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Testing the Scale Parameter of an Exponential Distribution with known Coefficient of Variation in a Type II Right Censored Situation — Conditional Approach

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SUMMARY

The two-parameter exponential distribution $E(\mu, \theta)$, $\theta > 0$, is a well known probability model used for life-length studies owing to the useful description of observed variation it gives for many real life situations. In the context of life-length studies, the location parameter μ and the scale parameter θ respectively represent the minimum guaranteed life and the average excess life of an equipment or system. The parameters μ and θ are functionally unrelated and the statistical inference about these parameters make use of the existence of complete minimal sufficient statistics. This brings about a substantial simplification in the inferential problems. There, however, exist situations where the average life θ depends on the guaranteed life μ and the functionally independent nature of the parameters no longer hold, resulting in the loss of optimal properties of the statistics. In this situation, the two-parameter model reduces to a one-parameter model $E(a\theta, \theta)$, where 'a' is known. Ironically though the reduced model looks simplified with a single parameter θ, however, from the inference point of view, the problem of inference about θ becomes complicated. Several authors including the present author, have studied this kind of inference problem. In the present paper, the problem of testing a simple hypothesis about θ in the reduced model $E(a\theta, \theta)$ has been studied in the type II right censored situation from a conditional view point.

Keywords : Conditional UMP test, Minimal sufficient statistic, Power of unconditional test.

1. INTRODUCTION

The exponential distribution $E(a\theta, \theta)$ with p.d.f. given by

$$f(x\ ;\ \theta) = \frac{1}{\theta}\exp\left[\frac{-(x - a\theta)}{\theta}\right], x \geq a\theta \qquad (1.1)$$

where $\theta > 0$ is the unknown parameter and $a > 0$ is a known constant, has generated interest in the recent past owing to the scope it has offered to the existing procedures of statistical inference. For this distribution the coefficient of variation is known and is given by $100/(1 + a)$. This arises as a result of a functional relationship between the location (μ) and scale (θ) parameters of the original two-parameter exponential distribution $E(\mu, \theta)$. The interest for inferentialists comes from the fact that in the reduced model $E(a\theta, \theta)$, the inferential procedures instead of getting simplified becomes more intricate. The property of completeness enjoyed by the statistics in the case of $E(\mu, \theta)$ distribution no longer holds for that of $E(a\theta, \theta)$, as the standard theory of UMVUE in not applicable in this case.

In this background, two distinct approaches have been distinguished with respect to the choice of the reference set against which performances have to be

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evaluated. These are the unconditional and conditional approaches having their own merits/limitations. Naturally therefore, various workers have approached differently the inference problem of $E(a\theta, \theta)$. The attractive feature of the conditional viewpoint is that conditional models tends to be simpler than the original unconditional ones and frequently brings about simplification of theory (Lehmann 1986).

While Ebrahimi (1985), Ghosh and Razmpour (1984), Joshi and Nabar (1991) and Joshi and Sathe (1983) used the unconditional approach, Handa *et al.* (2002) and Samanta (1985) followed the conditional approach to study the inference problems. In general the unconditional approaches adopted by these workers produced adhoc, approximate or sub-optimal procedures whereas the conditional approaches produced conditionally optimal procedures, probably because the conditional procedures make use of ancillary information. In the next section, we show how this could be done.

For testing the simple hypothesis

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta = \theta_1 \; (> \theta_0) \qquad (1.2)$$

Ebrahimi (1985) developed a test which was approximate because the distribution of the test statistic turned out to be complicated. Handa *et al.* (2002) using the conditional approach, developed a *conditional UMP test* for testing (1.2). They also investigated some interesting properties possessed by the distribution (1.1) like (i) the conditional superiority of the conditional test over its unconditional counterpart (Ebrahimi 1985), (ii) the property of conditional completeness, (iii) choice of ancillary statistics and (iv) large sample approximations. Their test, however, was based on a *complete sample* of size n.

There do exist numerous practical situations where complete samples are either unavailable or undesirable. For example, in life testing, fatigue testing and other kinds of tests of destructive nature, where data become available in an ordered manner, one can choose to discontinue experimentation after one has observed the first r observations. There are obvious advantages for choosing such a course, such as, one might be able to reach a decision in a shorter time or with fewer observations, than observing all items under test.

Let $X_{(1)} < X_{(2)} < \ldots < X_{(r)}$ denote the ordered statistics of a random sample of size n from the distribution (1.1) where $r \le n$. When only the first r observations are made or become available, the sample is usually termed a *right censored sample* (Epstein and Sobel (1954) and Tiku *et al.* (1986)). The objective of this paper is to extend the conditional UMP test developed by Handa *et al.* (2002) for complete samples to tests based on a right censored sample.

2. THE CONDITIONAL UMP TEST

It is well known that $(X_{(1)}, L)$ is a minimal sufficient statistic for θ, where

$$L = \sum_{i=2}^{r} (X_{(i)} - X_{(1)}) + (n-r)(X_{(r)} - X_{(1)})$$

Define

$$C = \frac{X_{(1)}}{X_{(1)} + r^{-1} L} \qquad (1.3)$$

Then C is an ancillary statistic and $(X_{(1)}, C)$ is also minimal sufficient. We now explore the existence of a conditional monotone likelihood ratio (MLR) for the family of conditional densities of $X_{(1)}$, given the ancillary C. For this purpose, we derive the conditional distribution of $X_{(1)}$, given C, in the following lemma.

Lemma 1. The conditional p.d.f. of $X_{(1)}$, given C is

$$f_{X_{(1)}|C}\left(x|c,\theta\right) = \begin{cases} \dfrac{\left[\dfrac{r+(n-r)c}{c\theta}\right]^r \exp\left[-\left\{\dfrac{r+(n-r)c}{c\theta}\right\}x\right] x^{r-1}}{J\left[\dfrac{a}{c}\left\{r+(n-r)c\right\}, r\right]}, & x \ge a\theta \\ 0, & \text{otherwise} \end{cases} \qquad (1.4)$$

where $J(\alpha, \beta) = \int_{\alpha}^{\infty} e^{-t} t^{\beta-1} dt = \tau(\beta) \Pr\left\{\chi^2(2\beta) > 2\alpha\right\}$

and $\tau(\beta)$ is the gamma function of β.

Proof. The joint p.d.f. of $X_{(1)}$ and L is given by

$$f_{X_{(1)},L}(x, y|\theta) = \frac{n}{\theta^r \tau(r-1)} \exp\left[-\frac{n(x-a\theta)}{\theta} + \frac{y}{\theta}\right] y^{r-2}, \quad x \ge a\theta, \; y > 0$$

Applying the transformation

$$U = X_{(1)},\ \ C = X_{(1)} \big/ (X_{(1)} + r^{-1}L)$$

the joint p.d.f. of U and C is obtained as follows:

$$f_{U,C}(u, c/\theta) = \frac{nr^{r-1}e^{na}(1-c)^{r-2}}{\theta^r c^r \tau(r-1)} \exp\left[-\left\{\frac{r+(n-r)c}{c\theta}\right\}u\right] u^{r-1},$$

$$u \geq a\theta,\ \ 0 < c < 1$$

This yields the marginal p.d.f. of C as

$$f_C(c) = \frac{nr^{r-1}e^{na}(1-c)^{r-2}}{\tau(r-1)\{r+(n-r)c\}^r} J\left[\left(\frac{r+(n-r)c}{c}\right)a,\ r\right],\quad 0 < c < 1 \qquad (1.6)$$

The lemma now follows from (1.5) and (1.6).

Now in the next lemma, we prove that the family of conditional densities given by (1.4) has an MLR in $X_{(1)}$.

Lemma 2. The family of conditional densities $\{f_{X_{(1)}|C}(x/c,\theta),\ \theta > 0\}$ given by (1.4) has an MLR in $X_{(1)}$, given C.

Proof. For any $\hat{\theta} > \theta$, we have, on using Lemma 1:

$$\frac{f_{X_{(1)}|C}(x/c,\hat{\theta})}{f_{X_{(1)}|C}(x/c,\theta)} = \frac{\left(\frac{r+(n-r)c}{c\hat{\theta}}\right)^r e^{-\left(\frac{r+(n-r)c}{c\hat{\theta}}\right)u} I(u > a\hat{\theta})}{\left(\frac{r+(n-r)c}{c\theta}\right)^r e^{-\left(\frac{r+(n-r)c}{c\theta}\right)u} I(u > a\theta)}$$

$$= \left(\frac{\theta}{\hat{\theta}}\right)^r e^{\frac{r+(n-r)c}{c}\left(\frac{1}{\theta} - \frac{1}{\hat{\theta}}\right)u} b(u)$$

where I(•) is the indicator function of the set (•) and

$$b(u) = I(u > a\hat{\theta}) / I(u > a\theta)$$

$$= \begin{cases} 1, & \text{if } u > a\hat{\theta} \\ 0, & \text{if } a\theta < u < a\hat{\theta} \end{cases}$$

Define $b(u) = 0$ if $u < a\theta$. Then it follows that $f_{X_{(1)}|C}(u|c,\hat{\theta}) \big/ f_{X_{(1)}|C}(u|c,\theta)$ is a non-decreasing function of $X_{(1)}$ and the family of conditional densities given by (1.4) has an MLR in $X_{(1)}$ conditionally on C. The existence of a UMP conditional test for testing (1.2) follows by Theorem 2, pp. 78, Lehmann (1986). We state this result in the following theorem.

Theorem 1. The conditional size α test, given C, for testing

$$H_0\colon \theta \leq \theta_0 \ \text{ against } \ H_1\colon \theta > \theta_0$$

given by

reject H_0 if $X_{(1)} > K(c)$ is a UMP test.

3. THE POWER OF THE CONDITIONAL UMP TEST

The implementation of this UMP conditional test requires the computation of $K(c)$ as well as the test's power function. The next lemma presents these.

Lemma 3. For the testing problem stated in Theorem 1,

(i) for a UMP test of size α, $0 < \alpha < 1$, $K(c)$ is the solution of the equation

$$J\left[\left(\frac{r+(n-r)c}{c\theta_0}\right)K(c),\ r\right] = \alpha\, J\left[\left(\frac{r+(n-r)c}{c}\right)a,\ r\right] \qquad (1.7)$$

(ii) the power function of the UMP conditional test at θ is given by

$$\gamma(\theta|c) = \begin{cases} \dfrac{J\left[\left(\frac{r+(n-r)c}{c\theta}\right)K(c),\ r\right]}{J\left[\left(\frac{r+(n-r)c}{c}\right)a,\ r\right]}, & \text{if } \theta < \dfrac{K(c)}{a} \\ 1, & \text{if } \theta \geq \dfrac{K(c)}{a} \end{cases} \qquad (1.8)$$

where $K(c)$ is the solution of equation (1.7).

Proof.

(i) We have $P_r(X_{(1)} > K_{(c)} \mid c, \theta_0) = \alpha$

Using Lemma 1, we have

$$\int_{K(c)}^{\infty} \left(\frac{r+(n-r)c}{c\theta_0}\right)^r e^{-\left(\frac{r+(n-r)c}{c\theta_0}\right)u} u^{r-1} du$$

$$= \alpha \, J\left[\left(\frac{r+(n-r)c}{c}\right)a,\ r\right]$$

Change of variable by $t = \left(\frac{r+(n-r)c}{c\theta_0}\right)u$

yields the required result.

(ii) The power at θ is given by

$$\gamma(\theta|c) = \begin{cases} \Pr\left(X_{(1)} > K(c) \mid c, \theta\right), & \text{if } \theta < \frac{K(c)}{a} \\ \Pr\left(X_{(1)} > a\theta \mid c, \theta\right), & \text{if } \theta \geq \frac{K(c)}{a} \end{cases}$$

$$= \begin{cases} \dfrac{\int_{K(c)}^{\infty} \left(\frac{r+(n-r)c}{c\theta}\right)^r e^{-\left(\frac{r+(n-r)c}{c\theta}\right)u} u^{r-1} du}{J\left[\left(\frac{r+(n-r)c}{c}\right)\text{a, r}\right]}, & \text{if } \theta < \frac{K(c)}{a} \\ 1, & \text{if } \theta \geq \frac{K(c)}{a} \end{cases}$$

Transformation $t = \left(\frac{r+(n-r)c}{c\theta}\right)u$ yields (1.8).

4. COMPARISON OF CONDITIONAL AND UNCONDITIONAL TESTS

Comparison is done numerically between our proposed conditional test with the existing unconditional test of Ebrahimi (1985) for the censored case in terms of the power criterion. (The comparison of the tests in the case of complete samples has already been dealt with by Handa *et al.* (2002). Before carrying out the comparison, however, the critical point $K(c)$ of the conditional test is to be obtained from equation (1.7). Since the equation involves incomplete gamma functions, the solution for $K(c)$ has been obtained numerically by using the Newton-Raphson's method for which standard computer routines are available. Next, we have computed range of values of c for which the power of the conditional test dominates the power of the unconditional test. The power values were extensively computed for various values of the parameters for the UMP conditional test from (1.8) and for the unconditional test, from Theorem 1 of Ebrahimi (1985). The numerical comparison revealed that there exists an interval $(0, c')$ such that the power $\gamma(\theta \mid c)$ of the conditional test uniformly exceeds the power of $\gamma(\theta)$ of the unconditional test when $c \in (0, c')$ and the upper end point of $(0, c')$ moves closer to unity as a increases, implying that higher the value of a (which is the same as a smaller coefficient of variation), the more effective is an ancillary statistic in providing conditional inference about the parameter θ. It was also noted that lesser the value of c, more was the power of the conditional test.

Table 1 gives the value of c' for some chosen values of n, r, a, θ_0 and α. It also gives the values of

Table 1. Values of c' for interval $(0, c')$ of domination of conditional test over unconditional test

$\theta_0 = 1.0$, $\alpha = 0.05$

n	r	a	c'	c''
3	2	0.4	0.8	-
		0.7	*	-
	3	0.4	*	0.3
		0.7	*	0.5
5	2	0.4	*	-
		0.7	*	-
	3	0.4	*	-
		0.7	*	-
	4	0.4	*	-
		0.7	*	-
	5	0.4	*	0.3
		0.7	*	0.4
7	2	0.4	*	-
		0.7	*	-
	3	0.4	*	-
		0.7	*	-
	4	0.4	*	-
		0.7	*	-
	5	0.4	*	-
		0.7	*	-
	6	0.4	*	-
		0.7	*	-
	7	0.4	*	0.3
		0.7	*	0.4

* The whole interval (0,1)

Table 2. Comparison of power values of conditional and unconditional tests

$\theta_0 = 1.0,\ \theta_1 = 1.1(0.9)1.9,\ \alpha = 0.05$

n	r	a	c	Test	Power values at various values of θ_1								
					1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
3	2	0.4	0.8	Conditional test	0.0753	0.1053	0.1392	0.1762	0.2156	0.2565	0.2983	0.3405	0.3827
			-	Unconditional test	0.0750	0.1041	0.1363	0.1707	0.2064	0.2427	0.2788	0.3144	0.3492
		0.7	0.9	Conditional test	0.0799	0.1170	0.1610	0.2110	0.2657	0.3244	0.3859	0.4494	0.5141
			-	Unconditional test	0.0781	0.1116	0.1494	0.1901	0.2325	0.2755	0.3182	0.3600	0.4005
	3	0.4	0.9	Conditional test	0.0770	0.1092	0.1454	0.1847	0.2258	0.2679	0.3102	0.3521	0.3931
			-	Unconditional test	0.0763	0.1073	0.1418	0.1789	0.2173	0.2563	0.2952	0.3333	0.3704
		0.7	0.9	Conditional test	0.0804	0.1181	0.1621	0.2111	0.2637	0.3188	0.3753	0.4322	0.4888
			-	Unconditional test	0.0788	0.1133	0.1524	0.1945	0.2383	0.2827	0.3268	0.3699	0.4114
5	2	0.4	0.9	Conditional test	0.7890	0.1145	0.1563	0.2033	0.2545	0.3090	0.3659	0.4244	0.4837
			-	Unconditional test	0.0775	0.1102	0.1469	0.1863	0.2273	0.2688	0.3102	0.3507	0.3900
		0.7	0.9	Conditional test	0.0891	0.1426	0.2114	0.2951	0.3928	0.5030	0.6242	0.7547	0.8927
			-	Unconditional test	0.0824	0.1224	0.1682	0.2179	0.2697	0.3219	0.3734	0.4231	0.4704
	3	0.4	0.9	Conditional test	0.0796	0.1159	0.1580	0.2045	0.2543	0.3061	0.3589	0.4119	0.4644
			-	Unconditional test	0.0830	0.1121	0.1503	0.1913	0.2340	0.2772	0.3202	0.3622	0.4027
		0.7	0.9	Conditional test	0.0876	0.1379	0.2007	0.2747	0.3583	0.4497	0.5469	0.6481	0.7518
			-	Unconditional test	0.0827	0.1232	0.1696	0.2200	0.2725	0.3254	0.3774	0.4276	0.4754
	4	0.4	0.9	Conditional test	0.0813	0.1198	0.1643	0.2131	0.2645	0.3173	0.3701	0.4222	0.4727
			-	Unconditional test	0.0795	0.1152	0.1556	0.1993	0.2446	0.2906	0.3361	0.3804	0.4231
		0.7	0.9	Conditional test	0.0872	0.1361	0.1959	0.2645	0.3401	0.4203	0.5033	0.5874	0.6711
			-	Unconditional test	0.0833	0.1247	0.1722	0.2239	0.2776	0.3317	0.3849	0.4361	0.4846
	5	0.4	0.9	Conditional test	0.0835	0.1254	0.1738	0.2267	0.2821	0.3382	0.3936	0.4473	0.4985
			-	Unconditional test	0.0811	0.1191	0.1624	0.2093	0.2581	0.3074	0.3560	0.4032	0.4484
		0.7	0.9	Conditional test	0.0876	0.1367	0.1958	0.2624	0.3342	0.4087	0.4839	0.5582	0.6304
			-	Unconditional test	0.0841	0.1267	0.1758	0.2292	0.2876	0.3405	0.3952	0.4477	0.4973
7	4	0.4	0.9	Conditional test	0.0837	0.1264	0.1769	0.2334	0.2940	0.3571	0.4212	0.4851	0.5477
			-	Unconditional test	0.0814	0.1198	0.1636	0.2110	0.2604	0.3103	0.3595	0.4072	0.4527
		0.7	0.9	Conditional test	0.0948	0.1587	0.2419	0.3433	0.4603	0.5901	0.7294	0.8753	1.0000
			-	Unconditional test	0.0840	0.1341	0.1890	0.2488	0.3108	0.3727	0.4328	0.4899	0.5431
	5	0.4	0.9	Conditional test	0.0850	0.1295	0.1817	0.2395	0.3005	0.3630	0.4252	0.4859	0.5442
			-	Unconditional test	0.0825	0.1225	0.1684	0.2182	0.2700	0.3222	0.3736	0.4232	0.4704
		0.7	0.9	Conditional test	0.0937	0.1545	0.2316	0.3227	0.4247	0.5344	0.6486	0.7645	0.8798
			-	Unconditional test	0.0875	0.1353	0.1911	0.2520	0.3150	0.3779	0.4389	0.4966	0.5503
	6	0.4	0.9	Conditional test	0.0870	0.1343	0.1898	0.2508	0.3146	0.3789	0.4420	0.5024	0.5595
			-	Unconditional test	0.0838	0.1258	0.1742	0.2268	0.2814	0.3364	0.3903	0.4421	0.4912
		0.7	0.9	Conditional test	0.0934	0.1527	0.2265	0.3115	0.4044	0.5016	0.6000	0.6972	0.7911
			-	Unconditional test	0.0881	0.1369	0.1939	0.2560	0.3204	0.3845	0.4465	0.5050	0.5594
	7	0.4	0.9	Conditional test	0.0893	0.1402	0.2000	0.2655	0.3335	0.4013	0.4668	0.5286	0.5859
			-	Unconditional test	0.0852	0.1295	0.1807	0.2364	0.2943	0.3523	0.4090	0.4631	0.5141
		0.7	0.9	Conditional test	0.0938	0.1531	0.2257	0.3079	0.3959	0.4858	0.5747	0.6603	0.7411
			-	Unconditional test	0.0888	0.1387	0.1972	0.2609	0.3268	0.3923	0.4555	0.5150	0.5700

c'' corresponding to the cases when $r = n$, *i.e.* when samples are complete. Table 2 presents the power comparison of the conditional and the unconditional tests for various levels of censoring r. For the conditional test, only the power values corresponding to $c = 0.9$ is shown, since for values of $c < 0.9$, the power automatically exceeds the power at $c = 0.9$.

5. CONCLUSION

It is clear from Table 2 that the power of the conditional test completely dominates the power of the unconditional test for *all values of the parameters* when the samples are censored. It is pointed out here that in the complete sample case also, where the comparison was with the most powerful (MP) unconditional test of Joshi and Nabar (1991), an interval $(0, c')$ was found such that the power of the conditional test dominated the power of the unconditional MP test. Thus, it has been possible to establish the supremacy of our conditional UMP test over the unconditional test of Ebrahimi (1985) for all values of c, thus demonstrating the effectiveness of the conditional test.

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