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Non-existence of Affine α-Resolvable Triangular Designs under $1 \le \alpha \le 10$

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SUMMARY

The existence on affine α-resolvability with some properties has been discussed for block designs in literature since 1942 for $\alpha = 1$ and in particular since 1963 for $\alpha \ge 2$. Non-existence of affine α-resolvable group divisible designs of regular type is known (Kageyama 2008). Also, Kageyama (2007) disproved the validity of such concept for triangular designs when $\alpha = 1, 2$. In this paper, for $3 \le \alpha \le 10$, the non-existence will be shown.

Key words: Triangular design; α-resolvable; Affine α-resolvable.

1. INTRODUCTION

A block design with parameters v, b, r and k is said to be α-resolvable if its b blocks can be grouped into t resolution sets of β blocks each such that every treatment appears in each resolution set precisely α times. So $b = \beta t$ and $r = \alpha t$. A 1-resolvable block design is simply called a-resolvable block design. An α-resolvable block design is said to be affine α-resolvable if any pair of blocks belonging to the same resolution set contains q_1 treatments in common, whereas any pair of blocks belonging to different resolution sets contains q_2 treatments in common (Shrikhande and Raghavarao 1963). It is known that in an affine α-resolvable block design $q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = \alpha k/\beta = k^2/v$, both of which must be integers. Note that when $\alpha = 1$, this definition of (affine) 1-resolvability coincides with the traditional definition (Raghavarao 1988).

A triangular design is a block design with $v = n(n - 1)/2$ treatments, having a triangular association scheme for $n \ge 4$, which is one of popular classes of 2-associate partially balanced incomplete block designs (Raghavarao 1988).

By use of the Hasse-Minkowski invariant of some matrix, some necessary conditions for the existence of affine α-resolvable triangular designs can be derived in Raghavarao (1988; Theorems 12.6.3 and 12.6.4). But they are not sufficient.

In this paper, it will be shown that there does not exist an affine α-resolvable triangular design when $\alpha \le 10$ completely, whereas when $\alpha \ge 11$ partially. The author believes that there does not exist an affine α-resolvable triangular design for any $\alpha \ge 1$.

2. PRELIMINARY

This section is essentially from Kageyama (2007).

Lemma 2.1. The matrices XY and YX have the same non-zero eigenvalues with the same multiplicities, where the matrices X and Y are of appropriate sizes.

Let N be the $v \times b$ incidence matrix of a triangular design with parameters $v = n(n - 1)/2$, b, r, k, λ_1, λ_1, where $\lambda_1 \ne \lambda_2$. (When $\lambda_1 = \lambda_2$, the design becomes a balanced incomplete block design.) Then the following is known.

Lemma 2.2. (Raghavarao 1988). In a triangular design, the matrix NN' has eigenvalues rk, $r + (n - 4)\lambda_1 - (n - 3)\lambda_2$ and $r - 2\lambda_1 + \lambda_2$ with multiplicities 1, $n - 1$ and $n(n - 3)/2$, respectively.

Furthermore, when N is the $v \times b$ incidence matrix of an affine α-resolvable triangular design with parameters $v = n(n - 1)/2$, $b = \beta t$, $r = \alpha t$, k, λ_1, λ_2, we have the following.

Lemma 2.3. (Shrikhande and Raghavarao 1963). In an affine α-resolvable block design, the matrix $N'N$ has eigenvalues rk, $k\{1 - (\alpha - 1)/(\beta - 1)\}$ and 0, with multiplicities 1, $b - t$ and $t - 1$, respectively.

Lemmas 2.1, 2.2 and 2.3 can produce the following.

Theorem 2.1. In an affine α-resolvable triangular design,

(i) when $r + (n-4)\lambda_1\ (n-3)\lambda_2 > 0$ and $r - 2\lambda_1 + \lambda_2 > 0$, the affine α-resolvability does not hold;

(ii) when $r + (n-4)\lambda_1 - (n-3)\lambda_2 = 0$ and $r - 2\lambda_1 + \lambda_2 > 0$, an identity $b = v + t - 1 - (n-1)$ holds;

(iii) when $r + (n-4)\lambda_1\ (n-3)\lambda_2 > 0$ and $r - 2\lambda_1 + \lambda_2 = 0$, an identity $b = v + t - 1 - n(n-3)/2$ holds.

3. STATEMENT

In an affine α-resolvable triangular design with parameters $v = n(n-1)/2$, $b = \beta t$, $r = \alpha t$, k, λ_1, λ_2, by Theorem 2.1(i), two cases have to be considered.

(I) When $r + (n-4)\lambda_1 - (n-3)\lambda_2 = 0$, it follows from Theorem 2.1(ii) that all parameters are expressed as

$$v = \frac{n(n-1)}{2}, b = \frac{\beta n(n-3)}{2(\beta-1)}, r = \frac{\alpha n(n-3)}{2(\beta-1)}, k = \frac{\alpha n(n-1)}{2\beta}$$

$$\lambda_1 = \frac{\alpha n(n-3)(\alpha n - \alpha - \beta)}{2\beta(\beta-1)(n-2)}, \lambda_2 = \frac{\alpha n[\alpha(n-1)(n-4) + 2\beta]}{2\beta(\beta-1)(n-2)}$$

where $t = n(n-3)/[2(\beta-1)]$. Here

$$\frac{\alpha n(\alpha n - \alpha - \beta)}{n-2} = \alpha^2 n + \alpha(\alpha - \beta) - \frac{2\alpha(\beta-\alpha)}{n-2} \tag{3.1}$$

must be an integer, since $2\beta(\beta-1)\lambda_1$ is an integer.

(II) When $r - 2\lambda_1 + \lambda_2 = 0$, it follows from Theorem 2.1(iii) that all parameters are expressed as

$$v = \frac{n(n-1)}{2}, b = \frac{\beta(n-1)}{\beta-1}, r = \frac{\alpha(n-1)}{\beta-1}, k = \frac{\alpha n(n-1)}{2\beta}$$

$$\lambda_1 = \frac{\alpha(n-1)[\alpha n + \beta(n-4)]}{2\beta(\beta-1)(n-2)}, \lambda_2 = \frac{\alpha(n-1)(\alpha n - 2\beta)}{\beta(\beta-1)(n-2)}$$

where $t = (n-1)/(\beta-1)$. Here

$$\frac{\alpha(\alpha n - 2\beta)}{n-2} = \alpha^2 - \frac{2\alpha(\beta-\alpha)}{n-2} \tag{3.2}$$

must be an integer, since $\beta(\beta-1)\lambda_2$ is an integer.

Now, by (3.1) and (3.2)

$$\frac{2\alpha(\beta-\alpha)}{n-2} \text{ is a positive integer.} \tag{3.3}$$

There are two more observations (Kageyama and Tsuji 1977) that in the case of $r + (n-4)\lambda_1 - (n-3)\lambda_2 = 0$, $2k/n$ is an integer, while in the case of $r - 2\lambda_1 + \lambda_2 = 0$, $2k/(n-1)$ is an integer. (This information will be used crucially in the proof of Theorem 3.1 along with (3.3).) Hence, in an affine α-resolvable triangular design, since $2k/n = \alpha(n-1)/\beta$ or $2k/(n-1) = \alpha n/\beta$, additional conditions for the non-existence are obtained. For example, when $r + (n-4)\lambda_1 - (n-3)\lambda_2 = 0$, if both $n-1$ and β are prime, then the design does not exist, while, when $r - 2\lambda_1 + \lambda_2 = 0$, if both n and β are prime, then the design does not exist.

It can be further shown that when n, $n-1$ or $n-2$ is prime, there does not exist an affine α-resolvable triangular design with $v = n(n-1)/2$ and $r - 2\lambda_1 + \lambda_2 = 0$ for any $\alpha \geq 1$. Here, since $t = (n-1)/(\beta-1) \geq 2$, we have $n > \beta$. Hence, when n is a prime, the integrality of $\alpha n/\beta$ implies β/α, which is a contradiction. When $n-1$ is a prime, t is not an integer. When $n-2$ is a prime, for $n = 4$ the non-integrality of λ_1 is shown, and in the case of $n \geq 5$ we can show the non-integrality of t after some evalutation on parameters. Anyway, the above mentioned result, after some calculation, shows the non-existence of the design, for example, for $n \leq 60$, i.e. $v \leq 1770$.

Now the following result has been established.

Theorem 3.1. There does not exist an affine α-resolvable triangular design for $\alpha \leq 10$.

Proof. When $\alpha = 1, 2$, the result has been shown by Kageyama (2007) who also shows $\beta \geq 11$ for the existence. Further note that $n \geq 4$, $t \geq 2$ and $\beta > \alpha$.

Case α = 3

Case 1: $r - 2\lambda_1 + \lambda_2 = 0$. By (3.3), $6(\beta-3)/(n-2)(= x$, say) must be a positive integer. Since $t = (n-1)/(\beta-1)$, $x = 6(\beta-3)/(t\beta - t - 1)$. If $t \geq 6$, then it is shown that $x < 1$, which is a contradiction. Hence $t = 2, 3, 4, 5$ since $t \geq 2$. When $t = 2, 3, 4$, it is clear that x is not an integer. When $t = 5$, $x = 1 + (\beta - 12)/(5\beta - 6)$, in which when $\beta \geq 13$, x is not an integer. When $\beta = 12$, we have $n = 56$ and then $k = 385$ and k^2/v is not an integer. When $\beta \leq 11$, $x = 1 - (12 - \beta)/(5\beta - 6)$, in which

$|1 - x| < 1$ since $\beta \geq 11$. Thus, there is no design for Case 1.

Case 2: $r + (n - 4)\lambda_1 - (n - 3)\lambda_2 = 0$. By (3.3), $6(\beta - 3)/(n - 2)$(= x, say) must be a positive integer. Now it is known that $2k/n$(= θ, say) is a positive integer. Then $\theta = 3(n - 1)/\beta = (18n - 18)/(nx - 2x + 18)$, in which it is shown that when $x \geq 19$, θ is not an integer. When $x = 18$, $\theta = 1$, i.e., $n = 2k$. In this case, $k^2/v = n/[2(n - 1)]$ is not an integer since $n \geq 4$. Hence $x \leq 17$. Now $\theta = 1 + (18 - x)(n - 2)/(xn - 2x + 18)$. Here, when $x \geq 19$, $|\theta - 1| < 1$. Hence it holds that $x \leq 8$. When $x = 8$, $\theta = 2 + (n - 11)/(4n + 1)$, in which when $n > 11$, $|\theta - 2| < 1$. When $n = 11$, $\theta = 2$, i.e, $k = \theta n/2 = 11$ and then k^2/v is not an integer. When $n \leq 10$, $\theta = 2 - (11 - n)/(4n + 1)$ which is not an integer since $n \geq 4$. Next, when $x = 7$, $\theta = 2 + (4n - 26)/(7n + 4)$, in which when $n \geq 7$, $|\theta - 2| < 1$. When $n \leq 6$, $\theta = 2 - (26 - 4n)/(7n + 4)$ which is not an integer since $n \geq 4$. Next, when $x = 6$, $\theta = 3 - 6/(n + 1)$ which implies that $n = 5$ and $\theta = 2$, i.e., $k = \theta n/2 = 5$ and then k^2/v is not an integer. Next, when $x = 5$, $\theta = 3 + (3n - 42)/(5n + 8)$, in which when $n \geq 15$, $|\theta - 3| < 1$. When $n = 14$, $\theta = 3$, i.e., $k = \theta n/2 = 21$ and then k^2/v is not an integer. When $n \leq 13$, $\theta = 3 - (42 - 3n)/(5n + 8)$ which is not an integer if $n \geq 5$. Then, when $n = 4$, $|\theta - 3|$ is not an integer. Next, when $x = 4$, $\theta = 4 + (n - 29)/(2n + 5)$, in which when $n \geq 30$, $|\theta - 4| < 1$. When $n = 29$, $\theta = 4$, i.e., $k = \theta n/2 = 58$ and then k^2/v is not an integer. When $n \leq 28$, $\theta = 4 - (29 - n)/(2n + 5)$ which is not an integer if $n \geq 9$. Then, when $n = 8$, $\theta = 3$, i.e., $k = 12$ and then v^2/v is not an integer. When $n = 7, 6, 5, 4$, $|\theta - 4|$ is not an integer. Next, when $x = 3$, $\theta = 6 - 30/(n + 4)$ which implies that $n = 6, 11, 26$ since $n \geq 4$. When $n = 6$, $\theta = 3$, i.e., $k = 9$ and then k^2/v is not an integer. When $n = 11$, $\theta = 4$, i.e., $k = 22$ and then k^2/v is not an integer. When $n = 26$, $\theta = 5$, i.e., $k = 65$ and then $\beta = 15$ since $\alpha = 3$. In this case, $t = n(n - 3)/[2(\beta - 1)]$ is not an integer. Next, when $x = 2$, $\theta = 9 - 72/(n + 7)$ which implies that $n = 5, 11, 17, 29, 65$ since $n \geq 4$. When $n = 5, 11, 29$, k is not an integer. When $n = 17$, $\theta = 6$, i.e., $k = 51$ and then k^2/v is not an integer. When $n = 65$, $\theta = 8$, i.e., $k = 260$ and then k^2/v is not an integer. Finally, when $x = 1$, $\theta = 18 - 306/(n + 16)$ which implies that $n = 18, 35, 86, 137, 290$ since $n \geq 4$, and then $\theta = 9, 12, 15, 16, 17$, respectively. However, since $\beta = 3(n - 1)/\theta$, we have only two cases, i.e., $n = 86$, $\beta = 17$ and $n = 290$, $\beta = 51$. In the former case, it is seen that k^2/v is not an integer. In the latter case, k and k^2/v are all integers, but t is not an integer.

Thus, there is no affine 3-resolvable triangular design.

Case $\alpha = 4$.

Case 1: $r - 2\lambda_1 + \lambda_2 = 0$. By (3.3), $8(\beta - 4)/(n - 2)$ (= x, say) must be a positive integer. Since $t = (n - 1)/(\beta - 1)$, $x = 8(\beta - 4)/(t\beta - t - 1)$. If $t \geq 8$, then it is shown that $x < 1$, which is a contradiction. Hence $2 \leq t \leq 7$ since $t \geq 2$. When $t = 7$, $x = 1 + (\beta - 24)/(7\beta - 8)$, in which when $\beta \geq 25$, x is not an integer. When $\beta = 24$, we have $n = 162$, but k is not an integer. When $\beta \leq 23$, $x = 1 - (24 - \beta)/(7\beta - 8)$, in which $|1 - x| < 1$ since $\beta \geq 11$. Next, when $t = 6$, $x = 1 + (2\beta - 25)/(6\beta - 7)$, in which when $\beta \geq 13$, $|x - 1| < 1$. When $\beta \leq 12$, $x = 1 - (25 - 2\beta)/(6\beta - 7)$, in which $|1 - x| < 1$ since $\beta \geq 11$. Next, when $t = 5$, $x = 1 + (3\beta - 26)/(5\beta - 6)$, in which $|x - 1| < 1$ since $\beta \geq 11$. Next, when $t = 4$, $x = 2 - 22/(4\beta - 5)$, which implies $\beta = 4$ but now $\alpha = 4$ which is a contradiction since $\beta > \alpha$. Next, when $t = 3$, $x = 2 + (2\beta - 24)/(3\beta - 4)$, in which when $\beta \geq 13$, $|x - 2| < 1$. When $\beta = 12$, $n = 34$ and then $k = 187$, but k^2/v is not an integer. When $\beta \leq 11$, $x = 2 - (24 - 2\beta)/(3\beta - 4)$, in which $|2 - x| < 1$ since $\beta \geq 11$. Next, when $t = 2$, $x = 4 - 20/(2\beta - 3)$, which implies $\beta = 4$ since $n \geq 4$. Here when $\beta = 4$, we have $v = k = 21$, which is a contradiction. Finally, when $t = 1$, $n = \beta$ and $x = 8 - 16/(\beta - 2)$, which implies that $\beta = 6, 10, 18$, since $\beta > \alpha$. In each case, k^2/v is not an integer. Thus, there is no design for Case 1.

Case 2: $r + (n - 4)\lambda_1 - (n - 3)\lambda_2 = 0$. By (3.3), $8(\beta - 4)/(n - 2)$(= x, say) must be a positive integer. Now it is known that $2k/n$(= θ, say) is a positive integer. Then $\theta = 4(n - 1)/\beta = (32n - 32)/(nx - 2x + 32)$, in which it is shown that when $x \geq 33$, θ is not an integer. When $x = 32$, $\theta = 1$, i.e., $n = 2k$. In this case, $k^2/v = k/(2k - 1)$ which is not an integer. Hence it holds that $x \leq 31$. Now $\theta = 1 + (32 - x)(n - 2)/(nx - 2x + 32)$. When $x \geq 16$, $|\theta - 1| < 1$. Hence it holds that $x \leq 15$. When $x = 15$, $\theta = 2 + (2n - 36)/(15n + 2)$, in which when $n \geq 19$, $|\theta - 2| < 1$. When $n = 18$, $\theta = 2$, i.e, $k = n$ and then k^2/v is not an integer. When $n \leq 17$, $\theta = 2 - (36 - 2n)/(15n + 2)$ which is not an integer since $n \geq 4$. Next, when $x = 14$, $\theta = 2 + (2n - 20)/(7n + 2)$, in which when $n \geq 11$, $|\theta - 2| < 1$. When $n = 10$, $\theta = 2$, i.e, $k = n$ and then k^2/v is not an integer. When $n \le 9$, $\theta = 2 - (20 - 2n)/(7n + 2)$ which is not an integer since $n \ge 4$. Next, when $x = 13$, $\theta = 2 + (6n - 44)/(13n + 6)$, in which when $n \ge 8$, $|\theta - 2| < 1$. When $n \le 7$, $\theta = 2 - (44 - 6n)/(13n + 6)$ which is not an integer since $n \ge 4$. Next, when $x = 12$, $\theta = 2 + (2n - 12)/(3n + 2)$, in which when $n \ge 7$, $|\theta - 2| < 1$. When $n = 6$, $\theta = 2$, i.e., $k = n$ and then k^2/v is not an integer. When $n = 5$ and 4, θ is not an integer. Next, when $x = 11$, $\theta = 2 + (10n - 52)/(11n + 10)$, in which when $n \ge 6$, $|\theta - 2| < 1$. When $n = 5$ and 4, θ is not an integer. Next when $x = 10$, $\theta = 3 + (n - 34)/(5n + 6)$ which is not an integer if $n \ge 35$. Then, when $n = 34$, $\theta = 3$, i.e., $k = 51$ and then v^2/v is not an integer. When $n \le 33$, $\theta = 3 - (34 - n)/(5n + 6)$ which is not an integer when $n \ge 5$. When $n = 4$, θ is not an integer. Next when $x = 9$, $\theta = 3 + (5n - 74)/(9n + 14)$ which is not an integer if $n \ge 15$. When $n \le 14$, $\theta = 3 - (74 - 5n)/(9n + 14)$ which is not an integer when $n \ge 5$. When $n = 4$, θ is not an integer. Next when $x = 8$, $\theta = 4 - 12/(n + 2)$ which implies that $n = 4, 10$ since $n \ge 4$. If $n = 4$, then $n = k$ and then k^2/v is not an integer. If $n = 10$, then $k = 15$ but t is not an integer. Next when $x = 7$, $\theta = 4 + (4n - 104)/(7n + 18)$ which is not an integer if $n \ge 27$. When $n = 26$, $\theta = 4$, i.e., $k = 2n$ and $k^2/v = 8 + 8/(n - 1)$ which implies that $n = 5, 9$ since $n \ge 4$. When $n = 5$, $v = k$. When $n = 9$, β is not an integer. When $n \le 25$, $\theta = 4 - (104 - 4n)/(7n + 18)$ which is not an integer when $n \ge 8$. When $n = 7, 6, 5, 4$, θ is not an integer. Next when $x = 6$, $\theta = 5 + (n - 66)/(3n + 10)$ which is not an integer if $n \ge 67$. When $n = 66$, $\theta = 5$, i.e., $k = 165$ and then k^2/v is not an integer. Hence when $n \ge 65$, $\theta = 5 - (66 - n)/(3n + 10)$ which is not an integer when $n \ge 15$. Hence it holds that $n \le 14$. When $n = 14$, $\theta = 4$, i.e., $k = 28$ and then k^2/v is not an integer. When $4 \le n \le 13$, θ is not an integer. Next when $x = 5$, $\theta = 6+(2n - 164)/(5n + 22)$ which is not an integer if $n \ge 83$. When $n = 82$, $\theta = 6$, i.e., $k = 246$ and then k^2/v is not an integer. Hence when $n \le 81$, $\theta = 6 - (164 - 2n)/(5n + 22)$ which is not an integer when $n \ge 21$. Hence it holds that $n \le 20$. When $n = 10$, $\theta = 4$, i.e., $k = 20$ and then k^2/v is not an integer. For other $4 \le n \le 20$, θ is not an integer. Next when $x = 4$, $\theta = 8 - 56/(n + 6)$ which implies that $n = 8, 22, 50$ since $n \ge 4$. When $n = 8$ and 22, we can show that k^2/v is not an integer. When $n = 50$, $\theta = 7$, i.e., $k = 175$, $\beta = 28$, but t is not an integer. Next when $x = 3$, $\theta = 10 + (2n - 292)/(3n + 26)$ which is not an integer if $n \ge 147$. When $n = 146$, $\theta = 10$, i.e., $k = 730$ and then k^2/v is not an integer. Hence when $n \le 145$, $\theta = 10 - (292 - 2n)/(3n + 26)$ which is not an integer when $n \le 54$. Hence it holds that $n \le 53$. When $n = 30$, $\theta = 8$, i.e., $k = 120$ and then k^2/v is not an integer. For other $4 \le n \le 53$, θ is not an integer. Next when $x = 2$, $\theta = 16 - 240/(n + 14)$ which implies that $n = 6, 10, 16, 26, 34, 46, 66, 106, 226$ since $n \ge 4$. Accordingly, $\theta = 4, 6, 8, 10, 11, 12, 13, 14, 15$ and $k = 12, 30, 64, 130, 187, 276, 429, 742, 1695$, respectively. When $n = 16$, β is not an integer. For $n = 6, 26, 46, 34, 66$ and 106, k^2/v is not an integer, while for $n = 10$ and 226, λ_1 is not an integer. Finally, when $x = 1$, $\theta = 32 - 992/(n + 30)$ which implies that $n = 32, 94, 218, 466, 962$ since $n \ge 4$. For the first two n, β is not an integer. When $n = 218$ and 466, k^2/v is not an integer. For $n = 962$, λ_1 is not an integer. Thus, there is no affine 4-resolvable triangular design.

For $5 \le \alpha \le 10$, some tedious and painful process similar to the above proof of Cases $\alpha = 3$ and 4 will show the non-existence of an affine α-resolvable triangular design. Then their proofs are omitted here. But, there is another practical technique for the case $r + (n - 4)\lambda_1 - (n - 3)\lambda_2 = 0$, to prove the non-existence. Therefore, it is described for $\alpha = 10$ in particular.

Case $\alpha = 10$

Case 1: $r - 2\lambda_1 + \lambda_2 = 0$. By (3.3), $20(\beta - 10)/(n - 2)$(= x, say) must be a positive integer. Since $t = (n - 1)/(\beta - 1)$, $x = (20\beta - 200)/(t\beta - t - 1)$. In this case, it is shown that when $t \ge 20$, $x < 1$ which is a contradiction. Hence it holds that $t \le 19$. When $t = 19$, $x = 1 + (\beta - 180)/(19\beta - 20)$, in which when $\beta \ge 181$, $|x - 1| < 1$. When $\beta = 180$, $n = 3402$ but k is not an integer. When $\beta \le 179$, $x = 1 - (180 - \beta)/(19\beta - 20)$ which shows that $|1 - x| < 1$ since $\beta \ge 11$. When $t = 18$, $x = 1 + (2\beta - 181)/(18\beta - 19)$, in which when $\beta \ge 91$, $|x - 1| < 1$. When $\beta \le 90$, $x = 1 - (181 - 2\beta)/(18\beta - 19)$ which shows that $|1 - x| < 1$ since $\beta \ge 11$. Next, when $t = 17, 16, 15, 14, 13, 12, 11$, we can get the same result on non-existence. Next, when $t = 10$, $x = 2 - 178/(10\beta - 11)$ which implies that there is no integral β since $\beta \ge 11$. When $t = 9$, $x = 2 + (2\beta - 180)/(9\beta - 10)$, in which when $\beta \ge 91$, $|x - 2| < 1$. When $\beta = 90$, $n = 802$ but k^2/v is not an integer. When $\beta \le 89$, $|2 - x| < 1$ when $\beta \ge 18$. Hence when $\beta \le 17$, the integrality of x shows that $\beta = 8$ for $n \ge 4$, and then $n = 64$. In this case k^2/v is not an integer. The similar argument for $8 \ge t \ge 2$ shows the non-existence of the design. Finally, when t = 1, i.e., $n = \beta$, $x = 20 - 160/(\beta - 2)$ which implies that $\beta = 12, 18, 22, 34, 42, 82$ since $\beta \geq 11$. In each value of such β, k^2/v is not an integer. Thus, there is no design for Case 1.

Case 2: $r + (n - 4)\lambda_1 - (n - 3)\lambda_2 = 0$. By (3.3), $20(\beta - 10)/(n - 2)(= x$, say) must be a positive integer. Now it is known that $2k/n(= \theta$, say) is a positive integer. Then $\theta = 10(n - 1)/\beta = 200(n - 1)/(nx - 2x + 200)$, in which it is shown that θ is not an integer for $x \geq 201$. When x = 200, $\theta = 1$ and n = 2k. Hence k^2/v is not an integer. When $x \leq 199$, $\theta = 1 + (200 - x)(n - 2)/(nx - 2x + 200)$, in which when $x \geq 100$, $|\theta - 1| < 1$. Hence it holds that $x \leq 99$. Furthermore

$$\theta = 2 + [(200 - 2x)(n - 2) - 200]/[(n - 2)x + 200] \quad (3.4)$$

1. If $(200 - 2x)(n - 2) < 200$, then $|2 - \theta| < 1$, since $x \leq 99$.

2. If $(200 - 2x)(n - 2) = 200$, then $\theta = 2$, i.e., n = k and hence k^2/v is not integer.

3. If $(200 - 2x)(n - 2) > 200$, then in (3.4) $|\theta - 2| < 1$ when $x \geq 67$. Hence it holds that $x \leq 66$. Thus we have to consider this problem for (3.4) under $x \leq 66$ with the condition (3). This procedure will be taken sequentially.

   Next

   $$\theta = 3 + [(200 - 3x)(n - 2) - 400]/[(n - 2)x + 200] \quad (3.5)$$

4. If $(200 - 3x)(n - 2) < 400$, then $|3 - \theta| < 1$, by the condition (3).

5. If $(200 - 3x)(n - 2) = 400$, then $\theta = 3$, i.e., $k = 3n/2$ (n being even) and hence $k^2/v = 4 + (n + 8)/[2(n - 1)]$ which implies that n = 4, 10. When n = 4, we have v = k = 6 which is a contradiction. When n = 10, k = 15 and a relation $k = \alpha n(n - 1)/(2\beta)$ yields $\beta = 30$, but t is not an integer.

6. If $(200 - 3x)(n - 2) > 400$, then in (3.5) $|\theta - 3| < 1$ when $x \geq 50$. Hence it holds that $x \leq 49$. Note that the condition (6) implies the condition (3). Thus we have to consider this problem for (3.5) under $x \leq 49$ with the condition (6).

   Next

   $$\theta = 4 + [(200 \;\; 4x)(n - 2) - 600]/[(n - 2)x + 200] \quad (3.6)$$

7. If $(200 - 4x)(n - 2) < 600$, then $|4 - \theta| < 1$, by the condition (6).

8. If $(200 - 4x)(n - 2) = 600$, then $\theta = 4$, i.e., k = 2n and hence $k^2/v = 8 + 8/(n - 1)$ which implies that n = 5, 9 since $n \geq 4$. When n = 5, we have v = k = 10 which is a contradiction. When n = 9, k = 18 and a relation $k = \alpha n(n - 1)/(2\beta)$ yields $\beta = 20$, but t is not an integer.

9. If $(200 - 4x)(n - 2) > 600$, then in (3.6) $|\theta - 4| < 1$ when $x \geq 40$. Hence it holds that $x \leq 39$. Note that condition (9) implies the condition (6). Thus we have to consider this problem for (3.6) under $x \leq 39$ with the condition (9).

By taking this procedure sequentially, we can reduce the value of x up to 1. In general, we can describe the following procedure. For a positive integer 'a' let

$$\theta = a + [(200 - ax)(n - 2) - 200(a - 1)]/[(n - 2)x + 200] \quad (3.7)$$

(i) If $(200 - ax)(n - 2) < 200(a - \theta)$, then $|a - \theta| < 1$, since in the previous step the condition $[200 - (a - 1)x](n - 2) - 200(a - 2) > 0$ (corresponding to (3), (6) or (9)) is satisfied.

(ii) If $(200 - ax)(n - 2) = 200(a - 1)$, then $\theta = a$, i.e., $k = na/2$ and hence $k^2/v = na^2/[2(n - 1)]$ must be an integer. (Once we know the value of a, the values of n are determined and for such value n we may show the non-integrality of some parameters on the design like (2), (5) or (8)).

(iii) If $(200 - ax)(n - 2) > 200(a - 1)$, then in (3.7) $|\theta - a| < 1$ when $x \geq 200/(a + 1)$. Hence it holds that $x < 200/(a + 1)$. Thus as a next step we have to consider this problem for (3.7) under $x < 200/(a + 1)$ with the condition (iii). In fact, this procedure have to be taken till a = 199, since x is a positive integer.

Among the above three patterns (i), (ii) and (iii), two cases (i) and (iii) are quite routine, but the pattern (ii) may take some time. This process depends on the values of a and n. As a condition between such values we have a relation that a^2 is divisible by $n - 1$, which is powerful. For example, let $a = 5$. Then $k = 5n/2$ and $k^2/v = 25n/[2(n - 1)]$. Since n is even, the integrality of k^2/v implies $n = 6$ and 26. When $n = 6$, we have $v = k = 15$ which is a contradiction to $v > k$. When $n = 26$, we have $k = 15$ and $\beta = 50$ but t is not an integer.

Thus, the above procedure will show finally that there is no affine 10-resolvable triangular design.

Hence the proof is complete.

It seems that the approach utilized here shows the non-existence for a bigger value of α routinely. Possibly, the procedure used in the proof of Theorem 3.1 may reveal that a number-theoretic approach on integrality of design parameters only will be powerful to get the real final result.

Since the complement of an affine α-resolvable triangular design is an affine (β – α)-resolvable triangular design, Theorem 3.1 can lead the following.

Corollary 3.1. When $\beta - \alpha \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, an affine α-resolvable triangular design with $b = \beta t$ does not exist.

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