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# Using Ring Theory to Construct Complete Sets of Sum of Squares Orthogonal F-squares and Latin Squares

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**SUMMARY**

Ring theory is used to construct a complete set of sum of squares orthogonal F-squares (SSOFSs or SSOF-squares) for n = 6. Sum of squares orthogonality of this set is exhibited with a numerical example. Previous methods involved field theoretic methods together with trial and error with computer codes. Sum of squares orthogonal Latin squares have been constructed for any value of n, SSOLS (n, n – 1), not just prime powers as in Projective Geometry. Attempts to construct a SSOLS (6, 5) set using ring theory are discussed.

*Key words:* Computer code, Row frequency F-square, Column frequency F-square, Semi-F-square, F-rectangle, Factorial, Main effect, Interaction effect, Field theory.

## 1. INTRODUCTION

Raktoe and Federer (1985) have given a method for constructing F-squares using ring theory. The method applies to mixed prime numbers. It is shown here how to extend their results to obtain complete sets of sum of squares orthogonal F-squares for mixed primes.

In the next section, row (column) frequency F-squares are defined. A regular F-square is both a row and column frequency square. For convenience in this paper, the term F-square will be used to refer to any row frequency, column frequency, or regular F-square.

The concept of SSOFSs of order n and the completeness of such sets were defined by Federer (2004) and complete sets of SSOFSs were constructed for various values of n. Subsequently, Federer (2005, 2006) defined a field theoretic approach to obtain a sum of squares orthogonal geometry that may be used to construct complete sets of sum of squares orthogonal F-squares and F-rectangles for n, a product of distinct primes. As shown by the above authors, an r-row by c-column array may be related to the main effects and interactions of a factorial arrangement. Sum of squares orthogonality results when the sum of squares of the F-square(s) or F-rectangle(s) equals the sum of squares for the corresponding factorial effect. If the sum of the sums of squares for a set of F-squares and/or F-rectangles equals that for the row by column interaction, the set is said to be complete. All of the row by column interaction sum of squares and degrees of freedom are accounted for by the complete set of SSOF-squares and/or SSOF-rectangles. All of the variation for the row by column interaction is taken into account.

Methods for constructing complete sets of F-squares and F-rectangles have been given by Federer (2004, 2005, 2006). These results were used to obtain a class of sum of squares orthogonal fractional replicates (Federer 2007). Trial and error computer codes using field theoretic ideas were utilized for the construction. It is desirable to investigate other methods for constructing a complete set. This is the purpose of this paper. Ring theory will be used to accomplish this goal. A detailed explanation is given for the case of a 6 × 6 = (2 × 3) × (2 × 3) array for a factorial arrangement of four factors A, B, C, and D. The case of a complete set of sum of squares orthogonal Latin squares for n = 6, SSOLS (6, 5) is discussed using ideas

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from ring theory. Finally, some comments on the general case are presented.

## 2. COMPLETE SET OF SUM OF SQUARES ORTHOGONAL F-SQUARES FOR A n × n = 6 × 6 = (2 × 3) × (2 × 3) ARRAY FOR FACTORS (A AND B) AND (C AND D)

The purpose of this section is to develop a construction procedure which assigns to each factorial interaction pertaining to a (2 × 3) × (2 × 3) factorial arrangement a set of F-squares of order six with either two or three symbols such that each set so assigned is a set of SSOF-squares thereby resulting in a complete set of SSOF-squares of order six. We develop the construction procedure, which involves ring theoretic concepts, in the remainder of this section.

The rows and columns of a six-row by six-column arrangement are considered to be a four factor factorial arrangement with factors A = {0, 1} at two levels, B = {0, 1, 2} at three levels for rows, C = {0, 1} at two levels and D = {0, 1, 2} at three levels for columns. The row × column interaction S is composed of the following set of interactions

S = {A × C, A × D, A × C × D, B × C, B × D, B × C × D, A × B × C, A × B × D, A × B × C × D}

Let the ring R(6) = {0, 1, 2, 3, 4, 5} under modulus six arithmetic. The two ideals of R(6) that are of interest are I(3) = {0, 3} and I(4) = {0, 4, 2}. We shall see below that R(6) is isomorphic to the direct product I(3) + I(4) = {00, 30, 04, 34, 02, 32}, where the symbol + is used to denote the direct product of I(3) and I(4).

We now specifically display the isomorphism mentioned above. The elements of the ring R(6) map into the direct product of I(3) + I(4) as follows. 0 → 00, 1 → 34, 2 → 02, 3 → 30, 4 → 04, 5 → 32. The levels of the factors in the factorial map into the elements of I(3) and I(4) as follows

| Map : | A | B | C | D |
|---|---|---|---|---|
| | 0 → 0 | 0 → 0 | 0 → 0 | 0 → 0 |
| | 1 → 3 | 1 → 4 | 1 → 3 | 1 → 4 |
| | | 2 → 2 | | 2 → 2 |

Finally, the mapping f : R(6) → {0, 1, 2} defined by

f(0) = f(5) = 0, f(3) = f(4) = 1, and f(1) = f(2) = 2

will be used in the construction process. Note that facts on the elements of I(3) reducing them modulus two and on the elements of I(4) reducing them modulus three.

**Table 1.** Combinations of

I(6) × I(6) = [I(3) + I(4)] × [I(3) + I(4)]

| Row | Column | | | | | |
|---|---|---|---|---|---|---|
| | 00 | 30 | 04 | 34 | 02 | 32 |
| 00 | 0000 | 0030 | 0004 | 0034 | 0002 | 0032 |
| 30 | 3000 | 3030 | 3004 | 3034 | 3002 | 3032 |
| 04 | 0400 | 0430 | 0404 | 0434 | 0402 | 0432 |
| 34 | 3400 | 3430 | 3404 | 3434 | 3402 | 3432 |
| 02 | 0200 | 0230 | 0204 | 0234 | 0202 | 0232 |
| 32 | 3200 | 3230 | 3204 | 3234 | 3202 | 3232 |

The row effects may be partitioned into the factorial main effects A and B and the interaction effect A × B. Likewise, the column effects may be partitioned into the main effects C and D and the interaction effect C × D. The row × column interaction may be partitioned into the factorial interactions A × C, A × D, A × C × D, B × C, B × D, B × C × D, A × B × C, A × B × D, A × B × C × D. This partitioning for the degrees of freedom and the sums of squares is given in Table 2. Moreover, Table 2 also lists the F-square(s) assigned to each of the preceding set of factorial interactions making up the row by column interaction.

To form F-squares using ring theory, the following steps are involved:

**Step 1:** Construct Table 1.

**Step 2:** Let $z_a$ correspond to the elements of I(3), $z_b$ the elements of I(4), $z_c$ the elements of I(3), and $z_d$ to the elements of I(4). For any choice of a quartet, $(z_a, z_b, z_c, z_d)$, with at least one non-zero z, construct the 6 × 6 square by replacing each quartet $(x_a x_b x_c x_d)$ in Table 1 by $z_a x_a + z_b x_b + z_c x_c + z_d x_d$, the arithmetic being modulus six.

**Step 3:** Apply to each entry of the square obtained in Step 2 the map f defined above thereby obtaining a square of order six in at most three symbols. A two symbol square will occur if all the entries of the original square are in the set {0, 3, 4, 5}.

A square obtained by using Steps 1 to 3 will be denoted as $L(x_a x_b x_c x_d)$. We shall show below that $L(x_a x_b x_c x_d)$ is an F-square in either two or three symbols. There are precisely three F-squares in two symbols, namely, L(3000), L(0030), and L(3030). The remaining F-squares have three symbols. Below we list 19 F-squares together with each defining quartet.

F1 is obtained from $3x_a$, L(3000).
F2 is obtained from $4x_b$, L(0400).
F3 is obtained from $3x_a + 4x_{bm}$, L(3400).
F4 is obtained from $3x_c$, L(0030).
F5 is obtained from $4x_d$, L(0004).
F6 is obtained from $3x_c + 4x_d$, L(0034).
F7 is obtained from $3x_a + 3x_c$, L(3030).
F8 is obtained from $3x_a + 4x_d$, L(3004).
F9 is obtained from $3x_a + 3x_c + 4x_d$, L(3034).
F10 is obtained from $4x_b + 3x_c$, L(0430).
F11 is obtained from $4x_b + 4x_d$, L(0404).
F12 is obtained from $4x_b + 2x_d$, L(0402).
F13 is obtained from $4x_b + 3x_c + 4x_d$, L(0434).
F14 is obtained from $4x_b + 3x_c + 2x_d$, L(0432).
F15 is obtained from $3x_a + 4x_b + 3xc$, L(3430).
F16 is obtained from $3x_a + 4x_b + 4x_d$, L(3404).
F17 is obtained from $3x_a + 4x_b + 2x_d$, L(3402).
F18 is obtained from $3x_a + 4x_b + 3x_c + 4x_d$, L(3434).
F19 is obtained from $3x_a + 4x_b + 3x_c + 2x_d$, L(3432).

Thirteen of these F-squares (F7 to F19) correspond to the factorial interactions making up the row by column interaction and F1, F2, and F3 correspond to rows and F4, F5, and F6 correspond to columns. These 19 F-squares are given in the data set and code presented in Appendix 1. F-squares 7 to 19 form a complete set of SSOF-squares as the sum of the sums of squares for this set of the F-squares adds to that for the row by column interaction. Note in Table 2 the equality of the sums of squares for the factorial effects and those of the corresponding F-squares. This satisfies the definition of sum of squares orthogonality. A regular F-square has the elements occurring an equal number of times in rows and in columns. F7, F11, F12, F13, F14, F16, F17, F18, and F19 are regular F-squares. A semi-F-square (Federer 2004), or a row (column) frequency F-square (Pesotan *et al.* 2005) is one with the symbols appearing equally frequent in the rows (columns) but not in the columns (rows). F8 and F9 are row frequency F-squares. F10 and F15 are column frequency F-squares. This set differs from the one obtained by Federer (2004) indicating that more than one complete set is possible.

The Type I and Type III sums of squares for F1, F3, F4, F7, F8, F9, F10, F15, F17, F18, and F19 are identical. Type I is a nested analysis eliminating only the categories listed above a category and Type III eliminates the effect of all other categories. The degrees of freedom and the sum of squares for the last category listed in Type I and Type III analyses are always identical.

We conclude this section by showing that a square $L(z_a z_b z_c z_d)$ produced by employing steps 1-3 is a F-square. Let s and n be positive integers with s dividing n and let T be a set of s symbols. For convenience, we will refer to a row(column) frequency F-square of order n with s symbols from T as a RF(n, s)-square [CF(n, s)-square] on T, and a regular F-frequency square of order n as a F(n, s)-square on T.

**Table 2.** Degrees of freedom and sums of squares for a row-column, a factorial, and a F-square arrangement

| Source of variation | Deg. of freedom | Sum of squares | F-sq. | Deg. of freed. | Sum of Squares |
|---|---|---|---|---|---|
| Row | 5 | 24.1389 | | | |
| A | 1 | 0.0278 | F1 | 1 | 0.0278 |
| B | 2 | 8.7222 | F2 | 2 | 8.7222 |
| A×B | 2 | 15.3889 | F3 | 2 | 15.3889 |
| Column | 5 | 12.4722 | | | |
| C | 1 | 1.3611 | F4 | 1 | 1.3611 |
| D | 2 | 7.3889 | F5 | 2 | 7.3889 |
| C×D | 2 | 3.7222 | F6 | 2 | 3.7222 |
| Row × Col | 25 | 96.3611 | | | |
| A×C | 1 | 6.2500 | F7 | 1 | 6.2500 |
| A×D | 2 | 1.7222 | F8 | 2 | 1.7222 |
| A×C×D | 2 | 3.1667 | F9 | 2 | 3.1667 |
| B×C | 2 | 3.7222 | F10 | 2 | 3.7222 |
| B×D | 4 | 6.6111 | F11 | 2 | 1.5556 |
| | | | F12 | 2 | 5.0556 |
| B×C×D | 4 | 21.9444 | F13 | 2 | 8.8556 |
| | | | F14 | 2 | 13.0889 |
| A×B×C | 2 | 9.5000 | F15 | 2 | 9.5000 |
| A×B×D | 4 | 28.6111 | F16 | 2 | 4.2222 |
| | | | F17 | 2 | 24.3889 |
| A×B×C×D | 4 | 14.8333 | F18 | 2 | 8.6667 |
| | | | F19 | 2 | 6.1667 |

Let U be any square of order three on R(6) and let x e R(6). By U + x, f(U) we mean the squares obtained from U by replacing each entry u in U by u + x, f(u) respectively, where f is the mapping defined above. Lemma 1 is easily verified and Lemma 2 follows from it.

**Lemma 1:** Let U be a RF(3, 3)-square[CF(3, 3)-square]on the ideal I(4) = {0, 4, 2}. Then for any element x in R(6) – I(4), U + x and f(U) are RF(3, 3)-squares[CF(3,3)-squares)] on the coset x + I(4) = {1, 3, 5} of I(4) and on the set {0, 1, 2} respectively.

**Lemma 2:** Let U be a RF(3, 3)-square[CF(3, 3)-square] on I(4) and let

$$V = \begin{bmatrix} f(U) & f(U+x) \\ f(U+y) & f(U+(x+y)) \end{bmatrix}$$

for any x, y in I(3) = {0, 3). Then V is a RF(6, 3)-square [CF(6, 3)-square] on the set {0, 1, 2).

We subdivide the lattice square of 36 treatment combinations presented in Table 1 into four subsquares as follows:

$$G^{**} = \begin{array}{c|ccc} & 00 & 04 & 02 \\ \hline 00 & 0000 & 0004 & 0002 \\ 04 & 0400 & 0404 & 0402 \\ 02 & 0200 & 0204 & 0202 \end{array}$$

$$H^{**} = \begin{array}{c|ccc} & 30 & 34 & 32 \\ \hline 00 & 0030 & 0034 & 0032 \\ 04 & 0430 & 0434 & 0432 \\ 02 & 0230 & 0234 & 0232 \end{array}$$

$$J^{**} = \begin{array}{c|ccc} & 00 & 04 & 02 \\ \hline 30 & 3000 & 3004 & 3002 \\ 34 & 3400 & 3404 & 3402 \\ 32 & 3200 & 3204 & 3202 \end{array}$$

$$K^{**} = \begin{array}{c|ccc} & 30 & 34 & 32 \\ \hline 30 & 3030 & 3034 & 3032 \\ 34 & 3430 & 3434 & 3432 \\ 32 & 3230 & 3234 & 3232 \end{array}$$

Let $(z_a z_b z_c z_d)$ be any quartet with at least one z nonzero and with $z_a$ and $z_c$ in I(3) and $z_b$ and $z_d$ in I(4). Use this quartet together with step 2 of the construction procedure on each of the above four subsquares to obtain the corresponding subsquares G*, H*, J*, and K*. Finally let G = f(G*), H = f(H*), J = f(J*), and K = f(K*). We note the following:

(a) Since I(4) is an ideal, all the entries of G* are from I(4)

(b) $H^* = G^* + 3z_c$, $J^* = G^* + 3z_a$, $K^* = G^* + (3z_a + 3z_c)$

(c) Up to a rearrangement of its rows and columns

$$L = L(z_a z_b z_c z_d) = \begin{bmatrix} G & H \\ J & K \end{bmatrix}$$

**Theorem:** $L(z_a z_b z_c z_d)$ is a F-square of order six on {0, 1, 2} when at least one of the $z_b$ or $z_d$ is nonzero and it is a F-square on {0, 1} when $z_b = z_d = 0$. Further

(i) when $z_b \neq 0$, $z_d \neq 0$, then L is a regular F(6, 3)-square on {0, 1, 2}

(ii) when $z_b = 0$, $z_d \neq 0$, then L is RF(6, 3)-square and when $z_b \neq 0$, $z_d = 0$, then L is a CF(6, 3)-square on the set {0, 1, 2}

(iii) when $z_b = z_c = z_d = 0$, then L is a RF(6, 2)-square on the set {0, 1}, and when $z_a = z_b = z_d = 0$, then L is a CF(6, 2)-square each on the set {0, 1}, and

(iv) finally, if $z_b = z_d = 0$ but $z_a \neq 0$, $z_c \neq 0$, then L is a regular F(6, 2)-square on {0, 1}

**Proof:** Due to (a), (b), (c) and Lemma 2 to prove the theorem, it suffices to show that G* is a F-square of order three on I(4) and that in case (i) of the theorem, G* is a regular F(3,3)-square, in case (ii) that G* is a RF(3, 3)-square[CF(3, 3)-square] depending on the assumptions on $z_b$, $z_d$, and that in case (iii) G* is the zero square all of whose elements are zero. All of this is easily verified by imposing various conditions in (i) to (iv) on the entries of G* which we now fully display

$$G^* = \begin{bmatrix} 0 & 4Z_d & 2Z_d \\ 4Z_b & 4Z_b + 4Z_d & 4Z_b + 2Z_d \\ 2Z_b & 2Z_b + 4Z_d & 2Z_b + 2Z_d \end{bmatrix}$$

This completes the proof.

**Remark:** An equivalent set of sum of squares orthogonal F-squares is given by the set L(3030), L(3002), L(3032), L(0230), L(0204), L(0234), L(0202), L(0232), L(3230), L(3204), L(3202), L(3234), and L(3232). The set of F-squares is given in Appendix 2. The set merely permutes the rows of F-squares F7 to F19 and this does not change the sum of squares orthogonality.

## A SSOLS(6, 5) SET

Federer (2005) has shown how to construct a complete set of sum of squares orthogonal Latin squares and illustrated the method for SSOLS(6, 5) and SSOLS(10, 9) sets. The method given is to start with any Latin square and cyclically permute the last n × 1 rows to obtain the n × 1 Latin squares of the set. This field theoretic method produces a connected incomplete block design for any pair of Latin squares with parameters $v = n$, $k = 2$, and $b = n^2$. An SSOLS(15, 14) set is available from the author.

Using the sums for $3x_a + 4x_b + 3x_c + 4x_d$ and $3x_a + 4x_b + 3x_c + 2x_d$ , modulus 6, results in two Latin squares. Likewise, using the sums for $3x_a + 2x_b + 3x_c + 4x_d$ and $3x_a + 2x_b + 3x_c + 2x_d$ , modulus 6, results in another pair of Latin squares. The four Latin squares obtained are

| $3x_a + 4x_b + 3x_c + 4x_d$ | $3x_a + 4x_b + 3x_c + 2x_d$ |
|---|---|
| 0 3 4 1 2 5 | 0 3 2 5 4 1 |
| 3 0 1 4 5 2 | 3 0 5 2 1 4 |
| 4 1 2 5 0 3 | 4 1 0 3 2 5 |
| 1 4 5 2 3 0 | 1 4 3 0 5 2 |
| 2 5 0 3 4 1 | 2 5 4 1 0 3 |
| 5 2 3 0 1 4 | 5 2 1 4 3 0 |

| $3x_a + 2x_b + 3x_c + 4x_d$ | $3x_a + 2x_b +3x_c + 2x_d$ |
|---|---|
| 0 3 4 1 2 5 | 0 3 2 5 4 1 |
| 3 0 1 4 5 2 | 3 0 5 2 1 4 |
| 2 5 0 3 4 1 | 2 5 4 1 0 3 |
| 5 2 3 0 1 4 | 5 2 1 4 3 0 |
| 4 1 2 5 0 3 | 4 1 0 3 2 5 |
| 1 4 5 2 3 0 | 1 4 3 0 5 2 |

These Latin squares are not sum of squares orthogonal as the first pair has two identical columns. The first and third and the second and fourth Latin squares have the same rows when permuted. Such conditions do not result in a set of sum of squares orthogonal Latin squares. If row 1 and row 2 of one Latin square of a pair are interchanged, the two Latin squares are sum of squares orthogonal. However, this method only produces two distinct Latin squares and five are required for the set SSOLS(6, 5). None of the other combinations of factors modulus 6 resulted in a Latin square. An open question is whether or not ring theory can be used to produce a set of five sum of squares orthogonal Latin squares.

## COMMENTS

Ring theory and field theory can both be used to construct complete sets of sum of squares orthogonal F-squares and F-rectangles. Ring theory is more general. An open question pertains to which method is optimal for constructing fractional replicates and/or codes. Are there other methods for constructing complete sets? How many such sets of SSOFSs are there for various values of n? The sum of squares orthogonal geometry involves statistical concepts. What are the equivalent mathematical concepts?

The procedures discussed herein are extendable for values of n in general. The results of Federer (2004, 2005, 2006) may be extended in a straightforward manner to include cubes and hypercubes. The ideals of a ring with n elements are required as a starting point.

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## APPENDIX 1

### SAS PROC GLM CODES AND DATA FOR OBTAINING TABLE 2

```
DATA RING6;

/*F1 = A3, F2 = B4, F3 = A3B4, F4 = C3, F5 = D4, F6 = C3D4, F7 = A3C3, F8 = A3D4, F9 = A3C3D4, F10 = B4C3, F11 = B4D4, F12 = B4D2, F13 = B4C3D4,F14 = B4C3D2, F15 = A3B4C3, F16 = A3B4D4, F17 = A3B4D2, F18 = A3B4C3D4,F19 = A3B4C3D2, L1= A3B4C3D4, L2 = A3B4C3D2*/

input ROW COL Y A B C D F1 F2 F3 F4 F5 F6 F7 F8 F9 F10 F11 F12 F13 F14 F15 F16 F17 F18 F19 L1 L2; lines;

1 1 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 2 3 0 0 1 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 0 0 1 1 3 3
1 3 7 0 0 0 1 0 0 0 0 1 1 0 1 1 0 1 2 1 2 0 1 2 1 2 4 2
1 4 8 0 0 1 1 0 0 0 1 1 2 1 1 2 1 1 2 2 0 1 1 2 2 0 1 5
1 5 5 0 0 0 2 0 0 0 0 2 2 0 2 2 0 2 1 2 1 0 2 1 2 1 2 4
1 6 6 0 0 1 2 0 0 0 1 2 0 1 2 0 1 2 1 0 2 1 2 1 0 2 5 1
2 1 6 1 0 0 0 1 0 1 0 0 0 1 1 1 0 0 0 0 0 1 1 1 1 1 3 3
2 2 6 1 0 1 0 1 0 1 1 0 1 0 1 0 1 0 0 1 1 0 1 1 2 0 0 0
2 3 6 1 0 0 1 1 0 1 0 1 1 1 2 2 0 1 2 1 2 1 2 0 2 0 1 5
2 4 6 1 0 1 1 1 0 1 1 1 2 0 2 1 1 1 2 2 0 0 2 0 0 2 4 2
2 5 6 1 0 0 2 1 0 1 0 2 2 1 0 0 0 2 1 2 1 1 0 2 0 2 5 1
2 6 6 1 0 1 2 1 0 1 1 2 0 0 0 2 1 2 1 0 2 0 0 2 1 1 2 4
3 1 6 0 1 0 0 0 1 1 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 4 4
3 2 2 0 1 1 0 0 1 1 1 0 1 1 0 1 2 1 1 2 2 2 1 1 2 2 1 1
3 3 3 0 1 0 1 0 1 1 0 1 1 0 1 1 1 2 0 2 0 1 2 0 2 0 2 0
3 4 4 0 1 1 1 0 1 1 1 1 2 1 1 2 2 2 0 0 1 2 2 0 0 1 5 3
3 5 5 0 1 0 2 0 1 1 0 2 2 0 2 2 1 0 2 0 2 1 0 2 0 2 0 2
3 6 6 0 1 1 2 0 1 1 1 2 0 1 2 0 2 0 2 1 0 2 0 2 1 0 3 5
4 1 9 1 1 0 0 1 1 2 0 0 0 1 1 1 1 1 1 1 1 2 2 2 2 2 1 1
4 2 8 1 1 1 0 1 1 2 1 0 1 0 1 0 2 1 1 2 2 1 2 2 0 1 4 4
4 3 7 1 1 0 1 1 1 2 0 1 1 1 2 2 1 2 0 2 0 2 0 1 0 1 5 3
4 4 5 1 1 1 1 1 1 2 1 1 2 0 2 1 2 2 0 0 1 1 0 1 1 0 2 0
4 5 4 1 1 0 2 1 1 2 0 2 2 1 0 0 1 0 2 0 2 2 1 0 1 0 3 5
4 6 3 1 1 1 2 1 1 2 1 2 0 0 0 2 2 0 2 1 0 1 1 0 2 2 0 2
5 1 6 0 2 0 0 0 2 2 0 0 0 0 0 0 2 2 2 2 2 2 2 2 2 2 2 2
5 2 8 0 2 1 0 0 2 2 1 0 1 1 0 1 0 2 2 0 0 0 2 2 0 0 5 5
5 3 7 0 2 0 1 0 2 2 0 1 1 0 1 1 2 0 1 1 1 2 0 1 0 1 0 4
5 4 5 0 2 1 1 0 2 2 1 1 2 1 1 2 0 0 1 2 2 0 0 1 1 2 3 1
5 5 7 0 2 0 2 0 2 2 0 2 2 0 2 2 2 1 0 0 0 2 1 0 1 0 4 0
5 6 2 0 2 1 2 0 2 2 1 2 0 1 2 0 0 1 0 1 1 0 1 0 2 1 1 3
6 1 2 1 2 0 0 1 2 0 0 0 0 1 1 1 2 2 2 2 2 0 0 0 0 0 5 5
6 2 5 1 2 1 0 1 2 0 1 0 1 0 1 0 0 2 2 0 0 2 0 0 1 2 2 2
6 3 3 1 2 0 1 1 2 0 0 1 1 1 2 2 2 0 1 1 1 0 1 2 1 2 3 1
6 4 8 1 2 1 1 1 2 0 1 1 2 0 2 1 0 0 1 2 2 2 1 2 2 1 0 4
6 5 4 1 2 0 2 1 2 0 0 2 2 1 0 0 2 1 0 0 0 0 2 1 2 1 1 3
6 6 4 1 2 1 2 1 2 0 1 2 0 0 0 2 0 1 0 1 1 2 2 1 0 0 4 0
;

proc glm data = RING6;
    class A B C D;
    model Y = A B A*B C D C*D A*C A*D A*C*D
              B*C B*D B*C*D A*B*C A*B*D
              A*B*C*D;

run;
proc glm data = RING6;
    class ROW COL;
    model Y = ROW COL;
run;
proc glm data = RING6;
    class F1 F2 F3 F4 F5 F6 F7 F8 F9 F10 F11
          F12 F13 F14 F15 F16 F17 F18 F19;
    model Y = F1 F2 F3 F4 F5 F6 F7 F8 F9 F10
              F11 F12 F13 F14 F15 F16 F17
              F18 F19;

run;

proc glm data = RING6;
    class ROW COL L1 L2;
    model Y = ROW COL L1 L2;

run;
```

## APPENDIX 2
## SAS PROC GLM CODES AND DATA FOR AN EQUIVALENT SET OF SUM OF SQUARES ORTHOGONAL F-SQUARES

```
data RING62;

/* F1 = L(3030), F2 = L(3002), F3 = L(3032),
F4 = L(0230), F5 = L(0204), F6 = L(0202),
F7 = L(0234), F8 = L(0232),  F9 = L(3230),
F10 = L(3204), F11 = L(3202), F12 = L(3234),
F13 = L(3232)*/

input ROW COL Y A B C D F1 F2 F3 F4 F5 F6 F7
F8 F9 F10 F11 F12 F13;

lines;
1 1 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 2 3 0 0 1 0 1 0 1 1 0 0 1 1 1 0 0 1 1
1 3 7 0 0 0 1 0 2 2 0 1 2 1 2 0 1 2 1 2
1 4 8 0 0 1 1 1 2 0 1 1 2 2 0 1 1 2 2 0
1 5 5 0 0 0 2 0 1 1 0 2 1 2 1 0 2 1 2 1
1 6 6 0 0 1 2 1 1 2 1 2 1 0 2 1 2 1 0 2
2 1 6 1 0 0 0 1 1 1 0 0 0 0 0 1 1 1 1 1
2 2 6 1 0 1 0 0 1 0 1 0 0 1 1 0 1 1 0 0
2 3 6 1 0 0 1 1 0 0 0 1 2 1 2 1 2 0 2 0
2 4 6 1 0 1 1 0 0 2 1 1 2 2 0 0 2 0 1 2
2 5 6 1 0 0 2 1 2 2 0 2 1 2 1 1 0 2 0 2
2 6 6 1 0 1 2 0 2 1 1 2 1 0 2 0 0 2 2 1
3 1 6 0 1 0 0 0 0 0 2 2 2 2 2 2 2 2 2 2
3 2 2 0 1 1 0 1 0 1 0 2 2 0 0 0 2 2 0 0
3 3 3 0 1 0 1 0 2 2 2 0 1 0 1 2 0 1 0 1
3 4 4 0 1 1 1 1 2 0 0 0 1 1 2 0 0 1 1 2
3 5 5 0 1 0 2 0 1 1 2 1 0 1 0 2 1 0 1 0
3 6 6 0 1 1 2 1 1 2 0 1 0 2 1 0 1 0 2 1
4 1 9 1 1 0 0 1 1 1 2 2 2 2 2 0 0 0 0 0
4 2 8 1 1 1 0 0 1 0 0 2 2 0 0 2 0 0 2 2
4 3 7 1 1 0 1 1 0 0 2 0 1 0 1 0 2 2 1 2
4 4 5 1 1 1 1 0 0 2 0 0 1 1 2 2 2 2 0 1
4 5 4 1 1 0 2 1 2 2 2 1 0 1 0 0 1 1 2 1
4 6 3 1 1 1 2 0 2 1 0 1 0 2 1 2 1 1 1 0
5 1 6 0 2 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1
5 2 8 0 2 1 0 1 0 1 2 1 1 2 2 2 1 1 2 2
5 3 7 0 2 0 1 0 2 2 1 2 0 2 0 1 2 0 2 0
5 4 5 0 2 1 1 1 2 0 2 2 0 0 1 2 2 0 0 1
5 5 7 0 2 0 2 0 1 1 1 0 2 0 2 1 0 2 0 2
5 6 2 0 2 1 2 1 1 2 2 0 2 1 0 2 0 2 1 0
6 1 2 1 2 0 0 1 1 1 1 1 1 1 1 2 2 2 2 2
6 2 5 1 2 1 0 0 1 0 2 1 1 2 2 1 2 2 1 1
6 3 3 1 2 0 1 1 0 0 1 2 0 2 0 2 0 1 0 1
6 4 8 1 2 1 1 0 0 2 2 2 0 0 1 1 0 1 2 0
6 5 4 1 2 0 2 1 2 2 1 0 2 0 2 2 1 0 1 0
6 6 4 1 2 1 2 0 2 1 2 0 2 1 0 1 1 0 0 2
;

proc glm data = RING62;

        class A B C D;

        model Y= A B A*B C D C*D A*C A*D
                 A*C*D B*C B*D B*C*D A*B*C
                 A*B*D A*B*C*D;

run;

proc glm data = RING62;

        class ROW COL F1 F2 F3 F4 F5 F6 F7 F8 F9
                 F10 F11 F12 F13;

        model Y = ROW COL F1 F2 F3 F4 F5 F6 F7
                 F8 F9  F10 F11 F12 F13;

run;
```