J. Ind. Soc. Agril. Statist.
61(3), 2007 : 361-363

Construction of Ternary Group Divisible Designs

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(Received : October 2006)

SUMMARY

In this paper we present a new method for construction of ternary group divisible designs using singular and semiregular group divisible designs. Association scheme of the resulting ternary group divisible designs is same as the corresponding singular and semiregular group divisible designs.

Key words : Balanced Ternary design, Singular and Semiregular Group Divisible design, Ternary Group Divisible design, BIB design.

1. INTRODUCTION

First reference to n-ary designs was made by Tocher 1952. For an n-ary design each entry in the incidence matrix can take any of the n possible values usually 0, 1, 2,…, (n – 1). If n = 3 then we get a ternary block design. Two survey papers by Billington 1984, 1989 give results on balanced ternary designs. Balanced ternary designs may not exist for all parametric combinations or even if exist may require a large number of replications. In this paper we give a method of construction of Ternary Group Divisible Designs (TGD) using singular and semiregular group divisible designs.

Denig and Sarvate 1992 have defined TGD as follows.

Let A be a set of V treatments arranged in B blocks. Let $\{G_i \mid i = 1, 2, \ldots, m\}$ be a partition of A into m sets each of size n, called groups. The groups define an association scheme on A with two classes; two treatments are first associates if they belong to the same group and are second associates otherwise. A TGD with parameters $(m, n, V, B, \rho_1, \rho_2, R, K, \Lambda_1, \Lambda_2)$ is defined to be an incidence structure satisfying

$$\sum_{j=1}^{B} n_{ij} = R \text{ for each } i = 1, 2, \ldots, V \quad (1.1)$$

$$\sum_{i=1}^{V} n_{ij} = K \text{ for each } j = 1, 2, \ldots, B \quad (1.2)$$

and $\sum_{k=1}^{B} n_{ik} n_{jk} = \Lambda_1$ if treatments i and j $(i \neq j)$ are first associates

$= \Lambda_2$ otherwise

where n_{ij} = number of times i^{th} treatment occurs in j^{th} block, $n_{ij} \in \{0, 1, 2\}$

$i = 1, 2, \ldots, V;\ j = 1, 2, \ldots, B$

Each treatment occurs with multiplicity '1' in ρ_1 blocks and with multiplicity '2' in ρ_2 blocks.

For a TGD following two results hold.

$$BK = VR \quad (1.3)$$

$$\Lambda_1(n-1) + \Lambda_2(V-n) = R(K-1) - 2\rho_2 \quad (1.4)$$

2. CONSTRUCTION OF TGDs FROM SINGULAR AND SEMIREGULAR GDDs

Definitions of singular and semiregular group divisible designs (GDD) can be found in Bose *et al.* (1954). A new construction of TGDs from singular and semiregular GDDs is studied through two theorems given below.

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The blocks of TGD are obtained from blocks of existing singular GDD by adding pairs of first associate treatments which occur in those blocks, one at a time.

Theorem 2.1. From a singular GDD with parameters ($v = mn$, b, $r = \lambda_1$, k, λ_1, λ_2) (v treatments are divided into m groups of n each) with p groups of treatments occurring in each block where $p = k/n$ a TGD can be constructed with parameters

$$V = v = mn,\ B = pn(n-1)b/2$$

$$\rho_1 = \lambda_1(n-1)(np-2)/2,\ \rho_2 = \lambda_1(n-1)$$

$$R = \lambda_1(n-1)(np+2)/2,\ K = k+2$$

$$\Lambda_1 = \lambda_1(n^2p - np + 4n - 2)/2,\ \Lambda_2 = \lambda_2(n-1)(np+4)/2$$

Proof. For a singular GDD, $r = \lambda_1$, hence if a treatment occurs in a certain block then every treatment belonging to the group occurs in that block. Bose and Connor (1952) have proved that given any BIB design a singular GDD can always be obtained by replacing each treatment by a group of n treatments. Hence $p = k/n$ groups of treatments occur in each of the blocks of the singular GDD.

Applying the procedure mentioned above we get a TGD with parameters mentioned in Theorem 2.1. The parameters V, B and K need no explanation. Remaining parameters are explained below.

ρ_1: Consider a block containing treatment x and its first associates. The block contains $(p-1)$ groups of treatments other than this group. From this block we get $(p-1)\,{}^nC_2$ pairs of first associate treatments which are added to this block. Therefore we get $\lambda_1(p-1)\,{}^nC_2$ blocks containing treatment x with multiplicity '1'. Hence first term in expression for ρ_1 is $\lambda_1(p-1)^nC_2$.

In the group of treatment x and its first associates, we get $(n-1)$ treatments other than x which give rise to ${}^{(n-1)}C_2$ first associate treatment pairs. Hence we get $\lambda_1{}^{(n-1)}C_2$ blocks in which treatment x occurs with multiplicity '1'. Therefore second term in expression for ρ_1 is $\lambda_1{}^{(n-1)}C_2$.

Thus
$$\rho_1 = \lambda_1(p-1)\,{}^nC_2 + \lambda_1{}^{(n-1)}C_2$$
$$= \lambda_1(n-1)(np-2)/2$$

ρ_2: If $(n-1)$ first associate treatment pairs containing treatment x are added to each of the λ_1 blocks containing treatment x then we get $\lambda_1(n-1)$ blocks in which treatment x occurs with multiplicity '2'. Hence $\rho_2 = \lambda_1(n-1)$.

Replication number R for treatment x is $R = \rho_1 + 2\rho_2$. Hence $R = \lambda_1(n-1)(np+2)/2$.

Λ_1 : Consider a treatment pair (x, y) where x and y are first associates. To each of the λ_1 blocks of singular GDD containing treatment pair (x, y), if pair (x, y) is added we get λ_1 blocks in which the treatment pair (x, y) occurs 4 times. Therefore, first term in the expression for Λ_1 is $4\lambda_1$.

There are $(n-2)$ pairs of first associate treatments other than treatment pair (x, y) which contain treatment x. If these $(n-2)$ pairs are added to each of the λ_1 blocks then we get $\lambda_1(n-2)$ blocks in which treatment x occurs with multiplicity '2' and y occurs with multiplicity '1' i.e. treatment pair (x, y) occurs twice. Similarly by adding $(n-2)$ pairs of first associate treatments which contain treatment y other than the pair (x, y) to each of the λ_1 blocks, we get $\lambda_1(n-2)$ blocks in which treatment pair (x, y) occurs twice. Thus second term in the expression for Λ_1 is $4\,\lambda_1(n-2)$.

Each of the λ_1 blocks contain $(p-1)$ groups of treatments other than the group of treatment x and its first associates. ${}^nC_2$ pairs of first associate treatments from each of the $(p-1)$ groups when added to each of the λ_1 blocks we get $\lambda_1(p-1)^nC_2$ blocks in which treatment pair (x, y) occurs once. Therefore third term in the expression for Λ_1 is $\lambda_1(p-1)^nC_2$.

In each of the λ_1 blocks containing treatment pair (x, y), there are $(n-2)$ treatments from the same group as that of x and y. If the ${}^{(n-2)}C_2$ pairs of first associate treatments arising from these $(n-2)$ treatments are added to each of the λ_1 blocks then we get $\lambda_1{}^{(n-2)}C_2$ blocks in which treatment pair (x, y) occurs once. Therefore fourth term in the expression for Λ_1 is $\lambda_1{}^{(n-2)}C_2$.

Hence
$$\Lambda_1 = 4\lambda_1 + 4\lambda_1(n-2) + \lambda_1(p-1)^nC_2 + \lambda_1{}^{(n-2)}C_2$$
$$= \lambda_1(n^2p - np + 4n - 2)/2$$

Λ_2 : Let treatments x and u be second associates of each other. These treatments occur together in λ_2 blocks. By adding all the first associate treatment pairs to these λ_2 blocks we get λ_2 as

$$\Lambda_2 = 4\lambda_2(n-1) + 2\lambda_2{}^{(n-1)}C_2 + \lambda_2\,{}^nC_2(p-2)$$
$$= \lambda_2(n-1)(np+4)/2$$

It can easily be seen that the resulting TGD has the same association scheme as that of the singular GDD. Theorem 2.1 can be illustrated with help of following example.

Example 1: Consider a singular GDD S12 (Bose *et al.* (1954)) with parameters $v = 9$, $b = 3$, $r = 2$, $k = 6$, $m = 3$, $n = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$

Groups	Plan
1 4 7	1 4 7 2 5 8
2 5 8	2 5 8 3 6 9
3 6 9	3 6 9 1 4 7

Applying Theorem 2.1 we get the following blocks

B_1	1	1	4	4	7	2	5	8
B_2	1	1	4	7	7	2	5	8
B_3	1	4	4	7	7	2	5	8
B_4	1	4	7	2	2	5	5	8
B_5	1	4	7	2	2	5	8	8
B_6	1	4	7	2	5	5	8	8
B_7	2	2	5	5	8	3	6	9
B_8	2	2	5	8	8	3	6	9
B_9	2	5	5	8	8	3	6	9
B_{10}	2	5	8	3	3	6	6	9
B_{11}	2	5	8	3	3	6	9	9
B_{12}	2	5	8	3	6	6	9	9
B_{13}	1	1	4	4	7	3	6	9
B_{14}	1	1	4	7	7	3	6	9
B_{15}	1	4	4	7	7	3	6	9
B_{16}	1	4	7	3	3	6	6	9
B_{17}	1	4	7	3	3	6	9	9
B_{18}	1	4	7	3	6	6	9	9

Blocks B_1 to B_{18} form a TGD with parameters $V = v = 9$, $m = 3$, $n = 3$, $B = 18$, $\rho_1 = 8$, $\rho_2 = 4$, $R = 16$, $K = 8$, $\Lambda_1 = 22$, $\Lambda_2 = 10$

Association scheme of this TGD is same as that of the singular GDD. A series of TGDs which can be constructed from singular GDDs is given by Deshpande (2000).

A catalogue of TGDs obtainable from Theorem 2.1 using singular GDDs of Bose *et al*. (1954) is given in Table 2.1. Reference number of each singular GDD is also given.

Theorem 2.2. If a semiregular GDD with parameters $v = mn$, b, r, $k = cm$, λ_1, λ_2 (v treatments are divided into m groups of n each) with $c \geq 2$ exists then there exists a TGD with parameters $V = v = mn$, $B = mc(c-1)b/2$, $\rho_1 = r(c-1)(cm-2)/2$, $\rho_2 = r(c-1)$, $R = r(c-1)(cm+2)/2$, $K = k+2$, $\Lambda_1 = \lambda_1(c^2m - cm + 4c - 2)/2$, $\Lambda_2 = \lambda_2(c-1)(cm+4)/2$

Proof. Bose and Connor (1952) have proved that for a semiregular GDD, k = cm i.e. a constant number of treatments (c) from each of the m groups occur in each block. Proof of Theorem 2.2 follows on the same lines as that of Theorem 2.1.

Example 2. A TGD with parameters $V = 6$, $m = 2$, $n = 3$, $B = 18$, $\rho_1 = 6$, $\rho_2 = 6$, $R = 18$, $K = 6$, $\Lambda_1 = 15$, $\Lambda_2 = 16$ can be obtained from semiregular GDD SR4 (Bose *et al*. (1954) with parameters $v = 6$, $m = 2$, $n = 3$, $b = 9$, $r = 6$, $k = 4$, $\lambda_1 = 3$, $\lambda_2 = 4$. Blocks of the TGD can be obtained as shown in Example 1; by using association scheme given in SR4.

One major limitation of Theorem 2.2 is that it gives designs with large number of replications. Using semiregular GDDs given in Bose *et al*. (1954), no TGD with $R \leq 10$ could be obtained. Therefore, further efforts are required to obtain TGDs with smaller number of replications.

Table 2.1

Design (Bose et al. (1954))	V	m	B	ρ_1	ρ_2	R	K	Λ_1	Λ_2
S1	6	3	6	2	2	6	6	10	4
S6	8	4	12	3	3	9	6	15	4

ACKNOLEDGEMENTS

Authors are thankful to the referees for comments.

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