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Construction of Partially Balanced Incomplete Block Designs through Unreduced Balanced Incomplete Block Designs

Vijayaraghava Kumar
College of Agriculture, Thiruvananthapuram
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SUMMARY

A method of construction of partially balanced incomplete block designs with three associate classes making use of unreduced balanced incomplete block designs with block concurrency, $\lambda \geq 2$ is illustrated. Association scheme, parameters of the design and method of construction of such designs are studied. A list of designs so constructed is given. These designs may be constructed in block of size six, twelve or higher numbers with comparatively small number of replications.

Key words : Balanced incomplete block designs, PBIBD, Association scheme, Three-Associate classes.

1. INTRODUCTION

Construction of Partially Balanced Incomplete Block (PBIB) designs are attempted by making use of ordered pairs of treatment symbols from each block of unreduced Balanced Incomplete Block (BIB) designs. An unreduced BIB design with v treatments, in b blocks each of size k units is constructed by taking all possible combinations of k symbols $(k \leq v)$ in each of the blocks, to develop $b = \binom{v}{k}$ blocks, for any fixed k. Unreduced BIB designs with number of concurrency of any two treatments in a block, $\lambda \geq 2$ were used for this study. This condition $\lambda \geq 2$ ensured that when every pair of treatments will occur together in at least two blocks, then any three treatments, will occur together in at least one block of the unreduced BIB designs with number of concurrences of any two treatments in the design, $\lambda \geq 2$ are used to ensure that any three treatments occur together in at least one block of the chosen design. Additional treatments are generated from each pair of symbols of these unreduced BIB designs to get the new PBIB designs. The association scheme followed by these designs as well as a list of designs so developed is presented.

2. DEVELOPING ADDITIONAL TREATMENTS

Consider an unreduced BIB design with block size $k \geq 3$. Let any two treatment symbols in a block be denoted by symbols i and j, $(i \neq j)$. There are two ordered pairs for these two symbols, viz. ij and ji. Now to deduce treatments of the PBIB design, combine the same symbol i.e. i with i or j with j, a fixed number of times in an ordered pair ij, so that no new ordered pair of distinct symbols occur together by this process. As an example, from the pair of symbols ij, two ordered triplets iij and ijj and three quadruplets iiij, iijj and ijjj can be developed. We make use of triplets, quadruplets or the combinations of symbols in higher numbers (developed from an ordered pair of two treatment symbols) in constructing a PBIB design. Each of the combinations represents a treatment symbol in the new design. Further, it was observed that this procedure of obtaining additional treatment symbols leads to an association scheme.

3. DISTINCT PAIR ASSOCIATION SCHEME

This is a three class association scheme defined on a pair of symbols i and j, $(i \neq j)$ from among a set of v symbols (i, j = 1, 2, ...v). Let there be $v^* = pv(v-1)$ new treatments (symbols) developed from these v symbols (p is any positive integer) by the following method.

1. When p = 1; take the v(v – 1) distinct ordered pairs based on v symbols as the new set of treatments, so that the total number of symbols can be expressed as $v^* = pv(v-1)$.

2. When p = 2; with each of the v(v – 1) distinct ordered pairs of symbols like ij, ij, $(i \neq j)$ associate two triplets of the type iij and ijj so as to get a total of $v^* = 2v(v-1)$ or $v^* = pv(v-1)$ triplets (or new treatments).

3. When p = 3; with each of the v(v – 1) distinct ordered pairs of symbols like ij, $(i \neq j)$ associate three quadruplets making use of the symbol i and j containing the ordered pair ij occurring only once in them as ijjj, iijj and iiij, to develop a total of $v^* = pv(v-1)$ treatments.

If we continue the process of combining additional symbols on same pair ij for any p, $v^* = pv(v-1)$ new treatments can be developed from v symbols, the number of treatments associated with a pair of ordered symbols being p. Ignoring the order of ij (taking ij as an unordered pair of two symbols), a set of 2p treatments can be developed from an unordered pair 'ij' by this method, and we call it as the 'ij group' of treatment symbols, for any given value of p.

The three class Distinct Pair (D.P.) association scheme (on v symbols) is now defined as follows:

1. For any given symbol belonging to the ' ij group' its first associates are the remaining (2p – 1) symbols in that ' ij group' to which it belongs.

2. The second associate of any treatment symbol belonging to an 'ij group' are all those treatments in the 'ik group' and 'jk group' for all i, j, k = 1, 2,..., v and $i \neq j \neq k$.

3. The third associate of any treatment in the 'ij group' will be all those treatments belonging to the 'kl group' where $i \neq j \neq k \neq l$ and for all i, j, k, l = 1, 2, ... v.

The parameters of this association scheme can be worked out as

$$v^* = 2p\binom{v}{2} = p(v)(v-1); \quad n_1 = (2p-1)$$
$$n_2 = 4p(v-2) \quad \text{and} \quad n_3 = p(v-2)(v-3)$$

where n_1, n_2 and n_3 denote the number of treatments in the first, second and third associate classes. The permutation matrices P^1, P^2 and P^3 of this scheme can be worked out as follows

$$\mathbf{P}^1 = \begin{pmatrix} 2p-2 & 0 & 0 \\ 0 & 4p(v-2) & 0 \\ 0 & 0 & p(v-2)(v-3) \end{pmatrix}$$

$$\mathbf{P}^2 = \begin{pmatrix} 0 & 2p-1 & 0 \\ 2p-1 & 2p(v-2) & 2p(v-3) \\ 0 & 2p(v-3) & p(v-3)(v-4) \end{pmatrix}$$

$$\mathbf{P}^3 = \begin{pmatrix} 0 & 0 & 2p-1 \\ 0 & 8p & 4p(v-4) \\ 2p-1 & 4p(v-4) & p(v-4)(v-5) \end{pmatrix}$$

Proof : Consider any treatment belonging to the 'ij group'. 2p treatments can be found in this group for any positive integer p. For a given treatment of this group the treatments other than itself are (2p – 1) which are its first associates. Obviously $n_1 = (2p-1)$.

The second associate of a treatment belonging to a given 'ij group' are those treatments belonging to the 'ik group' and 'jk group', i, j, k = 1, 2, ...,v and $i \neq j \neq k$. Since one can obtain (v – 2) distinct ordered pairs of symbols from each of the ik and jk groups (for all values of $k \neq i \neq j$), the number of treatments which are second associates of any treatment in the ij group is $n_2 = (2p)\,2(v-2) = 4p(v-2)$.

The third associate of a treatment in the 'ij group' are those belonging to the kl group (for all k and $l \neq i$ and $\neq j$; $k \neq l$; k, l = 1, 2, ... v). Obviously, the number of treatments in this associate class is $n_3 = 2p\binom{v-2}{2} = p(v-2)(v-3)$.

The elements of the permutation matrices $\mathbf{P}^1$, $\mathbf{P}^2$ and $\mathbf{P}^3$ can also be worked out following the same arguments. For example the element P^1_{11} of the matrix $\mathbf{P}^1$ denote the number of symbols common between the first associate classes of two symbols, which are first associates, which is 2p minus the given two symbols is equal to (2p – 2). The other two elements of the first row being zero, since for two symbols which are first associates the number of treatments common between the first associate one symbol and the second or third associate of another symbol being zero. P^1_{22} gives the number of elements common between the second associates of two symbols which are first associates. The treatments coming in the second associate class of any two symbols say, i and j which are first associates are the same (since these treatments are based on ik and jk groups) numbering $4p(v-2)$. Value of P^1_{23} is also zero as there are no common treatments between second and third associate of two treatments which are first associates. P^1_{33} is the number of common elements in the third associate class of two symbols which are first associates, the number being $p(v-2)(v-3)$ same as n_3. The elements of the $\mathbf{P}^2$ and $\mathbf{P}^3$ matrices can also be worked out in a similar manner.

4. CONSTRUCTION OF PBIB DESIGNS

From each block of an unreduced BIB design, a new block of a PBIB design can be developed by utilising all pairs of symbols and corresponding symbols developed through their groups. The new treatment symbols can be given identification numbers in the following way.

Numbering of the New Treatments

The 2p symbols of each 'ij group' can be numbered as $(ij)_1, (ij)_2, \ldots, (ij)_{2p}$. If p = 1 then there will be two symbols only in each 'ij group', that can be simply written as (ij) and (ji). This type of numbering will help in identifying the parameters of the permutation matrix. However, when this association scheme is used for constructing new PBIB designs and also for carrying out the analysis, the newly developed treatments can be serially numbered from 1 to v^* in any order.

Using notations for the parameters of the BIB and PBIB designs as given below, the following theorem is proved, from which the parameters of the PBIB design are derived.

Consider a BIB design (v, b, r, k and λ), where v denotes number of treatments, b number of blocks, r number of replications of the treatments, k size of the block, and λ block concurrencies of two treatments, and let the parameters of PBIB design be (v^*, b^*, r^*, k^*, λ_1, λ_2 and λ_3) where v^* denote number of treatments, b^* number of blocks, r^* number of replications of the treatments, k^* size of the block and λ_1, λ_2 and λ_3 block concurrencies of two treatments belonging to first, second and third associate classes respectively.

Theorem 4.1 : An unreduced BIB design with parameters $v = v, b = \binom{v}{k}, r = \binom{v-1}{k-1}, k = k$ and $\lambda = \binom{v-2}{k-2}$ will provide a 3 associate class PBIB design with the following parameters under the D.P. association scheme (for block size $k \geq 3$, number of treatments $v \geq 4$ and for positive integer p) :

$$v^* = 2p\binom{v}{2};\ b^* = b = \binom{v}{k};\ r^* = \binom{v-2}{k-2};\ \lambda_1 = \binom{v-2}{k-2};$$

$$\lambda_2 = \binom{v-3}{k-3} \text{ and } \lambda_3 = \binom{v-4}{k-4} \text{ for } k \geq 4\ (\lambda_3 = 0 \text{ for } k = 3)$$

Proof : From the v symbols of a BIB design, $2p\binom{v}{2}$ symbols are obtained by associating the 2p symbols belonging to the 'ij group' to all $\binom{v}{2}$ distinct unordered pairs. The number of blocks for both BIB design and PBIB design will remain the same since each block of the BIB design is used to develop a block of the PBIB design. The block size of the PBIB design will be 2p times the number of unordered pairs of symbols within a block of the BIB design hence $k^* = \binom{k}{2}.2p$. The number of replications of each treatment in the PBIB design will be the number of times each pair of symbols will occur (λ), in the blocks of the BIB design. The number of concurrencies of any two treatments in different blocks can also be worked out as $\lambda_1 = \binom{v-2}{k-2}$, $\lambda_2 = \binom{v-3}{k-3}$ and $\lambda_3 = \binom{v-4}{k-4}$.

As an illustration, let us consider an unreduced BIB design $v = 4, b = 4, r = 3, k = 3, \lambda = 2$ with contents of the four blocks as (1, 2, 3); (1, 2, 4); (1, 3, 4) and (2, 3, 4). For v = 4 and p = 1 there will be $v^* = 12$ permutations of pairs of symbols among 1, 2, 3 and 4; represented as (12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34) and (43). (These can also be numbered as new treatments of a PBIB design, from 1 to 12 in any order).

Now from each block of the BIB design a new block of the PBIB design can be developed by using all pairs of symbols in that block. So the blocks of the new design are

Block I : (12), (21), (13), (31), (23), (32)

Block II : (12), (21), (14), (41), (24), (42)

Block III : (13), (31), (14), (41), (24), (42)

and Block IV : (23), (32), (24), (42), (34), (43)

This will be a PBIB design with $v^* = 12$, $b^* = 4$, $r^* = 2$, $k^* = 6$, $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 0$. Now $k^* = k(k-1) = 6$. Since $\lambda = 2$, number of replications of treatments in the PBIB design is $r^* = 2$. Now λ_1, λ_2 and

Table 5.1. Selected PBIB designs for $v^* = 12$ to 84 with $r^* \leq 5$; developed from BIB design (v, b, r, k, λ) when p = 1 and p = 2

Parameters of BIB design when k = 3					Parameters of PBIB design when p = 1							Parameters of PBIB design when p = 2						
v	b	r	k	λ	v^*	b^*	r^*	k^*	λ_1	λ_2	λ_3	v^*	b^*	r^*	k^*	λ_1	λ_2	λ_3
4	4	3	3	2	12	4	2	6	2	1	0	24	4	2	12	2	1	0
5	10	6	3	3	20	10	3	6	3	1	0	40	10	3	12	3	1	0
5	5	4	4	3	20	5	3	12	3	2	1	40	5	3	24	3	2	1
6	20	10	3	4	30	20	4	6	4	1	0	60	20	4	12	4	1	0
6	6	5	5	4	30	6	4	20	4	3	2	60	6	4	40	4	3	2
7	35	15	3	5	42	35	5	6	5	1	0	84	35	5	12	5	1	0
7	7	6	6	5	42	7	5	30	5	4	3	84	7	5	60	5	4	3

λ_3 can be worked out as 2, 1 and 0 respectively. Similarly from the same BIB design, when p = 2 there will be 4 treatments from each pair of symbols (for example the pair '12' will give the treatments $112 = (12)_1$, $122 = (12)_2$, $211 = (12)_3$, and $221 = (12)_4$ so that there will be 24 treatments for the PBIB design.

5. APPLICATIONS

Fixing the size of the blocks of the BIB design as k = 3, 4 etc. and for integer values of $v \geq 4$ a number of PBIB designs can be developed from unreduced BIB designs for different positive integer selections of p. A list of selected designs are presented in Table 5.1.

A series of PBIB designs having block sizes like 6 and 12 with practically small number of replications like 3, 4 etc. can be developed by the above method which are very useful for practical workers. Since b^*, r^*, λ_1, λ_2 and λ_3 are invariant under fixed v and k, selection can be made based on the block size k.

This association scheme and the designs obtained have potential application in Type III mating designs of Griffing (1956). In this type of mating design, for v inbred lines there will be $\binom{v}{2}$, F_1 hybrids and $\binom{v}{2}$ reciprocals. In the above PBIB designs with p = 1 we can identify a treatment symbol ij as a cross $i \times j$ of two parents i and j; so that all the v(v – 1) crosses among v parents (excluding the parental crosses) can be accommodated in b^* incomplete blocks. The analysis of these designs can be attempted in the lines of PBIB designs having three associate classes as in Dey (1986).

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