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# Optimal Fractional Factorial Plans using Projective Geometry

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**SUMMARY**

Optimal fractional factorial plans for estimating mean, all main effects and specified two factor interactions using finite projective geometry are given in Dey and Suen (2002). In this paper we obtain optimal fractional factorial plans for estimating mean, all main effects, specified two factor interactions and specified three factor interactions using the concept of spread in finite projective geometry.

*Key words :* Galois field, Finite projective geometry, Universal optimality, Spread.

## 1. INTRODUCTION

Fractional factorial designs are commonly used in industrial experiments where a large number of factors has to be studied. Optimality of fractional factorial plans has been studied by many researchers in recent years. The issue of estimability and optimality in the context of two-level factorials has been studied in Chiu and John (1998), Hedayat and Pesotan (1992, 1997) and Wu and Chen (1992). Also optimality results for arbitrary factorials including asymmetric ones are given in Dey and Mukerjee (1999). Three main types of optimal fractional factorial plans for the estimation of mean, all main effects and specified set of two factor interactions with far fewer runs are given in Dey and Suen (2002):

(i) $(F_1, F_2; F_3, F_4; \ldots ; F_{2u-1}, F_{2u})_1$ ; a plan allowing the optimal estimation of the mean, 2u main effects $F_1, F_2,\ldots, F_{2u}$ and u two factor interactions $F_1F_2, F_3F_4,\ldots, F_{2u-1}F_{2u}$

(ii) $(F_1,\ldots, F_u; F_{u+1},\ldots F_{u+v})_2$ ; a plan allowing the optimal estimation of the mean, u + v main effects $F_1, F_2,\ldots, F_{u+v}$ and uv two factor interactions $F_iF_j (1 \le i \le u, u+1 \le j \le u+v)$

(iii) $(F_1,\ldots, F_u)_3$ ; a plan allowing the optimal estimation of the mean, u main effects $F_1, F_2,\ldots, F_u$ and u two factor interactions $F_1F_2, F_3F_4,\ldots, F_{u-1}F_u, F_uF_1$

and also hybrid models arising from these using finite projective geometry.

Here we will construct new optimal fractional factorial plans which are hierarchical in nature for estimation of mean, all main effects and specified two-factor interactions using the concept of spread in finite projective geometry. We will also construct optimal fractional factorial plans which are hierarchical in nature for estimation of mean, all main effects, specified two factor interactions and specified three factor interations using the concept of spread in finite projective geometry.

In Section 2, we give some preliminaries of finite projective geometry. In Section 3, we construct optimal fractional factorial plans using the concept of spread in finite projective geometry.

## 2. FINITE PROJECTIVE GEOMETRY

A finite peojective geometry of (r – 1) dimension PG(r – 1, m) over GF(m), Galois field of order m, m is a prime power, consists of the ordered set $(x_0, x_1, \ldots, x_{r-1})$ of points where $x_i$ (i = 0, 1, ..., r – 1) are elements of GF(m) and all of them are not simultaneously zero. For any $\lambda \in GF(m)$, $(\lambda \ne 0)$, the point $(\lambda x_0,\ldots,\lambda x_{r-1})$

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represents the same point as that of $(x_0,...,x_{r-1})$. All those points which satisfy a set of (r – t – 1) linearly independent homogeneous equations with coefficients from GF(m) (all of them are not simultaneously zero with the same equation) is said to represent a t-flat in PG(r – 1, m).

In particular a 0-flat, a 1-flat...and a (r – 2)-flat respectively in PG(r – 1, m) are known as a point, a line...and a hyperplane of PG(r – 1, m). The number of points lying on any (t – 1)-flat is $\frac{m^t - 1}{m - 1}$ and the number of independent points lying on a (t – 1)-flat is t.

**Definition.** A (s – 1)-spread F of PG(r – 1, m) is a set of (s – 1)-spaces which partitions PG(r – 1, m), that is, every point of PG(r – 1, m) lies exactly in one (s – 1)-space of F. Hence any two (s – 1)-spaces of F are disjoint. In other words, the projective space PG(r – 1, m) has (s – 1)-spread if and only if s|r. If projective space PG(r – 1, m) has (s – 1)-spread then there are $\frac{m^r - 1}{m^s - 1}$ disjoint (s – 1)-spaces which partitions the point set of PG(r – 1, m).

**Example.** Consider PG(3, 2) over GF(m). Here s|r if s = 2 since r = 4. Thus PG(3, 2) have 1-spreads (line).

There are 5 disjoint 1-spreads (lines) which partitions PG(3, 2) say $L_1,..., L_5$.

$L_1$ : 1000, 0110, 1110; $L_2$ : 0100, 0011, 0111;

$L_3$ : 0010, 1101, 1111; $L_4$ : 0001, 1010, 1011;

$L_5$ : 1100, 0101, 1001

One may refer to Hirschfeld (1998) for more details.

## 3. CONSTRUCTION OF OPTIMAL PLANS

### 3.1 Optimal Fractional Factorial Plan for Estimating the Mean, all Main Effects and Specified Two Factor Interactions

Consider a (r – 1)-dimensional finite projective geometry PG(r – 1, m) over GF(m). Let F be a set of distinct points of PG(r – 1, m) given by $F_1, F_2,..., F_f$. The point $F_i$ in finite projective geometry is interchangeably written as a factor or a main effect. A combinatorial characterization for a plan to be universally optimal under a hierarchical model is given in Dey and Mukerjee (1999). The result has been modified in Dey and Suen (2002) to construct optimal plans under hierarchical model for estimation of mean, all main effects and specified two factor interactions. They have stated that one can obtain a universally optimal fractional factorial plan for an $m^f$ experiment involving $m^r$ runs generated by the row space of matrix P of order r × f; where columns of P are the points of F, provided Theorems 3.1 and 3.2 given in their paper are satisfied.

**Theorem 1.** For any prime power m and any integers s(> 1) and r(> 1), one can construct a universally optimal saturated plan (we will abbreviate it in remainder as UOSP) for an $m^{f_s}$ (where $f_s = \frac{m^{sr} - 1}{m^s - 1}$) experiment involving $m^{sr}$ runs for estimation of sf main effects $F_{ij}$ (i = 1...f; j = 1... s), $\binom{s}{2}$f two factor interactions $F_{ij_1}\ F_{ij_2}$ $(1 \le j_1 < j_2 \le s)$, $\binom{s}{3}f_s$ three factor interactions $F_{i_1} F_{ij_2} F_{ij_3}$ $(1 \le j_1 < j_2 < j_3 \le s)$ ..., and $f_s$ s-factorinteractions.

**Proof.** Consider PG(r – 1, $m^s$) over GF($m^s$).

The number of points in PG(r – 1, $m^s$) i.e.

$$|PG\ (r-1,\ m^s)| = \frac{m^{sr} - 1}{m^s - 1}\ (= f_s \text{ say})$$

The $f_s$ points can be used to construct an $(m^s)^{f_s}$ saturated orthogonal main effect plans.

We collapse each $m^s$-level factors in PG(r – 1, $m^s$) into s m-level factors. Hence we have $f_s$ groups each having s m-level factors. The s m-level factors can also interact each other within a group.

Let P be a matrix of order sr × $sf_s$ whose columns are the points of $f_s$ groups each having s m-level factors. Using the result of Dey and Suen (2002), we can see that the row space of matrix P generates a UOPS for an $m^{sf_s}$ experiment involving $m^{sr}$ runs for estimation of $sf_s$ main effects, $\binom{s}{2}f_s$ two factor interactions, $\binom{s}{3}f_s$ three factor interactions... and $f_s$ s-factor interactions.

It may be remarked that the points of PG(r – 1, $m^s$) i.e. $f_s$ groups of s m-level factors can be related to (s – 1)-spread (s = 2, 3...) in PG(r – 1, m).

We will rewrite the above theorem using the concept of spread in finite projective geometry.

**Theorem 2.** Consider PG(r – 1, m) over GF(m).

Let there exist a (s – 1) spread (s = 2, 3...) in PG(r – 1, m). Then the whole set of points in PG(r – 1, m) can be divided into $\frac{m^r-1}{m^s-1}$ disjoint (s – 1)-spaces or (s – 1)-spreads.

Let $\frac{m^r-1}{m^s-1}$ (= $f_s$ say) disjoint (s – 1)-spreads. This implies s|r (or r is a multiple of s).

Hence r = ps, say (p is an integer).

Let these $f_s$ spaces be assigned as $s_1, \ldots, s_{f_s}$. Let $s_{i_1}, s_{i_2}, \ldots, s_{i_s}$ be the s independent points on (s – 1)-space $s_i$ for i = 1,..., $f_s$. We denote F as set of points of $f_s$ groups each having s points i.e. $|F| = sf_s$. P is a matrix of order ps × $sf_s$, whose columns are the points of F. The points of each group in F can interact within each other. Hence there are $sf_s$ main effects, $\binom{s}{2}f_s$ two factor interactions, $\binom{s}{3}f_s$ three factor interactions...and $f_s$ s-factor interactions.

**Note.** When s = 2 and s = 3 then ∃ (s – 1)=1 spreads (or lines) and (s – 1) = 2-spreads (or planes) in PG(r – 1, m) respectively.

We can get following two theorems from Theorem1 for s = 2 and s = 3 repectively.

**Theorem 3.** For any prime power m, one construct a universally optimal saturated plan for an $m^{2f_2}$ experiment involving $m^r$ runs for estimation of the mean, $2f_2$-main effect and $f_2$ two factor interactions.

**Proof.** Consider PG(r – 1, m) over GF(m) and r is even. Let $(m+1) \left| \frac{m^r-1}{m-1} \right.$, then there are $\frac{m^r-1}{m^2-1}$ (= $f_2$ say) disjoint lines in PG(r – 1, m). Assign these lines as $L_1, \ldots, L_{f_2}$. Let $F_{i1}$ and $F_{i2}$ be the two independent points on $L_i$ for i = 1,...,$f_2$. Hence we have 2f points. F is a set of $2f_2$ points. Let P be a matrix of order r × $2f_2$, whose columns are the points of F. It can be checked that the matrix P satisfies the conditions given in Dey and Suen (2002), as applicable in the present context. This completes the proof.

**Example 1.** Consider PG(3, 3) over GF(3). There are 10 disjoint lines $L_1, \ldots, L_{10}$ in PG(3, 3). Choose any two independent points from each of the line. We have 10 sets, each having 2 points. There is no interaction between the factors of the sets but there are interactions within the factors of sets. P is a matrix or order 4 × 20. The matrix P generates an optimal plan for estimating the mean, 20 main effects and 10 two-factor interactions involving $3^4$ runs.

**Example 2.** Consider PG(5, 2) over GF(2). There are 21 disjoint lines. Now, proceeding as in Example 1, we have 21 sets, each having 2 points. P is a matrix of order 6 × 42. The matrix P generates an optimal plan for estimating the mean, 42 main effects and 21 two-factor interactions involving $2^6$ runs.

The linear graph of the plan is given by

| | | |
|---|---|---|
| $F_1$——$F_2$ | $F_{15}$——$F_{16}$ | $F_{29}$——$F_{30}$ |
| $F_3$——$F_4$ | $F_{17}$——$F_{18}$ | $F_{31}$——$F_{32}$ |
| $F_5$——$F_6$ | $F_{19}$——$F_{20}$ | $F_{33}$——$F_{34}$ |
| $F_7$——$F_8$ | $F_{21}$——$F_{22}$ | $F_{35}$——$F_{36}$ |
| $F_9$——$F_{10}$ | $F_{23}$——$F_{24}$ | $F_{37}$——$F_{38}$ |
| $F_1$——$F_{12}$ | $F_{25}$——$F_{26}$ | $F_{39}$——$F_{40}$ |
| $F_{13}$——$F_{14}$ | $F_{27}$——$F_{28}$ | $F_{41}$——$F_{42}$ |

In Table 1, we give some optimal fractional factorial plans using the concept of spread.

**Table 1**

| S. No. | PG(r – 1, m) | Optimal Plan for |
|---|---|---|
| 1. | PG(3, 2) | 10 me's and 5 2fi's |
| 2. | PG(3, 3) | 20 me's and 10 2fi's |
| 3. | PG(3, 4) | 34 me's and 17 2fi's |
| 4. | PG(5, 2) | 42 me's and 21 2fi's |
| 5. | PG(7, 2) | 170 me's and 85 2fi's |

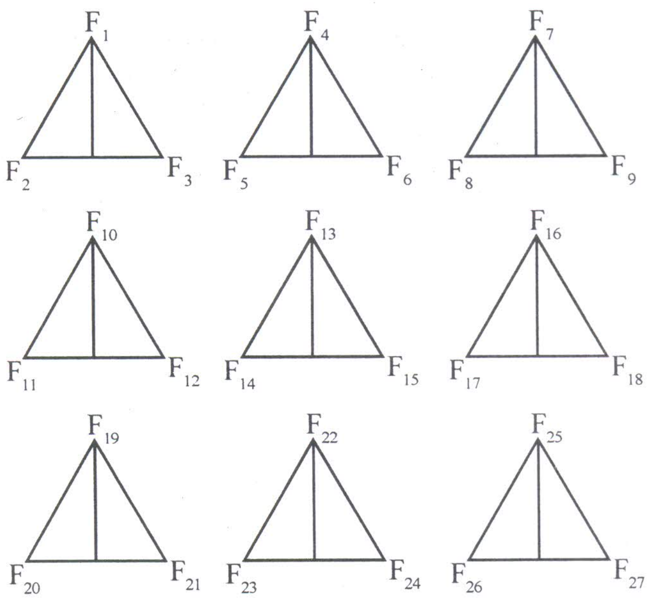
**Theorem 4.** For any prime power m, one can construct a UOSP saturated plan for an $m^{3f_3}$ experiment involving $m^r$ runs for estimating the mean, $3f_3$ main-effects, $3f_3$ two-factor interactions and $f_3$ three-factor interactions.

**Proof.** Consider PG(r – 1, m) over GF(m) and 3|r. Let $(m^3 - 1) \mid (m^r - 1)$, then there are $\frac{m^r - 1}{m^3 - 1}$ (= $f_3$ say) disjoint planes in PG(r – 1, m). Assign these planes as $P_1, P_2, \ldots, P_{f_3}$. Let $F_{i1}, F_{i2}, F_{i3}$ be the three independent points in $P_i$ for i = 1, ..., $f_3$. Now, F is a set of $3f_3$ points. Let P be matrix or order $r \times 3f_3$, whose columns are the points of F. Row space of matrix P generates UOSP for an $m^{3f_3}$ factorial experiment in $m^r$ runs for estimation of the mean, $3f_3$ main effects, $3f_3$ two-factor interactions and $f_3$ three factor interactions.

**Example 3.** Consider PG(5, 2) over GF(2). Number of points in any plane in PG(5, 2) = 7. There are 9 dispoint planes say $P_1, \ldots, P_9$. Choose any 3 independent points from each of the plane. We have 9 sets, each having 3 points. There is no interaction between the factors of the sets but there are initeractions within the factors of the sets. P is a matrix of order $6 \times 27$.

The matrix P generates an optimal plan for estimating the mean, 27 main effects, 27 two-factor interactions and 9 three factor interactions involving $2^6$ runs.

The linear graph of the plan is given by




In Table 2, we give some optimal fractional factorial plans using the concept or spreads.

**Table 2**

| S. No. | PG(r – 1, m) | Optimal Plan for |
|---|---|---|
| 1. | PG(5, 2) | 27 me's, 27 2fi's and 9 3fi's |
| 2. | PG(5, 3) | 84 me's, 84 2fi's and 28 3fi's |
| 3. | PG(8, 2) | 219 me's, 219 2fi's and 73 3fi's |
| 4. | PG(8, 3) | 2271 me's, 2271 2fi's and 757 3fi's |

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