.0	0.0	0.189	0.323	0.0	0.0	0.270	0.446	0.553	0.0	0.0	0.0	0.125	0.263
$\theta_A/8$	0.0	0.0	0.141	0.332	0.453	0.0	0.084	0.406	0.560	0.651	0.0	0.0	0.077	0.273	0.400
$\theta_A/10$	0.0	0.0	0.169	0.357	0.475	0.0	0.112	0.429	0.579	0.667	0.0	0.0	0.105	0.298	0.423

Table 2a. Illustrative lower bound of α_B satisfying (3.13) for assured gain in efficiency of e_{S2} over e

		p_1	p_2	Range of T		T	p_1	p_2	Range of T		T	p_1	p_2	Range of T		T
		0.3	0.45	≥ 0.377		0.40	0.5	0.3	≥ 0.286		0.30	0.5	0.6	Any		0.2
		n					n					n				
θ_{B0}	θ_{A0}	10	25	50	75	100	10	25	50	75	100	10	25	50	75	100
θ_A = 0.20 and θ_B = 0.15																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.130	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.300	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	—	0.0	0.0	0.0	0.0	0.205	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	—	0.375	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0	—	0.690	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.760	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0	—	—	0.0	0.0	0.0
θ_A = 0.30 and θ_B = 0.15																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.530	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.855	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	—	0.0	0.0	0.0	0.0	0.230	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.0	0.0	0.0	.740	0.0	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
θ_A = 0.40 and θ_B = 0.15																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.055	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	—	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
θ_A = 0.20 and θ_B = 0.25																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.085	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.185	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.795	0.0	0.0	0.0	0.0	0.155	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	—	0.300	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0	—	0.485	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.555	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0	—	0.780	0.0	0.0	0.0
θ_A = 0.30 and θ_B = 0.25																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.345	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.540	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	—	0.0	0.0	0.0	0.0	0.225	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.0	0.0	0.0	0.0	0.555	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
θ_A = 0.40 and θ_B = 0.25																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	0.865	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0

Table 2b. Illustrative lower bound of α_B satisfying (3.13) for assured gain in efficiency of e_{S2} over e

		p_1	p_2	Range of T		T	p_1	p_2	Range of T		T	p_1	p_2	Range of T		T
		0.3	0.45	≥ 0.377		0.40	0.5	0.3	≥ 0.286		0.30	0.5	0.6	Any		0.2
		n					n					n				
θ_{B0}	θ_{A0}	10	25	50	75	100	10	25	50	75	100	10	25	50	75	100
$\theta_A = 0.20$ and $\theta_B = 0.55$																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.090	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.380	0.0	0.0	0.0	0.0	0.095	0.0	0.0	0.0	0.0	0.450	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	—	0.195	0.0	0.0	0.0	0.535	0.0	0.0	0.0	0.0	—	0.275	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.325	0.0	0.0	0.0	0.750	0.0	0.0	0.0	0.0	—	0.425	0.0	0.0	0.0
$\theta_A = 0.30$ and $\theta_B = 0.55$																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.180	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.265	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	0.770	0.0	0.0	0.0	0.0	0.185	0.0	0.0	0.0	0.0	0.915	0.0	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.0	0.0	0.0	0.0	0.355	0.0	0.0	0.0	0.0	—	0.055	0.0	0.0	0.0
$\theta_A = 0.40$ and $\theta_B = 0.55$																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/4$	0.445	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.615	0.0	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	0.690	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.895	0.0	0.0	0.0	0.0
$\theta_A = 0.20$ and $\theta_B = 0.70$																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.070	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.300	0.0	0.0	0.0	0.0	0.080	0.0	0.0	0.0	0.0	0.355	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	0.810	0.165	0.0	0.0	0.0	0.435	0.0	0.0	0.0	0.0	0.910	0.230	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	—	0.270	0.0	0.0	0.0	0.605	0.0	0.0	0.0	0.0	—	0.350	0.0	0.0	0.0
$\theta_A = 0.30$ and $\theta_B = 0.70$																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.150	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.215	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	0.620	0.0	0.0	0.0	0.0	0.170	0.0	0.0	0.0	0.0	0.735	0.0	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	0.850	0.0	0.0	0.0	0.0	0.310	0.0	0.0	0.0	0.0	—	0.075	0.0	0.0	0.0
$\theta_A = 0.40$ and $\theta_B = 0.70$																
θ_B	θ_A	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$3\theta_B/4$	$3\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/2$	$\theta_A/2$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/4$	$\theta_A/4$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\theta_B/8$	$\theta_A/8$	0.365	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.500	0.0	0.0	0.0	0.0
$\theta_B/10$	$\theta_A/10$	0.565	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.720	0.0	0.0	0.0	0.0

Table 3a. Illustrative upper bound of α_A for assured gain in efficiency of e_{S2} over e for a particular value α_B = 0.4

		p_1	p_2	Range of T		T	p_1	p_2	Range of T		T	p_1	p_2	Range of T		T
		0.3	0.45	≥ 0.377		0.40	0.5	0.3	≥ 0.286		0.30	0.5	0.6	Any		0.2
		n					n					n				
θ_{B0}	θ_{A0}	10	25	50	75	100	10	25	50	75	100	10	25	50	75	100
θ_A = 0.20 and θ_B = 0.15																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	—	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.655	0.515	—	1.0	1.0	1.0	1.0
$\theta_B/8$	$\theta_A/8$	—	1.0	1.0	1.0	0.460	—	1.0	0.625	0.305	0.235	—	—	1.0	1.0	0.630
$\theta_B/10$	$\theta_A/10$	—	—	1.0	1.0	0.355	—	1.0	0.485	0.230	0.165	—	—	1.0	1.0	0.485
θ_A = 0.30 and θ_B = 0.15																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.790	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	—	1.0	1.0	0.625	0.500	1.0	1.0	0.540	0.435	0.395	—	1.0	1.0	0.755	0.545
$\theta_B/8$	$\theta_A/8$	—	1.0	0.570	0.295	0.230	1.0	1.0	0.255	0.195	0.170	—	1.0	1.0	0.340	0.250
$\theta_B/10$	$\theta_A/10$	—	1.0	0.440	0.220	0.165	—	0.720	0.190	0.135	0.115	—	.1.0	0.715	0.260	0.185
θ_A = 0.40 and θ_B = 0.15																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.750	0.695	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	0.620	0.460	0.410	1.0	0.755	0.425	0.380	0.355	1.0	1.0	0.760	0.495	0.430
$\theta_B/8$	$\theta_A/4$	—	1.0	0.620	0.460	0.185	1.0	0.335	0.195	0.165	0.155	—	1.0	0.340	0.230	0.195
$\theta_B/10$	$\theta_A/10$	—	1.0	0.220	0.155	0.130	1.0	0.260	0.140	0.120	0.105	—	1.0	0.260	0.170	0.140
θ_A = 0.20 and θ_B = 0.25																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	—	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.580	—	1.0	1.0	1.0	1.0
$\theta_B/8$	$\theta_A/8$	—	1.0	1.0	1.0	0.775	—	1.0	1.0	0.365	0.245	—	—	1.0	1.0	1.0
$\theta_B/10$	$\theta_A/10$	—	—	1.0	1.0	0.565	—	1.0	1.0	0.270	0.160	—	—	1.0	1.0	1.0
θ_A = 0.30 and θ_B = 0.25																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.830	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	1.0	0.715	0.520	1.0	1.0	0.580	0.440	0.390	—	1.0	1.0	1.0	0.585
$\theta_B/8$	$\theta_A/8$	—	1.0	1.0	0.305	0.215	1.0	1.0	0.250	0.170	0.140	—	1.0	1.0	0.375	0.245
$\theta_B/10$	$\theta_A/10$	—	1.0	0.605	0.220	0.140	—	1.0	0.170	0.105	0.075	—	1.0	1.0	0.280	0.170
θ_A = 0.40 and θ_B = 0.25																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	0.910	1.0	1.0	1.0	0.760	0.695	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	0.650	0.455	0.395	1.0	0.840	0.415	0.360	0.335	1.0	1.0	0.855	0.495	0.420
$\theta_B/8$	$\theta_A/8$	—	1.0	0.275	0.180	0.145	1.0	0.330	0.165	0.130	0.115	—	.1.0	0.335	0.205	0.160
$\theta_B/10$	$\theta_A/10$	—	1.0	0.200	0.115	0.090	1.0	0.245	0.105	0.075	0.060	.—	.1.0	0.245	0.135	0.100

Table 3b. Illustrative upper bound of α_A for assured gain in efficiency of e_{S2} over e for a particular value α_B = 0.4

		p_1	p_2	Range of T		T	p_1	p_2	Range of T		T	p_1	p_2	Range of T		T
		0.3	0.45	≥ 0.377		0.40	0.5	0.3	≥ 0.286		0.30	0.5	0.6	Any		0.2
		n					n					n				
θ_{B0}	θ_{A0}	10	25	50	75	100	10	25	50	75	100	10	25	50	75	100
θ_A = 0.20 and θ_B = 0.15																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	—	1.0	1.0	1.0	1.0
$\theta_B/8$	$\theta_A/8$	—	1.0	1.0	1.0	1.0	—	1.0	1.0	—	0.420	—	1.0	1.0	1.0	1.0
$\theta_B/10$	$\theta_A/10$	—	1.0	1.0	1.0	1.0	—	1.0	1.0	—	0.295	—	—	1.0	1.0	1.0
θ_A = 0.30 and θ_B = 0.15																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	1.0	1.0	0.665	1.0	1.0	0.845	0.485	0.400	1.0	1.0	1.0	1.0	0.925
$\theta_B/8$	$\theta_A/8$	—	1.0	1.0	0.450	0.230	.1.0	1.0	0.295	0.130	0.075	—	1.0	1.0	0.740	0.305
$\theta_B/10$	$\theta_A/10$	—	1.0	1.0	0.320	0.125	1.0	1.0	0.185	0.0	0.0	—	.1.0	.1.0	0.520	0.190
θ_A = 0.40 and θ_B = 0.15																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.790	0.705	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	0.815	0.455	0.370	1.0	1.0	0.400	0.320	0.285	1.0	1.0	1.0	0.515	0.400
$\theta_B/8$	$\theta_A/4$	—	1.0	0.260	0.105	0.055	1.0	0.355	0.080	0.0	0.0	—	1.0	0.365	0.140	0.075
$\theta_B/10$	$\theta_A/10$	—	1.0	0.155	0.0	0.0	1.0	0.240	0.0	0.0	0.0	—	1.0	0.245	0.050	0.0
θ_A = 0.20 and θ_B = 0.25																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/8$	$\theta_A/8$	—	.1.0	1.0	1.0	1.0	—	1.0	1.0	1.0	0.635	—	1.0	1.0	1.0	1.0
$\theta_B/10$	$\theta_A/10$	—	1.0	1.0	1.0	1.0	—	1.0	1.0	1.0	0.465	—	1.0	1.0	1.0	1.0
θ_A = 0.30 and θ_B = 0.25																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	1.0	1.0	0.825	1.0	1.0	1.0	0.520	0.415	1.0	1.0	1.0	1.0	1.0
$\theta_B/8$	$\theta_A/8$	—	1.0	1.0	0.645	0.270	1.0	1.0	0.360	0.130	0.055	—	1.0	1.0	1.0	0.380
$\theta_B/10$	$\theta_A/10$	—	1.0	1.0	0.455	0.150	1.0	1.0	0.235	0.0	0.0	—	1.0	1.0	1.0	0.250
θ_A = 0.40 and θ_B = 0.25																
θ_B	θ_A	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$3\theta_B/4$	$3\theta_A/4$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\theta_B/2$	$\theta_A/2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.815	0.715	1.0	1.0	1.0	1.0	1.0
$\theta_B/4$	$\theta_A/4$	1.0	1.0	1.0	0.460	0.360	1.0	1.0	0.395	0.305	0.265	1.0	1.0	1.0	0.535	0.400
$\theta_B/8$	$\theta_A/8$	1.0	1.0	0.270	0.080	0.0	1.0	0.390	0.050	0.0	0.0	—	.1.0	0.410	0.120	0.0
$\theta_B/10$	$\theta_A/10$	—	1.0	0.155	0.0	0.0	1.0	0.260	0.0	0.0	0.0	.—	.1.0	0.265	0.0	0.0

admitting the real solution of the equation (3.12) and our solution ranges of α_B are presented in Tables 2*a* and 2*b*. Then corresponding to a particular illustrative value of $\alpha_B = 0.4$ falling in most of the solution ranges, we obtain the ranges of α_A for assured gain in efficiency of e_{S2} over e.

For numerical illustration, we use three sets of values of p_1 and p_2 and the value of T satisfying (2.5) for assured gain in efficiency in combining direct and randomized responses irrespective of how a sample of respondents is chosen. These three sets of device parameters are

(i) $p_1 = 0.3$, $p_2 = 0.45$, Range of $T \geq 0.377$ and the used value of $T = 0.4$,

(ii) $p_1 = 0.5$, $p_2 = 0.3$, Range of $T \geq 0.286$ and the used value of $T = 0.3$ and

(iii) $p_1 = 0.5$, $p_2 = 0.6$, Range of T = any value in between (0.0, 1.0) and the used value of T = 0.2.

For the sake of simplicity, we use the sample sizes as n = 10, 25, 50, 75, 100; θ_A = 0.2, 0.3, 0.4 and θ_B = 0.15, 0.25, 0.55, 0.70. We also use the prior guess values $\theta_{A0} = \theta_A, 3\theta_A / 4, \theta_A / 2, \theta_A / 4, \theta_A / 8, \theta_A / 10$ and $\theta_{B0} = \theta_B, 3\theta_B / 4, \theta_B / 2, \theta_B / 4, \theta_B / 8, \theta_B / 10$.

It may be observed from Tables 1a and 1b that our proposed estimator e_{S1} in (3.1) is better than the usual estimator e in (2.3) for full range of δ in most situations showing the lower bound as zero specially when the sample sizes are small (≤ 25). The range of δ decreases as the sample size increases and also as the prior guess value θ_{A0} departs much from the true value θ_A.

It is observed from Tables 2*a* and 2*b* that here also in most cases α_B attains the full range to assure the availability of real solution in (3.12) showing the lower bound as zero. The cases when sample sizes are small (≤ 25) and when the prior gueses θ_{A0} and θ_{B0} depart much from their true value, the findings are divided in two types of results: the solution being not available (presenting as — in respective boxes) in some cases and in some cases, the ranges become smaller than the full range.

Based on the results of full range of α_B from the two tables (Tables 2*a* and 2*b*), we decide to present the illustrative solution ranges of α_A in terms of upper bound, corresponding to $\alpha_B = 0.4$ in Tables 3*a* and 3b. From Tables 3*a* and 3*b*, it is clear that here also, in most situations, α_A attains the full range (showing upper bound as 1.0) to ensure the gain in efficiency of e_{S2} in (3.8) over e in (2.3). Whenever the value $\alpha_B = 0.4$ is not consistent with Tables 2*a* and 2*b*, the ouput is presented as — in respective boxes of Tables 3a and 3b. However, the ranges become smaller for higher sample sizes and for prior guess values departing much from true values. Of course, in very few cases, we do not get the admissible solution ranges of α_A and in those cases the upper bounds are presented as zero in respective boxes.

5. CONCLUDING REMARKS

We observe from the results of above numerical illustration that after profitable application of Mangat and Singh (1990)'s technique in unrelated question model, there is enough scope to choose δ to generate more efficient estimators. In practice, since the survey on sensitive questions are very expensive and time consuming, in many frequent cases, the sample sizes are kept small. In those cases and also even when the prior guess values depart much from their true values, our proposed estimator performs better than the usual estimator.

REFERENCES

Abul-Ela, Abdel-Latif, A., Greenberg, B.G. and Horvitz, D.G. (1967). A multiproportions randomized response models. *J. Amer. Statist. Assoc.,* **62**, 990-1008.

Chaudhuri, A. (2001). Using randomized response from a complex survey to estimate a sensitive proportion in a dichotomous finite population. *J. Statist. Plann. Inf.,* **94**, 37-42.

Chaudhuri, A. and Mukerjee, R. (1988). *Randomized Response: Theory and Techniques.* Marcel Dekker, New York.

Dowling, T.A. and Shachtman, R. (1975). On the relative efficiency of randomized response models. *J. Amer. Statist. Assoc*., **70**, 84-87.

Folsom, R.E., Greenberg, B.G., Horvitz, D.G. and Abernathy, J.R. (1973). The two alternate questions randomized response model for human surveys. *J. Amer. Statist. Assoc*., **68**, 525-530.

Greenberg, B.G., Abul-Ela, Abdel-Latif, A., Simmons, W.R. and Horvitz, D.G. (1969). The unrelated question RR model: Theoretical framework. *J. Amer. Statist. Assoc*., **64**, 520-539.

Horvitz, D.G., Shah, B.V. and Simmons, W.R. (1967). The unrelated question RR model. *Proceedings of the Social Statistics Section, Amer. Statist. Assoc*., 65-72.

Lanke, J. (1975). Some contributions to the theory of survey sampling. Ph.D. thesis, University of Lund.

Mangat, N.S. and Singh, R. (1990). An alternative randomized response procedure. *Biometrika*, **77(2)**, 439-442.

Mangat, N.S., Singh, R., Singh, S. (1992). An improved unrelated question randomized response strategy. *Cal. Statist. Assoc. Bull.,* **42**, 277-281.

Moor, J.J.A. (1971). Optimization of the unrelated question randomized response model. *J. Amer. Statist. Assoc*., **66**, 627-629.

Singh, S. (1994). Unrelated question randomized response sampling using continuous distribution. *J. Ind. Soc. Agril. Statist.,* **46**, 349-361.

Singh, S., Mangat, N.S. and Singh, R. (1994). On estimation of mean/total of stigmatizing quantitative variable. *Statistica*, **54**, 383-386.

Singh, H.P, Shukla, S. and Mathur, N. (2007). Shrinkage estimation of proportion of population possessing stigmatizing character in unrelated question randomized response technique. *J. Amer. Statist. Assoc*., **61(1)**, 1-13.

Thompson, J.R. (1968). Some shrinkage techniques for estimating the mean. *J. Amer. Statist. Assoc*., **63**, 113-122.

Warner, S.L. (1965). Randomized response: A survey technique for eliminating evasive answer bias. *J. Amer. Statist. Assoc*., **60**, 63-69.