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Generation of Linear Trend-Free Designs for Factorial Experiments

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SUMMARY

The present article deals with generation of designs for 2^k factorial experiments (without and with confounding) that permit the estimation of main effects free from linear trend effects present in the experimental units. The proposed method exploits the use of component wise product of vectors to generate linear trend-free for main effects designs for two-level factorials. The procedure of identifying two- and three- factor interactions that are linear trend-free has also been incorporated in the method. The method has also been extended to generate designs for confounded factorial experiments with any number of factors k ($\geq$ 3) that are linear trend-free for main effects and identify two- and three- factor interactions for which the design is linear/nearly linear trend-free.

Keywords: Factorial experiments, Linear trend-free designs, Orthogonal polynomials, Run orders, Systematic designs.

1. INTRODUCTION

Designs for factorial experiments are very popular among the agricultural, biological and industrial experimenters. Designs for factorial experiments are used for identifying the important main effects and interactions. However, the experimental units in these designs may exhibit a trend over space or time. Such situations occur in agricultural experiments when there is a slope, in the field and there is sequential application of treatments to the same experimental unit over different time periods. Trends may also occur in the experimental units when the land is irrigated and the nutrients are supplied by the fertilizers but because of the slope, the distribution of nutrients is not uniform. In such experimental situations, a common polynomial trend within experimental units is likely to occur. The trend may be represented by a polynomial of appropriate degree smaller than the block size. The presence of trends among the experimental units within a block affects the inference problem on main effects and interactions of interest. In the presence of trends among the experimental units it may be desirable to estimate the main effects and interactions of interest free from trend effects. Generally, we consider the presence of a linear trends among the experimental units within a block. In the presence of linear trends among the experimental units within a block of a factorial experiment, it is desired to allocate the treatment combinations to experimental units in such a manner that the main effects and interactions of interest are estimated free from the linear trend effects. Such designs are called as *linear trend-free designs for factorial experiments* for estimating the effects of interest and the ordered application of treatments to experimental units is called *run order*.

Bradley and Yeh (1980) gave a rigorous treatment to the theory of trend-free block designs. Yeh and Bradley (1983) discussed the existence of trend-free block designs for specified trend polynomials under a homoscedastic model. For further research on trend-free

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block designs, one may refer to Dhall (1986), Bradley and Odeh (1988), Stufken (1988), Chai and Majumdar (1993), Jacroux *et al.* (1995, 1997), Majumdar and Martin (2002), Lal *et al.* (2005, 2007), among others. For factorial set up of treatments, Daniel and Wilcoxon (1966) developed plans for sequencing the treatment combinations of a two-level full or fractional factorial to experimental units. Cheng and Jacroux (1988) further introduced trend-free run orders of two-level designs using group theory. John (1990) introduced a fold over method and discussed the trend-free properties of systematic run orders based on their method. Coster (1993) extended the work of Coster and Cheng (1988) to mixed level factorials. For the construction of trend-free fractional factorial and response surface designs using computer, readers may visit the website of Nguyen (http://designcomputing.net/gendex/rat/). The approach used to generate trend-free designs is to minimize a proper objective function using the random seed.

The present article attempts to generate linear trend-free designs for full factorials both without and with confounding. The search is restricted to two-level factorials. The designs are generated by making use of a computer-aided search because this approach is relatively easy and fast as compared to generating designs through an algebraic treatment. The search is restricted to linear main effects trend-free designs but the search also identifies 2- and 3- factor interactions that are linear trend-free/nearly linear trend-free in these designs. The method is illustrated with suitable examples. Section 2 deals with general description of linear trend-free designs for two-level factorial experiments. Section 3 develops the method for generation of linear trend-free 2^k full factorial experiments in which main effects are estimated free from linear trend effects and 2- and 3- factor interactions are identified that are estimable free from linear trend effects. The method is illustrated in Section 4. Section 5 modifies the method to generate confounded factorial experiments that are linear trend-free for main effects and identifies 2- and 3-factor interactions that are also estimable free from linear trend effects. The working of the algorithm for generating linear trend-free confounded factorial designs has been described in Section 6. Catalogues of linear trend-free 2^k factorial experiments for (k = 3, …, 7) are available with the authors.

2. LINEAR TREND-FREE DESIGNS FOR TWO LEVEL FACTORIAL EXPERIMENTS

In factorial experiments, generally the interest of the experimenter is to estimate the lower order factorial effects precisely. In general, the treatment combinations are applied to experimental units randomly. If it is known or assumed that experimental units exhibit a linear trend over space or time, then it is advantageous to choose a systematic run order so that the run order is linear trend-free for main effects and is linear/nearly linear trend-free for lower order interactions. For a design of 2^k factorial experiment the total number of treatment combinations is $n = 2^k$ and the model for factorial experiment with linear trend effect conducted in a single replication is

$$y = \boldsymbol{U}\beta + \boldsymbol{T}\theta + \boldsymbol{e} \qquad (1)$$
$$E(\boldsymbol{e}) = \boldsymbol{0},\ D(\boldsymbol{e}) = \sigma^2\boldsymbol{I}$$

where $\boldsymbol{y}$ is a $n \times 1$ vector of observations, $\boldsymbol{U}$ is an $n \times n$ design matrix and β is an n-component vector of general mean and treatment effects (treatment combinations written in lexicographic order from the left), $\boldsymbol{T}$ is the vector of coefficients of the first degree orthogonal polynomial of order n, θ is a regression coefficient and $\boldsymbol{e}$ is $n \times 1$ vector of independently and identically normally distributed errors. In a 2^k factorial experiment, the total number of treatment combinations is even. So vector $\boldsymbol{T}$ is

$$\boldsymbol{T} = (-(n-1), -(n-3), ..., -3, -1, 1, 3, (n-3), (n-1)) \qquad (2)$$

We may redefine

$$\mathbf{U} = [\mathbf{1} \vdots \boldsymbol{X}_1 \vdots \boldsymbol{X}_2 \vdots \cdots \vdots \boldsymbol{X}_s \vdots \cdots \vdots \boldsymbol{X}_k],$$

where $\mathbf{1}$ is an n-component vector of ones, $\mathbf{X}_s$ is an $n \times \binom{k}{s}$ matrix of coefficients of the s-factor factorial effects, $s = 1, 2, ..., k$. Based on general definition given by Bradley and Yeh (1980), all the s-factor factorial effects are linear trend-free if $\boldsymbol{X}_s'\boldsymbol{T} = \mathbf{0}$. Here $\mathbf{0}_t$ denotes a t-component vector of all zeros. For example, a design for factorial experiment would be linear trend free for main effects if $\boldsymbol{X}_1'\boldsymbol{T} = \mathbf{0}$.

It may not be possible always to generate a design for factorial experiment that is linear trend-free for all the effects of interest. This provides a motivation to go for nearly linear trend-free designs for some of the effects of interest. The condition given by Chai (1995) for a balanced incomplete block design to be nearly linear trend-free block design can be generalized for the factorial experiments. In a factorial experiment, the condition for the $n \times 1$ column vector of coefficients of contrast of interest, say **A**, to be nearly linear trend-free is

$$\mathbf{0} < \boldsymbol{A'T} \leq n \quad (3)$$

Our interest is to obtain designs for factorial experiments in which contrasts for the main effects are estimated free from linear trend effects and to identify/search two and three factor interactions that are estimable free from linear trend effect. To obtain such designs, we begin with following lemmas:

Lemma 2.1 {Cheng and Jacroux (1988)}. In any run order of a complete 2^k design, the number of mutually orthogonal linear trend-free factorial contrasts (main effects or interaction) are at most $2^k - k - 1$.

Let $s_1, s_2, ..., s_k$ denote the coefficients of the estimates of the main effect contrasts, defined in Section 3 as I3.

Lemma 2.2 {Cheng and Jacroux (1988)}. For $p = 1, ..., k$, let s_p denote the vector of coefficients of the estimated contrast of main effect of factor p and $\circ$ denote the component-wise product of vectors. For example, for $s_1' = (-1, 1, -1, 1, -1, 1, -1, 1)$,

$s_2' = (-1, -1, 1, 1, -1, -1, 1, 1)$ and

$s_3' = (-1, -1, -1, -1, 1, 1, 1, 1)$,

$s_1 \circ s_2 \circ s_3 = (-1, 1, 1, -1, 1, -1, -1, 1)$.

Then the component wise product

$$\prod_{p=1}^{t+1} \circ s_{j_p} = s_{j_1} \circ s_{j_2} \circ \cdots \circ s_{j_{t+1}}$$

is orthogonal to $\boldsymbol{T}_0, \boldsymbol{T}_1, ..., \boldsymbol{T}_t$, where $\boldsymbol{T}_i$ is the vector of coefficients of i^{th} order orthogonal polynomial for number of design runs n, $i = 0, 1, ..., t$.

Using these two lemmas, the algorithm to develop the factorial experiments that are linear trend-free for main effects is given in Section 3 and working of the algorithm is given in Section 4.

3. GENERATION OF TREND-FREE DESIGNS FOR FACTORIAL EXPERIMENTS

We shall henceforth denote by AL1 the algorithm that generates designs for complete factorial experiment in single replication that are linear trend-free for estimation of main effects and also identifies the 2- and 3- factor interactions that are estimable free from linear trend effect. AL1 is described in the sequel.

I. The Method AL1

I1 Let the number of factors be k; number of treatment combinations be $n = 2^k$, n is even.

I2 Generate an $n \times k$ array with $n/2$ symbols as +1 and $n/2$ symbols as −1 in each column. The generation of columns is explained in the following steps.

I3 The j^{th} column of the array contains 2^{j-1} replications of the symbols −1 and +1 alternatively; $j = 1, 2, ..., k$. Thus, the first column contains symbols −1 and +1 alternatively and the k^{th} column contains $n/2$ times symbol −1 and $n/2$ times symbol +1.

I4 In the generated $n \times k$ array, the k columns correspond to the coefficients of the contrasts corresponding to the k main effects.

I5 The n rows of $n \times k$ matrix represent n treatment combinations in lexicographic order.

Now we describe the steps to convert the full factorial generated in I3 into a linear trend-free for main effects design by using the steps given below. Let $s_1, s_2, ..., s_k$ denote the k columns of the array with each column denoting the coefficients of contrasts for main effects. We shall denote by $\prod_{j=1}^{k} \circ s_j$ the component wise product of symbols. For example, if $k = 2$, then $s_1' = (-1\ +1\ -1\ +1)$, $s_2' = (-1\ -1\ +1\ +1)$ and $(s_1 \circ s_2)' = (+1\ -1\ -1\ +1)$.

I6 **Case I: k is odd.**

For $i = 1, 2, ..., k-1$, define $A_i = \prod_{j(\neq i)=1}^{k} \circ s_j$, and $A_k = \prod_{j=1}^{k} \circ s_j$. Then

$$\boldsymbol{X} = [A_1 \quad A_2 \quad \cdots \quad A_i \quad \cdots \quad A_k]$$

is the required linear trend-free design for main effects for a 2^k factorial experiment.

I7 Case II: *k* is even.

For $i = 1, 2, ..., k$, define $A_i = \prod_{j(\neq i)=1}^{k} \circ s_j$. Then

$$\boldsymbol{X} = [A_1 \quad A_2 \quad \cdots \quad A_i \quad \cdots \quad A_k]$$

is the required linear trend-free design for main effects for a 2^k factorial experiment.

We further describe steps to identify linear trend-free two-factor and three-factor interactions in the design generated in I6 or I7.

I8 From the linear trend-free for main effects design generated in I6 or I7 generate a new design matrix $n \times \left[k + \binom{k}{2} + \binom{k}{3}\right]$ given by $\mathbf{Z} = [\boldsymbol{X} \quad \boldsymbol{X}^{(2)} \quad X^{(3)}]$. Here $\boldsymbol{X}^{(u)}$, $u = 2, 3$ contains columns corresponding to the coefficients of the contrasts of all the $\binom{k}{u}$ u-factors interactions obtained from X.

I9 For $u = 2, 3$, identify the columns in $\boldsymbol{X}^{(u)}$ that are linear trend-free. Then the corresponding u-factor interactions are linear trend-free. Further, identify the columns from the remaining columns in $\boldsymbol{X}^{(u)}$ that satisfy the condition in (3). Then the corresponding u-factor interactions are nearly linear trend-free. The remaining columns are not trend-free.

Remark 3.1: The algorithm AL1 is infact a combination of steps provided by Chen and Jacroux (1988) as stated above in Lemma 2.2 and clarification regarding odd and even number of factors given by Hinkelman and Jo (1998).

4. WORKING OF AL1

Consider the problem of constructing a linear trend-free design for main effects for a 2^4 factorial experiment with four factors as A, B, C and D. Using the method AL1, first obtain an 16 × 4 array using step I3. The four columns of the array correspond to

Table 4.1(a)

s_1	s_2	s_3	s_4	Treatment combinations
–1	–1	–1	–1	(1)
1	–1	–1	–1	*a*
–1	1	–1	–1	*b*
1	1	–1	–1	*ab*
–1	–1	1	–1	*c*
1	–1	1	–1	*ac*
–1	1	1	–1	*bc*
1	1	1	–1	*abc*
–1	–1	–1	1	*d*
1	–1	–1	1	*ad*
–1	1	–1	1	*bd*
1	1	–1	1	*abd*
–1	–1	1	1	*cd*
1	–1	1	1	*acd*
–1	1	1	1	*bcd*
1	1	1	1	*abcd*

Table 4.1(b)

A	*B*	*C* Design *X*	*D*	Treatment combinations
–1	–1	–1	–1	(1)
–1	1	1	1	*bcd*
1	–1	1	1	*acd*
1	1	–1	–1	*ab*
1	1	–1	1	*abd*
1	–1	1	–1	*ac*
–1	1	1	–1	*bc*
–1	–1	–1	1	*d*
1	1	1	–1	*abc*
1	–1	–1	1	*ad*
–1	1	–1	1	*bd*
–1	–1	1	–1	*c*
–1	–1	1	1	*cd*
–1	1	–1	–1	*b*
1	–1	–1	–1	*a*
1	1	1	1	*abcd*

the coefficients of the contrasts of the four main effects. The full description of the generated array is given below as Table 4.1(a). We denote the contrasts of main effects as s_1, s_2, s_3, and s_4.

In this example k (=4) is even. So we use Step I7 to generate linear trend-free design for main effects. This design is given in Table 4.1(b) as Design $\boldsymbol{X}$.

In Design $\boldsymbol{X}$, all the main effects are linear trend-free; some other factorial effects, however, may also be linear trend-free (Lemma 2.1). Since the interest of the experimenter is to estimate lower order interactions more precisely, we search for two and three factor interactions that are linear trend-free by using steps I8 and I9 of AL1. Using step I8, we first generate design matrix $\boldsymbol{Z}$ for main effects, two-factor and three factor interactions. The design $\boldsymbol{Z}$ is given in Table 4.2.

Using now Step I9 we identify two-factor and three-factor interactions that are linear trend-free or are nearly linear trend-free. It is found that all the six two-factor interactions *AB*, *AC*, *AD*, *BC*, *BD* and *CD* are linear trend-free. On the other hand, only one three factor interaction, *BCD*, satisfies condition (3) and is, therefore, nearly linear trend-free. Other three-factor interactions are neither linear nor nearly linear trend-free.

Table 4.3 gives an example of a design $\boldsymbol{Z}$ for 2^5 factorial which is linear trend-free for all main effects, all two-factor and three-factor interactions.

Table 4.2. Design Z for 2^4 factorial

Z													
X				$X^{(2)}$						$X^{(3)}$			
A	*B*	*C*	*D*	*AB*	*AC*	*AD*	*BC*	*BD*	*CD*	*ABC*	*ABD*	*ACD*	*BCD*
–1	–1	–1	–1	1	1	1	1	1	1	–1	–1	–1	–1
–1	1	1	1	–1	–1	–1	1	1	1	–1	–1	–1	1
1	–1	1	1	–1	1	1	–1	–1	1	–1	–1	1	–1
1	1	–1	–1	1	–1	–1	–1	–1	1	–1	–1	1	1
1	1	–1	1	1	–1	1	–1	1	–1	–1	1	–1	–1
1	–1	1	–1	–1	1	–1	–1	1	–1	–1	1	–1	1
–1	1	1	–1	–1	–1	1	1	–1	–1	–1	1	1	–1
–1	–1	–1	1	1	1	–1	1	–1	–1	–1	1	1	1
1	1	1	–1	1	1	–1	1	–1	–1	1	–1	–1	–1
1	–1	–1	1	–1	–1	1	1	–1	–1	1	–1	–1	1
–1	1	–1	1	–1	1	–1	–1	1	–1	1	–1	1	–1
–1	–1	1	–1	1	–1	1	–1	1	–1	1	–1	1	1
–1	–1	1	1	1	–1	–1	–1	–1	1	1	1	–1	–1
–1	1	–1	–1	–1	1	1	–1	–1	1	1	1	–1	1
1	–1	–1	–1	–1	–1	–1	1	1	1	1	1	1	–1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 4.3 Linear trend-free for main effects, 2-factors and 3-factors interactions design for Complete Factorial Experiment

	A	B	C	D	E
abcd	1	1	1	1	–1
ae	1	–1	–1	–1	1
be	–1	1	–1	–1	1
cd	–1	–1	1	1	–1
ce	–1	–1	1	–1	1
bd	–1	1	–1	1	–1
ad	1	–1	–1	1	–1
abce	1	1	1	–1	1
de	–1	–1	–1	1	1
bc	–1	1	1	–1	–1
ac	1	–1	1	–1	–1
abde	1	1	–1	1	1
ab	1	1	–1	–1	–1
acde	1	–1	1	1	1
bcde	–1	1	1	1	1
(1)	–1	–1	–1	–1	–1
e	–1	–1	–1	–1	1
bcd	–1	1	1	1	–1
acd	1	–1	1	1	–1
abe	1	1	–1	–1	1
abd	1	1	–1	1	–1
ace	1	–1	1	–1	1
bce	–1	1	1	–1	1
d	–1	–1	–1	1	–1
abc	1	1	1	–1	–1
ade	1	–1	–1	1	1
bde	–1	1	–1	1	1
c	–1	–1	1	–1	–1
cde	–1	–1	1	1	1
b	–1	1	–1	–1	–1
a	1	–1	–1	–1	–1
abcde	1	1	1	1	1

Remark 4.1: The method AL1 described in Section 3 can be operated in a very simple manner to generate the same design as generated by AL1. The two cases (I) k is even and (II) k is odd are dealt separately. First we describe the method for case (I).

Case I. $n = 2^k$; k is even

Step-I Generate full factorial design for $n = 2^k$ factorial experiment in standard (lexicographic) order using the steps I1 to I3 of AL1.

Step-II Retain the even numbered letters treatment combinations as such and in the same position wherever these occur.

Step-III Replace the odd numbered letters treatment combinations by the complement letters. In other words produce another treatment combination which contains those letters not present in the original treatment combination. Retain it in the same position as that of the original one. The new design generated is the same as the one produced by AL1. For application of this simplified algorithm, see Design 1 for 2^4 factorial experiments.

Step IV Steps 1 – 3 take care of steps I1 – I5 and I7 of AL1.

Case II. $n = 2^k$; k is odd

Step-I Generate full factorial design for $n = 2^{k-1}$ factorial experiment in standard (lexicographic) order using the steps I1 to I3 of AL1.

Step-II Replace the even numbered letters treatment combinations of 2^{k-1} factorial by the complement letters. In other words produce another treatment combination which contains those letters not present in the original treatment combination. Retain it in the same position as that of the original one.

Step-III For the odd numbered letters treatment combinations augment the k^{th} letter with each odd numbered letters treatment combinations in 2^{k-1} factorial experiment. Retain these in the same position as that of the original one.

Step-IV In this way first half of the treatments combinations (2^{k-1}) of 2^k factorial are obtained. The second half of the treatment combinations is obtained by writing the complement letters of first half treatments combinations. Thus this is the desired design.

Step-V Steps I – IV take care of steps I1 – I6 of AL1.

Remark 4.2: The designs obtained above are the component wise product of the contrasts of main effects. These designs are same as generated by AL1. Thus by Lemma 2.2 these designs are not only linear trend-free for main effects but are also linear trend-free for higher orders interactions. Lemma 2.1 will be applicable on these designs.

In the next section we give an improved version of the method AL1 to generate designs for confounded two-level factorial experiments that are linear trend-free for all the main effects and also for some of the 2- and 3-factors interactions.

5. METHOD AL2 FOR CONFOUNDED FACTORIAL EXPERIMENTS

In this section we consider the problem of obtaining confounded two-level factorial designs in the presence of linear trends within blocks. The designs obtained are trend-free for main effects and some 2-factor and 3-factor interactions. Consider a $(2^k, 2^{k-p})$ factorial experiment run in $b = 2^p$ blocks of size $m = 2^{k-p}$. Suppose that p independent factorial effects are confounded in each replication. We assume the following linear additive model:

$$\boldsymbol{y} = \boldsymbol{U}\beta + \boldsymbol{B}\gamma + \boldsymbol{T}\theta + \boldsymbol{e} \qquad (4)$$

$$E(\boldsymbol{e}) = 0,\ D(\boldsymbol{e}) = \sigma^2 \boldsymbol{I},$$

where $\boldsymbol{B}$ is the $n \times b$ design matrix of observations versus blocks and γ is a b-component vector of block parameters; the other symbols are same as defined in model (1). $\boldsymbol{T} = \boldsymbol{1}_b \otimes t$ and t is the $(m \times 1)$ linear trend vector for blocks of size m and b is the number of blocks. The condition for main effects to be linear trend-free is $\boldsymbol{X}_1' \boldsymbol{T} = \boldsymbol{0}$. For $s = 1, 2, ..., k$, the matrix $\boldsymbol{X}_s$ is partitioned as $\boldsymbol{X}_s = [\boldsymbol{X}_{s1}' \quad \boldsymbol{X}_{s2}' \cdots \ \boldsymbol{X}_{sb}']'$. To obtain the desired design that is linear trend-free for main effects and for some of the 2- and 3- factor interactions, we have the following method AL2:

I. The method AL2

L1 Generate full factorial design for $n = 2^k$ factorial experiment in standard (lexicographic) order using the steps I1 to I3 of AL1. Replace the symbols –1 and +1 in each column by 0 and 1, respectively.

L2 Fix the p factorial effects to be confounded.

L3 Solve the following 2^p equations for the p chosen contrasts (factorial effects) to be confounded

$$\boldsymbol{Ax} = \boldsymbol{0} \text{ and } \boldsymbol{Ax} = \boldsymbol{1}$$

where $\boldsymbol{x}' = (x_1, x_2, ..., x_k)$ denotes the k factors in the experiment and $\boldsymbol{A} = (a_{ij})$ is a $p \times k$ matrix of known coefficients a_{ij}'s, $i = 1, 2, ..., p$; $j = 1, 2, ..., k$. $a_{ij} = 1$ or 0 depending upon whether the j^{th} factor is present or absent in the ith factorial effect confounded. 0 is a p component vector of zeros and 1 is a p component vector of all ones.

L4 Step L3 generates $b = 2^p$ blocks of size 2^{k-p} each.

We now describe steps to convert the confounded design generated in step L2 into a linear trend-free design for main effects.

L5 The treatment combinations within the $b = 2^p$ blocks maintain the same order of sequence as in the lexicographic order of complete factorial experiment in step L1 with symbols as 0 and 1 instead of –1 and +1. Again replace the symbols 0 and 1 by –1 and +1, respectively.

L6 Let $s_1^l,\ s_2^l, ...,\ s_k^l$ denote the k columns of the l^{th} block generated in step L3, $l = 1, 2, ..., b$. Perform steps I6 and I7 on each of the b blocks separately. Let $\boldsymbol{X}_{(l)}$ denote the matrix of coefficients generated for the l^{th} block by using this step.

L7 Then $\boldsymbol{X} = [\boldsymbol{X}_{(1)}' \quad \boldsymbol{X}_{(2)}' \cdots \ \boldsymbol{X}_{(b)}']'$ is the required linear trend-free design for main effects.

We further describe steps to identify linear trend-free two-factor and three-factor interactions in the design generated in L6 and L7.

L8 From the linear trend-free for main effects design generated in L6 and L7 generate a new design $n \times \left[k + \binom{k}{2} + \binom{k}{3}\right]$ given by $Z = \left[X\ X^{(2)}\ X^{(3)}\right]$.

Here $X^{(u)}$, $u = 2, 3$ contains columns corresponding to the coefficients of the contrasts of all the $\binom{k}{u}$ u-factors interactions obtained from X.

L9 For $u = 2, 3$, identify the columns in $X^{(u)}$ that are linear trend-free. Then the corresponding u-factor interactions are linear trend-free. Further, identify the columns in $X^{(u)}$ that satisfy the condition in (3). Then the corresponding u-factor interactions are nearly linear trend-free. The remaining columns are not trend-free.

6. WORKING OF AL2

Consider again the problem of constructing a linear trend-free for all main effects design for a 2^4 confounded factorial experiment obtained by confounding the highest order interaction *ABCD*. Using step I3 of algorithm AL1, we can obtain an 16×4 array as in Table 4.1(a). Then use of step L3 of AL2 requires solving the following two equations

$$s_1 + s_2 + s_3 + s_4 = 0 \bmod(2)$$

$$s_1 + s_2 + s_3 + s_4 = 1 \bmod(2)$$

The two blocks obtained are given in Design 6.1.

Design 6.1

Block - 1

A	*B*	*C*	*D*	Treatment combinations
–1	–1	–1	–1	(1)
1	1	–1	–1	*ab*
1	–1	1	–1	*ac*
–1	1	1	–1	*bc*
1	–1	–1	1	*ad*
–1	1	–1	1	*bd*
–1	–1	1	1	*cd*
1	1	1	1	*abcd*

Block - 2

A	*B*	*C*	*D*	Treatment combinations
1	–1	–1	–1	*a*
–1	1	–1	–1	*b*
–1	–1	1	–1	*c*
1	1	1	–1	*abc*
–1	–1	–1	1	*d*
1	1	–1	1	*abd*
1	–1	1	1	*acd*
–1	1	1	1	*bcd*

Using steps L4, L5, L6 and L7 of AL2 gives the two blocks of a linear trend-free for main effects design for 2^4 factorial experiment in which *ABCD* is confounded. The design is given in Design 6.2.

Design 6.2

Block - 1

A	*B*	*C*	*D*	Treatment combinations
		$X_{(1)}$		
–1	–1	–1	–1	(1)
1	1	–1	–1	*ab*
1	–1	1	–1	*ac*
–1	1	1	–1	*bc*
1	–1	–1	1	*ad*
–1	1	–1	1	*bd*
–1	–1	1	1	*cd*
1	1	1	1	*abcd*

Block - 2

A	*B*	*C*	*D*	Treatment combinations
		$X_{(2)}$		
–1	1	1	1	*bcd*
1	–1	1	1	*acd*
1	1	–1	1	*abd*
–1	–1	–1	1	*d*
1	1	1	–1	*abc*
–1	–1	1	–1	*c*
–1	1	–1	–1	*b*
1	–1	–1	–1	*a*

and $X' = \left[X'_{(1)} \quad X'_{(2)} \right]$.

We now identify two- and three- factor interactions that are linear trend-free in Design 6.2. Using steps L7, L8 and L9 of AL2 gives that all the two factor interactions i.e. *AB*, *AC*, *AD*, *BC*, *BD* and *CD* are linear trend-free. Similarly, among three factor interactions, *BCD* is linear trend-free, *ACD* is nearly linear trend-free and the other two *ABC* and *ABD* are neither linear trend-free nor nearly linear trend-free.

Using AL1 and AL2, one can obtain designs of complete factorial and confounded factorial experiments for any number of factors k (≥ 3) that are linear trend-free for main effects and can identify two- and three- factor interactions that are linear/nearly linear trend-free.

We give below another example of 2^5 confounded design in which some of the two factor interactions are nearly trend-free.

Example 6.1: 2^5 Linear trend-free design for Confounded Factorial Experiment (*ABCDE* confounded)

Block - 1

A	*B*	*C*	*D*	*E*
–1	–1	–1	–1	–1
1	1	–1	–1	–1
1	–1	1	–1	–1
–1	1	1	–1	–1
1	–1	–1	1	–1
–1	1	–1	1	–1
–1	–1	1	1	–1
1	1	1	1	–1
1	–1	–1	–1	1
–1	1	–1	–1	1
–1	–1	1	–1	1
1	1	1	–1	1
–1	–1	–1	1	1
1	1	–1	1	1
1	–1	1	1	1
–1	1	1	1	1

Block - 2

A	*B*	*C*	*D*	*E*
1	–1	–1	–1	–1
–1	1	–1	–1	–1
–1	–1	1	–1	–1
1	1	1	–1	–1
–1	–1	–1	1	–1
1	1	–1	1	–1
1	–1	1	1	–1
–1	1	1	1	–1
–1	–1	–1	–1	1
1	1	–1	–1	1
1	–1	1	–1	1
–1	1	1	–1	1
1	–1	–1	1	1
–1	1	–1	1	1
–1	–1	1	1	1
1	1	1	1	1

Note: In this design all main effects, 2-factor and 3-factor interactions are linear trend-free except the interaction BE which is nearly linear trend-free. Interactions CE and DE are not linear/nearly linear trend-free.

Remark 6.1: For obtaining a linear trend-free for main effects design for complete factorial experiments, a simple version of AL1 is given in Remark 4.1. It needs to be further investigated whether such a simplification of AL2 is possible for obtaining a linear trend-free for main effects confounded factorial design.

7. DISCUSSION

The algorithms AL1 and AL2 have been translated in Microsoft Visual C++ program and these programmes have been used in computer aided generation of trend-free designs for the desired factorial experiment for any number of factors k (≥ 3). The catalogue of the designs obtained for 2^k factorial experiment, for $k = 3, \ldots, 7$ (for both without and with confounding factorial experiments) that are linear trend-free for main effects are available with the authors and can be obtained by sending an E-mail to klkalra@gmail.com.

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