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Canonical Efficiency Factors and Related Issues Revisited

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SUMMARY

The notion of canonical efficiency factors is re-examined in the context of arbitrary block designs, and a simple statistical interpretation of these is provided. Some related issues are also discussed.

Key words: Block designs, Efficiency factor, Orthogonal design, C-design.

1. INTRODUCTION

The notion of *canonical efficiency factors* was formally introduced by James and Wilkinson (1971), though it was at least implicitly discussed earlier by Jones (1959). Canonical efficiency factors are relevant in the context of all block designs, in particular in respect of efficiency-balanced designs and C-designs which have been studied quite extensively; see *e.g.,* Caliński (1971), Williams (1975), Puri and Nigam (1975, 1977) and Saha (1976) for initial developments and the two-volume book by Caliński and Kageyama (2000) for a more comprehensive treatment and an extensive bibliography. In this short note, we give a (possibly) more transparent statistical interpretation of the canonical efficiency factors in the context of arbitrary block designs with possibly unequal replicates and unequal block sizes. As a consequence, a statistical justification for studying C-designs is also provided.

2. RESULTS

Consider a block design d with v treatments and b blocks, wherein the i^{th} treatment is replicated r_{di} times, $1 \le i \le v$ and for $1 \le j \le b$, k_{dj} denotes the size of the j^{th} block of d. Throughout, for a positive integer s, I_s will denote an identity matrix of order s and 1_s, an s × 1 vector with all entries equal to 1. With reference to the design d, let

$$R_d = \operatorname{diag}(r_{d1}, \ldots, r_{dv}), R_d^{1/2} = \operatorname{diag}(r_{d1}^{1/2}, \ldots, r_{dv}^{1/2})$$

$$K_d = \operatorname{diag}(k_{d1}, \ldots, k_{db}), r_d = (r_{d1}, \ldots, r_{dv})'$$

$$k_d = (k_{d1}, \ldots, k_{db})', \rho_d = (r_{d1}^{1/2}, \ldots, r_{dv}^{1/2})' \tag{1}$$

Under a standard homoscedastic, fixed effects model, the coefficient matrix of the reduced intrablock normal equations for estimating linear functions of treatment effects (the well-known C-matrix) of d is given by

$$C_d = R_d - N_d K_d^{-1} N_d' \tag{2}$$

where $N_d = (n_{dij})$ is the v × b incidence matrix of d, n_{dij} being the number of times the i^{th} treatment appears in the j^{th} block of d.

It is well-known that all treatment contrasts are estimable using d if and only if d is *connected* or, equivalently, if and only if $\operatorname{Rank}(C_d) = v - 1$. Henceforth, only connected block designs are considered.

For a connected block design d with v treatments, define the v × v matrix A_d as

$$A_d = R_d^{-1/2} C_d R_d^{-1/2} \tag{3}$$

where $R_d^{-1/2}$ is the inverse of $R_d^{1/2}$. We then have the following result.

Lemma 1. For a connected design d, the following are true:

(i) $\operatorname{Rank}(A_d) = \operatorname{Rank}(C_d) = v - 1$

(ii) A_d is nonnegative definite.

(iii) The eigenvalues of A_d and $R_d^{-1}C_d$ are the same.

Proof. The result in (i) is trivially true. Let x be a v × 1 non-null vector and define $q = R_d^{-1/2}x$. Let $Q = T - N_d K_d^{-1} B$ be the vector of adjusted treatment totals, where T and B respectively denote the vector of treatment and block totals. If $\sigma^2 > 0$ is the variance of an observation, then

$$0 \le \sigma^{-2}\mathrm{Var}(q'Q) = x'R_d^{-1/2}C_d R_d^{-1/2}x = x'A_d x$$

This proves (ii). Finally, observe that for a scalar μ, $A_d - \mu I_v = R_d^{1/2}(R_d^{-1}C_d - \mu I_v)R_d^{-1/2}$; which proves (iii).

Let

$$A_d = \sum_{i=0}^{v-1} \lambda_{di}\xi_{di}\xi_{di}' \tag{4}$$

be a spectral decomposition of A_d, where $0 = \lambda_{d0} < \lambda_{d1} \le \lambda_{d2} \le ... \le \lambda_{d,v-1}$ are the eigenvalues of A_d and $\xi_{d0} = n^{-1/2}\rho_d, \xi_{d1}, ..., \xi_{d,v-1}$ are the corresponding orthonormal eigenvectors. Clearly, then

$$\sum_{i=0}^{v-1} \xi_{di}\xi_{di}' = I_v \Rightarrow \sum_{i=1}^{v-1} \xi_{di}\xi_{di}' = I_v - \rho_d\rho_d' / n \tag{5}$$

The positive eigenvalues of the matrix A_d, namely λ_{di}, $1 \le i \le v-1$ are known as the *canonical efficiency factors of* d. By part (iii) of Lemma 1, the canonical efficiency factors are also the positive eigenvalues of the matrix $R_d^{-1}C_d$. In particular, for equireplicate designs (i.e., when $r_{di} = r$ (say), for all i; $1 \le i \le v$), the canonical efficiency factors are simply 1/r times the positive eigenvalues of C_d, the C-matrix of d.

Using (3) and (4), we can write

$$C_d = R_d^{1/2}\left(\sum_{i=1}^{v-1} \lambda_{di}\xi_{di}\xi_{di}'\right)R_d^{-1/2} \tag{6}$$

The following result can be proved by invoking the definition of a generalized inverse.

Lemma 2. A generalized inverse of C_d is given by

$$C_d^- = R_d^{-1/2}\left(\sum_{i=1}^{v-1} \lambda_{di}^{-1}\xi_{di}\xi_{di}'\right)R_d^{-1/2} \tag{7}$$

For $1 \le i \le v-1$, let

$$p_i = R_d^{1/2}\xi_{di} \tag{8}$$

If $\tau = (\tau_1, ..., \tau_v)'$ is the vector of treatment effects, then for each i, $1 \le i \le v-1$, $p_i'\tau$ represents a treatment contrast as, $p_i' 1_v = \xi_{di}' R_d^{1/2} 1_v = \xi_{di}'\rho_d = 0$ by virtue of (4). Also, it is not hard to see that (i) the vectors $\{p_i\}$, $1 \le i \le v-1$ are linearly independent and, (ii) any arbitrary treatment contrast $l'\tau$ can be expressed as a linear combination of the contrasts $\{p_i'\tau\}, 1 \le i \le v-1$.

Consider the treatment contrasts $\{p_i'\tau, 1 \le i \le v-1\}$ and let the best linear unbiased estimator (BLUE) of $p_i'\tau$ be $p_i'\hat{\tau}$. The variance of $p_i'\hat{\tau}$ under the design d, using (7) is then given by

$$\mathrm{Var}(p_i'\hat{\tau})_d = \sigma^2 p_i' C_d^- p_i = \sigma^2 \lambda_{di}^{-1} \tag{9}$$

where σ^2 is the per observation variance.

Let d_1 be a (possibly hypothetical) orthogonal design with the same replication numbers as in d. Recall that a connected block design is called *orthogonal* if the BLUE of any treatment contrast is uncorrelated with the BLUE of any block contrast. Under d_1, the variance of the BLUE of $p_i'\tau$ is given by

$$\mathrm{Var}(p_i'\hat{\tau})_{d_1} = \sigma^2 p_i' R_d^{-1} p_i = \sigma^2 \tag{10}$$

As is customary, if we now define the efficiency factor of the contrast $p_i'\tau$ as the ratio of the variance of the BLUE of $p_i'\tau$ under d_1 to that under d, then one has

$$\text{Efficiency factor } (p_i'\tau) = \lambda_{di} \tag{11}$$

Thus, the canonical efficiency factors are really the efficiency factors of the contrasts $p_i'\tau$ relative to an orthogonal design with the same replication numbers. According to Pearce, Caliński and Marshall (1974), a contrast of treatment effects $s'\tau$ is a *basic* contrast if and only if $C_d R_d^{-1} s = \in s$ for some positive scalar $\in$. It is not hard to see that the contrasts $p_i'\tau, 1 \le i \le v-1$ are in fact a set of basic contrasts, and thus the canonical efficiency factors are the efficiency factors of basic contrasts relative to an orthogonal design. See also Ceranka and Mejza (1979), John and Williams (1995, pp. 38-39) and Caliński and Kageyama (2000, p. 84) in this context. The following two results are well-known in the context of equireplicate designs; the versions given in Lemmas 4 and 5 are for an arbitrary block design.

Lemma 4. For an arbitrary connected design d, all the canonical efficiency factors are in the interval (0, 1].

Proof. By definition, $\lambda_{di} > 0$ for each i, $1 \le i \le v - 1$. Also, from (9)-(11), $\lambda_{di} = p_i'R_d^{-1}p_i / p_i'C_d^{-}p_i$ so that

$$1-\lambda_{di} = \frac{p_i'(C_d^{-} - R_d^{-1})p_i}{p_i'C_d^{-}p_i} \qquad (12)$$

In order to show that $\lambda_{di} \le 1$, it suffices to show that the numerator of (12) is nonnegative. Now, since the design is connected, $p_i \in M(C_d)$, where $M(\cdot)$ represents the column span of a matrix. This in turn implies that there exists a vector λ such that $p_i = C_d\lambda$. Therefore

$$p_i'(C_d^{-} - R_d^{-1})p_i = \lambda'(C_dC_d^{-}C_d - C_dR_d^{-1}C_d)\lambda$$

$$= \lambda'(C_d - C_dR_d^{-1}C_d)\lambda$$

But

$$C_d - C_dR_d^{-1}C_d = R_d - N_dK_d^{-1}N_d' - (R_d - N_dK_d^{-1}N_d') \times (R_d - N_dK_d^{-1}N_d')$$

$$= N_dK_d^{-1}(K_d - N_d'R_d^{-1}N_d)K_d^{-1}N_d' \qquad (13)$$

The matrix on the right hand side of (13) is nonnegative definite, since the matrix $K_d - N_d' R_d^{-1}N_d$ is so, being the coefficient matrix of the reduced normal equations for block effects. It follows now that for each $p_i \in M(C_d)$, $p_i'(C_d^{-} - R_d^{-1})p_i \ge 0$. This completes the proof.

For which class of designs the canonical efficiency factors are each equal to unity? The following result characterizes such designs.

Lemma 5. Under an arbitrary connected block design d, each canonical efficiency factor equals unity if and only if d is an orthogonal design.

Proof. (i) "If" part: Recall that a connected design d is orthogonal if and only if $N_d = r_dk_d'/n$ where $n = \sum_{i=1}^{v} r_{di} = \sum_{j=1}^{b} k_{dj}$ is the total number of experimental units in d. For an orthogonal design d, $C_d = R_d - r_dr_d'/n \Rightarrow A_d = I_v - \rho_d\rho_d'/n$, which is a symmetric, idempotent matrix of rank v – 1. It follows then that the positive eigenvalues of A_d are each equal to unity.

(ii) "Only if" part: Suppose now that d is such that all the canonical efficiency factors are equal to unity, i.e., for $1 \le i \le v - 1$, $\lambda_{di} = 1$. Then, from (6)

$$C_d = R_d^{1/2}\left(\sum_{i=1}^{v-1}\xi_{di}\xi_{di}'\right)R_d^{1/2}$$

$$= R_d^{1/2}(I_v - \rho_d\rho_d'/n)R_d^{1/2}, \text{ by (5)}$$

$$= R_d - r_dr_d'/n \qquad (14)$$

Also, for an arbitrary design d, C_d is given by (2). Equating (2) and (14), we have

$$N_dK_d^{-1}N_d' = r_dr_d'/n$$

$$\Rightarrow \quad \text{Rank}(N_dK_d^{-1}N_d') = \text{Rank}(N_d) = 1 \qquad (15)$$

This implies that $N_d = ab'$ for some non-null vectors a, b where a is v × 1 and b is b × 1. Now, $k_d' = 1_v' N_d = 1_v' ab' = \alpha b'$ where $\alpha = 1_v' a$. Therefore, $b' = \alpha^{-1}k_d'$. Also, $r_d = N_d1_b = ab'1_b = \beta a$, where $\beta = b'1_b$. Note that both α and β are nonzero scalars. Hence, $a = \beta^{-1}r_d$. This shows that $N_d = (\alpha\beta)^{-1}r_dk_d'$. The proof is completed by recalling that $1_v'N_d1_b = n$.

If for a design d, all the canonical efficiency factors are equal, then d is called an efficiency-balanced design. A characterization of efficiency-balanced designs was provided by Williams (1975). Another class of designs, called C-designs (Saha 1976) or, simple partially efficiency-balanced designs (Puri and Nigam 1977) has also been studied in the literature. The main motivation for studying these designs appears to be the simplicity in the analysis as, for such designs, computation of a generalized inverse of the C-matrix is very simple. Following Caliński (1971), Saha (1976), and Puri and Nigam (1977) defined these designs through a v × v matrix

$$M_{0d} = R_d^{-1}N_dK_d^{-1}N_d' - 1_v r_d'/n \tag{16}$$

As we shall see presently, C-designs can be defined more directly in terms of the canonical efficiency factors.

Consider a connected block design d with v treatments, b blocks and other parameters as in (1). Suppose the matrix A_d given by (3) has only two distinct positive eigenvalues, $\lambda \in (0, 1)$ and unity and suppose the multiplicity of the eigenvalue λ is m where $0 \le m \le v-1$. This of course means that the design d has only two distinct canonical efficiency factors, viz., λ and unity. Without loss of generality, let the positive eigenvalues of A_d be $\lambda_{d1} = \lambda_{d2} .... = \lambda_{dm} = \lambda$ and for $m + 1 \le i \le v-1$, $\lambda_{di} = 1$. Also, as before, for $1 \le i \le v-1$, let the orthonormal eigenvector corresponding to λ_{di} be ξ_{di}. Then, it can be seen easily that the C-matrix of d is given by

$$C_d = R_d^{1/2}\left(\lambda\sum_{i=1}^{m}\xi_{di}\xi_{di}' + \sum_{i=m+1}^{v-1}\xi_{di}\xi_{di}'\right)R_d^{1/2}$$

$$= R_d^{1/2}\left(\lambda\sum_{i=1}^{v-1}\xi_{di}\xi_{di}' + (1-\lambda)\sum_{i=m+1}^{v-1}\xi_{di}\xi_{di}'\right)R_d^{1/2} \tag{17}$$

Using (16) and (17), we have

$$M_{0d} = R_d^{-1}N_dK_d^{-1}N_d' - 1r_d'/n$$

$$= R_d^{-1}(R_d - C_d) - 1r_d'/n$$

$$= (1-\lambda)\left(I_v - 1r_d'/n - R_d^{-1/2}\sum_{i=m+1}^{v-1}\xi_{di}\xi_{di}'R_d^{1/2}\right)$$

$$= (1-\lambda)R_d^{-1/2}\left(I_v - \rho_d\rho_d'/n - \sum_{i=m+1}^{v-1}\xi_{di}\xi_{di}'\right)R_d^{1/2}$$

$$= (1-\lambda)R_d^{-1/2}\left(\sum_{i=1}^{m}\xi_{di}\xi_{di}'\right)R_d^{1/2} \tag{18}$$

$$= (1-\lambda)\, L \tag{19}$$

where

$$L = R_d^{-1/2}\left(\sum_{i=1}^{m}\xi_{di}\xi_{di}'\right)R_d^{1/2} \tag{20}$$

is a symmetric idempotent matrix of rank m. Clearly in such a case, $1-\lambda$ is the unique nonzero eigenvalue of M_{0d} with multiplicity m. These facts were also observed by Caliński (1971).

Conversely, suppose for a connected block design d, the matrix M_{0d} is given by (18). Also from (16), for any arbitrary design d

$$M_{0d} = R_d^{-1}N_dK_d^{-1}N_d' - 1_v r_d'/n$$

$$= R_d^{-1/2}(I_v - R_d^{-1/2}C_dR_d^{-1/2} - \rho_d\rho_d'/n)R_d^{1/2}$$

$$= R_d^{-1/2}(I_v - A_d - \rho_d\rho_d'/n)R_d^{1/2} \tag{21}$$

Equating (18) to (21), we have, after some simplification

$$A_d = \lambda\sum_{i=1}^{m}\xi_{di}\xi_{di}' + \sum_{i=m+1}^{v-1}\xi_{di}\xi_{di}' \tag{22}$$

which shows that A_d has two distinct eigenvalues, λ with multiplicity m and unity, with multiplicity $v-1-m$.

From (19), it is easy to see that for every positive integer t

$$M_{0d}^t = (1-\lambda)^{t-1}M_{0d} \tag{23}$$

Condition (23) was used by Saha (1976) to define a C-design, i.e., a design d is a C-design if and only if (23) holds. The above analysis shows that equivalently, a connected block design d is a C-design if and only if d has only two distinct canonical efficiency factors, $\lambda \in (0, 1)$ with multiplicity m and unity with multiplicity $v-1-m$. Based on the statistical interpretation of the canonical efficiency factors, one can now provide a statistical justification for studying C-designs: under these designs, $v-1-m$ basic contrasts are estimated with full efficiency while the remaining m are all estimated with the same efficiency factor $\lambda \in (0, 1)$. Clearly, an orthogonal design is a C-design with m = 0 and a non-orthogonal efficiency-balanced design is a C-design with $m = v-1$. From an overall efficiency point of view, if one were to use a C-design then one should choose one for which m is as small as possible and/or λ is as close to 1 as possible. Even in such a case, a C-design may turn out to be less efficient than a non-C-design. The issue of optimality of non-orthogonal C-designs (including efficiency-balanced designs) has not been studied in its complete generality. For some results on the optimality of binary efficiency-balanced designs in certain restricted classes of competing designs, see Mukerjee and Saha (1990).

Finally, using (17) it can be shown that for a C-design d, a choice of a generalized inverse of C_d is

$$\begin{aligned} C_d^- &= R_d^{-1/2}\left(\lambda^{-1}\sum_{i=1}^{m}\xi_{di}\xi'_{di} + \sum_{i=m+1}^{v-1}\xi_{di}\xi'_{di}\right)R_d^{-1/2} \\ &= (\lambda^{-1}M_{0d} + I_v)R_d^{-1} - 1_v1'_v\ /\ n \qquad (24) \end{aligned}$$

an expression that is very similar to the one obtained by Caliński (1971) through a different approach.

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