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On Inverse Binomial Randomized Response Technique

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SUMMARY

For estimating π, the proportion of population belonging to a sensitive group; Mangat and Singh (1991) suggested an alternative procedure applicable to the situation where the Warner's (1965) randomized response procedure may result in zero 'yes' answers. Mangat and Singh (1991) suggested an unbiased estimator with its variance formula. The variance formula is difficult to calculate numerically if m, the predetermined number of 'yes' answers chosen by investigator is large. Keeping this in view, Mangat and Singh (1991) have suggested two upper bounds of the exact variance. In this paper, we have suggested several upper bounds to the exact variance and found that certain of these bounds are more closer to the exact variance than that the upper bound suggested by Mangat and Singh (1991).

Key words: Variance, Upper bound, Predetermined number, Sensitive group.

1. INTRODUCTION

Examples of non-response in sample surveys are in abundance. To solve this problem, during last three decades a number of efforts with varying degrees of success have been made in the literature. Randomized response is a technique used to elicit responses to sensitive questions which was first developed by Warner (1965). The Warner (1965) scheme is as follows: A proportion π of a population of individuals has the sensitive characteristic A; the proportion ($1-\pi$) has the non-sensitive characteristics $\bar{A}$. π is to be estimated by means of an interview survey based on a simple random sampling with replacement (SRSWR) of n individuals. The interview is carried out as follows: Each respondent is informed that the survey concerns the characteristics A and $\bar{A}$. Each interviewee is furnished with an identical spinner, a randomized response device, with the face marked so that the spinner points to the letter A with probability p and the letter $\bar{A}$ with probability (1-p). The respondent is asked to spin the spinner unobserved by the interviewer and report whether the spinner points to the characteristic – A and $\bar{A}$ – which the respondent has. The answers are recorded.

R=1, if the respondent reports that the spinner points to the characteristics which he has and R=0, if the respondent reports that the spinner points to the characteristic which he does not has.

Assuming that the reporting is completely truthful, θ, the probability of reply of the type R=1 (or the probability of 'yes' answer) is given by

$$\theta = p\pi + (1-p)(1-\pi) \tag{1.1}$$

Suppose the interviews provide n_1 replies of the type R=1 and $n_0 = n - n_1$ replies of the type R=0. Then it is given in Warner (1965) that

$$\hat{\pi} = \frac{\left[\left(\frac{n_1}{n}\right) - 1 + p\right]}{(2p-1)}, \quad p \neq 0.5 \tag{1.2}$$

is an unbiased estimate of π.

Mangat and Singh (1991) pointed out that in practice, it may happen that the value of π to be estimated and p in randomized response device are on the opposite extremes of 0.5. This will lead to a very small value of the probability of yes answer θ. For such cases, n_1 may take zero value for not so large values of n and thus the estimate so obtained may depend completely on p which is not agreeable. The frequency of $\hat{\pi}$ taking inadmissible values outside [0,1] is also increased. To avoid such problem Mangat and Singh

(1991) suggested the use of inverse binomial randomized response (IBRR) procedure. Using the result of Sathe (1977) they have given two upper bounds of the variance of the estimator of π proposed by them. In the present paper nine more upper bounds using the result of various authors are achieved and their sharpness examined through numerical illustrations.

2. THE ESTIMATOR USING IBRR MODEL AND UPPER BOUNDS OF ITS VARIANCE

In inverse binomial sampling procedure the sample size n is not fixed in advance. Instead, sampling is continued until a predetermined number m of respondents replying with R=1 (i.e. reporting yes answer) have been drawn. Thus, Mangat and Singh (1991) suggested an unbiased estimator of π as

$$\hat{\pi}_h = \left(\frac{\hat{\theta}-1+p}{2p-1}\right), p \neq 0.5 \qquad (2.1)$$

where

$$\hat{\theta} = \frac{(m-1)}{(n-1)} \qquad (2.2)$$

is an unbiased estimator of θ. Noting that m is a predetermined number of 'yes' answers selected by the investigator and $\hat{\theta}$ never achieve zero value like n_1/n.

The variance of $\hat{\pi}_h$ is given by

$$V\left(\hat{\pi}_h\right) = \frac{1}{(2p-1)^2} V\left(\hat{\theta}\right) \qquad (2.3)$$

Using the result given in Best (1974), the variance of $\hat{\theta}$ is given by

$$V(\hat{\theta}) = \left[\alpha(m-1)\left\{\left\{\sum_{r=2}^{m-1}\left(\frac{\theta}{\alpha}\right)^r \frac{(-1)^r}{(m-r)}\right\} - \left(\frac{\theta}{\alpha}\right)^m (-1)^m \log_e \theta\right\} - \theta^2\right] \qquad (2.4)$$

where $\alpha = (1-\theta)$

Substitution of (2.4) in (2.3) yields the variance of $\hat{\pi}_h$ as

$$V\left(\hat{\pi}_h\right) = \frac{1}{(2p-1)^2}\left[\alpha(m-1)\left\{\left\{\sum_{r=2}^{m-1}\left(\frac{\theta}{\alpha}\right)^r \frac{(-1)^r}{(m-r)}\right\} - \left(\frac{\theta}{\alpha}\right)^m (-1)^m \log_e \theta\right\} - \theta^2\right] \qquad (2.5)$$

for $m \geq 3$.

It is to be noted that with the increase in m, the expression (2.5) becomes difficult to calculate numerically. Keeping this in view, using the upper bounds of the variance $V(\hat{\theta})$

$$V_1(\hat{\theta}) = 2\theta^2\alpha[(m-2\alpha)+\{(m-2\alpha)^2+4\theta\alpha\}^{1/2}]^{-1} \qquad (2.6)$$

$$V_2(\hat{\theta}) = 2\theta^2\alpha(m-2\alpha)^{-1} \qquad (2.7)$$

reported by Sathe (1977) and Pathak and Sathe (1984) respectively. Mangat and Singh (1991) forwarded the following upper bounds of the variance $V(\hat{\pi}_h)$ as

$$V_1(\hat{\pi}_h) = 2\theta^2\alpha(2p-1)^{-2}[(m-2\alpha) + \{(m-2\alpha)^2+4\theta\alpha\}^{1/2}]^{-1} \qquad (2.8)$$

$$V_2(\hat{\pi}_h) = \theta^2\alpha(2p-1)^{-2}(m-2\alpha)^{-1} \qquad (2.9)$$

Mangat and Singh (1991) have shown that for large m, the upper bounds $V_i(\hat{\pi}_h)$, $i=1,2$, are sufficiently accurate.

Nine more upper bounds of the variance $V(\hat{\pi}_h)$ are given in Section 3.

3. SUGGESTED UPPER BOUNDS OF THE VARIANCE $V(\hat{\pi}_h)$

If $\hat{\theta} \sim IB(m,\theta)$ then various upper bounds of the variance $V(\hat{\theta})$ are given by

$$V_3(\hat{\theta}) = \theta^2\alpha(m+1-\alpha)^{-1} = \theta^2\alpha(m-2)^{-1} \qquad (3.1)$$

$$V_4(\hat{\theta}) = \frac{\theta^2\alpha}{(m-2)}\left[1 - \frac{2(m-4)\theta}{(m-3)(m-3\alpha-1)}\right] \qquad (3.2)$$

$$V_5(\hat{\theta}) = \frac{\theta^2}{4\alpha(m-2)}\{(m-1)(m-2\alpha-1)+4\alpha\}$$

$$-(m-1)\{(m-2\alpha-1)^2+8\theta\alpha\}^{1/2} \quad (3.3)$$

$$V_6(\hat{\theta}) = \frac{(m+1)(m-2\alpha)\theta}{4m}\left[\left\{1+\frac{8m\theta\alpha}{(m+1)(m-2\alpha)}\right\}^{1/2}-1\right] \quad (3.4)$$

$$V_7(\hat{\theta}) = \frac{\theta}{6m}[(A^2-12m\theta B)^{\frac{1}{2}}-A] \quad (3.5)$$

$$V_8(\hat{\theta}) = \frac{\theta^2\alpha}{(m-2)}[1-4\theta\{m-2\alpha+[(m-4\alpha)^2+12\alpha\theta]^{1/2}\}^{-1}] \quad (3.6)$$

$$V_9(\hat{\theta}) = \frac{\theta^2\alpha}{(m-2)}\left[1-\frac{2\theta}{(m-3)}+\frac{6\theta^2}{(m-3)(m-4)}\right.$$

$$\left.\times\left\{1-\frac{8\theta}{m-4\alpha+\{(m-6\alpha)^2+20\alpha\theta)\}^{1/2}}\right\}\right] \quad (3.7)$$

$$V_{10}(\hat{\theta}) = \frac{\theta\alpha}{m}[1-2\alpha(m-1)\{m+\theta$$

$$+[(m-3\alpha+1)^2+8\theta\alpha]^{1/2}\}^{-1}] \quad (3.8)$$

$$V_{11}(\hat{\theta}) = \frac{\theta^2\alpha}{m}\left[1+\frac{2\alpha}{(m-2)}\right.$$

$$\left.\times\left\{1-\frac{6\theta}{[(m-3\alpha+1)+\{(m-5\alpha+1)^2+16\alpha\theta\}^{1/2}]}\right\}\right] \quad (3.9)$$

where

$$A = \left[\{m^2+m(3\theta-1)-3\alpha\theta\}-\frac{6\alpha^2}{(m+1)}\right]$$

$$B = \alpha\{\alpha(m-1)(m+1)^{-1}-(m+2)\}$$

It is to be noted that the upper bounds $V_3(\hat{\theta})$, $V_4(\hat{\theta})$, $V_5(\hat{\theta})$, $V_i(\hat{\theta})$, i = 6, 7; $V_j(\hat{\theta})$, j = 8, 9, 10, 11; are respectively, given by Mikulski and Smith (1976), Ray and Sahai (1978), Prasad and Sahai (1982), Sahai (1983) and Pathak and Sathe (1984). We have not given Sahai's (1980) upper bounds into this investigation since the derivation of his upper bounds is erroneous, for instance, see Pathak and Sahai (1984).

Replacing $V(\hat{\theta})$ in (2.3) by $V_j(\hat{\theta})$, j = 3 to 11, we get the following upper bounds of the variance $V(\hat{\pi}_h)$ as

$$V_3(\hat{\pi}_h) = \theta^2\alpha(2p-1)^{-2}(m+1-\alpha)^{-1}$$

$$= \theta^2\alpha(2p-1)^{-2}(m-2)^{-1} \quad (3.10)$$

$$V_4(\hat{\pi}_h) = \frac{\theta^2\alpha}{(2p-1)^2(m-2)}\left[1-\frac{2(m-4)\theta}{(m-3)(m-3\alpha-1)}\right] \quad (3.11)$$

$$V_5(\hat{\pi}_h) = \frac{\theta^2}{4\alpha(m-2)(2p-1)^2}\left[\{(m-1)(m-2\alpha-1)+4\alpha\}\right.$$

$$\left.-(m-1)\{(m-2\alpha-1)^2+8\theta\alpha\}^{1/2}\right] \quad (3.12)$$

$$V_6(\hat{\pi}_h) = \frac{(m+1)(m-2\alpha)\theta}{4m(2p-1)^2}$$

$$\times\left[\left\{1+\frac{8m\theta\alpha}{(m+1)(m-2\alpha)}\right\}^{1/2}-1\right] \quad (3.13)$$

$$V_7(\hat{\pi}_h) = \frac{\theta}{6m(2p-1)^2}\left[(A^2-12m\theta B)^{1/2}-A\right] \quad (3.14)$$

$$V_8(\hat{\pi}_h) = \frac{\theta^2\alpha}{(m-2)(2p-1)^2}$$

$$\times\left[1-4\theta\left\{m-2\alpha+\left[(m-4\alpha)^2+12\alpha\theta\right]^{1/2}\right\}^{-1}\right] \quad (3.15)$$

$$V_9(\hat{\pi}_h) = \frac{\theta^2\alpha}{(m-2)(2p-1)^2}\left[1-\frac{2\theta}{(m-3)}+\frac{6\theta^2}{(m-3)(m-4)}\right.$$

$$\left.\times\left\{1-\frac{8\theta}{\left[m-4\alpha+\{(m-6\alpha)^2+20\alpha\theta\}^{1/2}\right]}\right\}\right] \quad (3.16)$$

$$V_{10}(\hat{\pi}_h) = \frac{\theta\alpha}{m(2p-1)^2}\Big[1 - 2\alpha(m-1)\{m+\theta + [(m-3\alpha+1)^2 + 8\theta\alpha]^{1/2}\}^{-1}\Big] \quad (3.17)$$

$$V_{11}(\hat{\pi}_h) = \frac{\theta^2\alpha}{m(2p-1)^2}\left[1 + \frac{2\alpha}{(m-2)} \times \left\{1 - \frac{6\theta}{\left[(m-3\alpha+1) + \{(m-5\alpha+1)^2 + 16\alpha\theta\}^{1/2}\right]}\right\}\right] \quad (3.18)$$

where A and B are same as defined earlier.

The values of the variance $V(\hat{\pi}_h)$ and its upper bounds $V_j(\hat{\pi}_m)$, j = 1 to 11 have been computed for different values of p, m, θ, π are shown in Table 1.

From Table 1, we observed that for $\pi = 0.1, p = 0.2, \theta = 0.74$, when $4 \le m \le 7$, the upper bound $V_7(\hat{\pi}_h)$ is nearest to exact variance $V(\hat{\pi}_h)$ followed by $V_{11}(\hat{\pi}_h)$. For m = 3 the upper bound $V_7(\hat{\pi}_h)$ is sharpest one but followed by $V_{10}(\hat{\pi}_h)$. For m = 8, 9, the upper bound $V_7(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ give the same value closest to the exact variance $V(\hat{\pi}_h)$ followed by $V_{10}(\hat{\pi}_h)$. The upper bounds $V_7(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ gives the same value equal to the exact variance followed by $V_9(\hat{\pi}_h)$. When $11 \le m \le 14$, the values of upper bounds $V_7(\hat{\pi}_h)$, $V_9(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ are same and match with the exact variance $V(\hat{\pi}_h)$ followed by $V_8(\hat{\pi}_h)$ and $V_{10}(\hat{\pi}_h)$ whose values are very close to the exact variance $V(\hat{\pi}_h)$. For $m \ge 15$, it is observed that the upper bounds $V_7(\hat{\pi}_h)$, $V_8(\hat{\pi}_h)$, $V_9(\hat{\pi}_h)$, $V_{10}(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ give the same value which equals to the exact variance $V(\hat{\pi}_h)$ followed by $V_5(\hat{\pi}_h)$. Thus, the upper bounds $V_7(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ appear to be the best approximation to the exact variance $V(\hat{\pi}_h)$.

When $\pi = 0.1, p = 0.8, \theta = 0.26$, we note that the upper bound $V_{11}(\hat{\pi}_h)$ is a good approximation of the exact variance $V(\hat{\pi}_h)$, except for m = 3 where $V_7(\hat{\pi}_h)$ is closest to that of $V(\hat{\pi}_h)$. It is observed that for m = 3, the upper bound $V_7(\hat{\pi}_h)$ is closest to the exact variance $V(\hat{\pi}_h)$ followed by $V_{10}(\hat{\pi}_h)$, while for m = 5, the upper bound $V_{11}(\hat{\pi}_h)$ is closed to the exact variance $V(\hat{\pi}_h)$ followed by $V_5(\hat{\pi}_h)$. For m = 6, the upper bound $V_{11}(\hat{\pi}_h)$ is nearest to the exact variance $V(\hat{\pi}_h)$ followed by $V_8(\hat{\pi}_h)$, while for m = 7, 8, the $V_{11}(\hat{\pi}_h)$ is closest to the exact variance $V(\hat{\pi}_h)$ followed by $V_9(\hat{\pi}_h)$. For m = 9, the upper bounds $V_9(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ yield the same value equal to the exact variance $V(\hat{\pi}_h)$ followed by $V_8(\hat{\pi}_h)$. For $10 \le m \le 13$, the common value of upper bounds $V_8(\hat{\pi}_h)$, $V_9(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ is equal to the exact variance $V(\hat{\pi}_h)$ followed by $V_5(\hat{\pi}_h)$. For m = 14, the upper bounds $V_7(\hat{\pi}_h)$, $V_8(\hat{\pi}_h)$, $V_9(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ give the same value and equal to the exact variance $V(\hat{\pi}_h)$ followed by $V_5(\hat{\pi}_h)$. For $m \ge 15$, the upper bounds $V_4(\hat{\pi}_h)$, $V_5(\hat{\pi}_h)$, $V_7(\hat{\pi}_h)$, $V_8(\hat{\pi}_h)$, $V_9(\hat{\pi}_h)$, $V_{10}(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ give the same value equal to the exact variance $V(\hat{\pi}_h)$. For small predetermined number of yes answer, $5 \le m \le 8$ and even for large predetermined number of yes answers, the upper bound $V_{11}(\hat{\pi}_h)$ appears to the good approximation to the exact variance $V(\hat{\pi}_h)$. However, for large values of m (≥ 15) other upper bounds as indicated above, also give value very close (or exact) to the exact variance $V(\hat{\pi}_h)$.

When $\pi = 0.5$, $p = 0.2$, $\theta = 0.5$; it is seen from Table 1 that for m = 3, $V_7(\hat{\pi}_h)$ is closest to the exact value of the variance $V(\hat{\pi}_h)$ followed by $V_{10}(\hat{\pi}_h)$. For $3 < m \le 7$, we see that $V_7(\hat{\pi}_h)$ is nearest to the exact

Table 1. The actual variance and its upper bounds

For $\pi = 0.1$, $p = 0.2$, $\theta = 0.74$, $\alpha = 0.26$

m	$V(\hat{\pi}_h)$	$V_1(\hat{\pi}_h)$	$V_2(\hat{\pi}_h)$	$V_3(\hat{\pi}_h)$	$V_4(\hat{\pi}_h)$	$V_5(\hat{\pi}_h)$	$V_6(\hat{\pi}_h)$	$V_7(\hat{\pi}_h)$	$V_8(\hat{\pi}_h)$	$V_9(\hat{\pi}_h)$	$V_{10}(\hat{\pi}_h)$	$V_{11}(\hat{\pi}_h)$
3	0.152212	0.154772	0.159471	0.395489	-	0.201204	0.357816	0.152278	0.159471	-	0.152382	0.153119
5	0.086833	0.087448	0.088279	0.131830	0.101533	0.088660	0.370628	0.086843	0.087088	0.088965	0.086869	0.086862
6	0.071352	0.071713	0.072170	0.098872	0.075755	0.072010	0.374181	0.071357	0.071437	0.071578	0.071372	0.071361
7	0.060527	0.060755	0.061032	0.079098	0.062278	0.060816	0.376832	0.060529	0.060561	0.060571	0.060538	0.060530
8	0.052539	0.052692	0.052873	0.065915	0.053368	0.052684	0.378890	0.052540	0.052555	0.052551	0.052546	0.052540
9	0.046406	0.046514	0.046638	0.056498	0.046847	0.046486	0.380536	0.046407	0.046414	0.046410	0.046410	0.046407
10	0.041551	0.041629	0.041718	0.049436	0.041807	0.041599	0.381882	0.041551	0.041555	0.041552	0.041554	0.041551
11	0.037613	0.037672	0.037737	0.043943	0.037771	0.037643	0.383005	0.037613	0.037615	0.037613	0.037615	0.037613
12	0.034355	0.034400	0.034450	0.039549	0.034458	0.034375	0.383955	0.034355	0.034356	0.034355	0.034356	0.034355
13	0.031615	0.031651	0.031690	0.035954	0.031685	0.031629	0.384771	0.031615	0.031616	0.031615	0.031616	0.031615
14	0.029279	0.029308	0.029339	0.032957	0.029329	0.029289	0.385479	0.029280	0.029280	0.029279	0.029280	0.029279
15	0.027265	0.027288	0.027313	0.030422	0.027300	0.027272	0.386098	0.027265	0.027265	0.027265	0.027265	0.027265
16	0.025509	0.025528	0.025548	0.028249	0.025535	0.025514	0.386645	0.025509	0.025509	0.025509	0.025509	0.025509
17	0.023965	0.023981	0.023998	0.026366	0.023985	0.023969	0.387132	0.023965	0.023965	0.023965	0.023966	0.023965
18	0.022598	0.022611	0.022625	0.024718	0.022613	0.022601	0.387568	0.022598	0.022598	0.022598	0.022598	0.022598
19	0.021378	0.021389	0.021401	0.023264	0.021390	0.021380	0.387961	0.021378	0.021378	0.021378	0.021378	0.021378
20	0.020282	0.020292	0.020302	0.021972	0.020292	0.020284	0.388316	0.020282	0.020282	0.020282	0.020283	0.020282

For $\pi = 0.1$, $p = 0.8$, $\theta = 0.26$, $\alpha = 0.74$

m	$V(\hat{\pi}_h)$	$V_1(\hat{\pi}_h)$	$V_2(\hat{\pi}_h)$	$V_3(\hat{\pi}_h)$	$V_4(\hat{\pi}_h)$	$V_5(\hat{\pi}_h)$	$V_6(\hat{\pi}_h)$	$V_7(\hat{\pi}_h)$	$V_8(\hat{\pi}_h)$	$V_9(\hat{\pi}_h)$	$V_{10}(\hat{\pi}_h)$	$V_{11}(\hat{\pi}_h)$
3	0.079528	0.084859	0.091418	0.138956	–	0.083077	0.119457	0.080928	0.091418	–	0.081031	0.084831
5	0.038244	0.038881	0.039476	0.046319	0.039553	0.038161	0.128183	0.038357	0.038374	0.038880	0.038394	0.038296
6	0.030148	0.030458	0.030742	0.034739	0.030407	0.030114	0.130071	0.030192	0.030176	0.030185	0.030213	0.030159
7	0.024844	0.025016	0.025173	0.027791	0.024924	0.024829	0.131379	0.024864	0.024853	0.024849	0.024876	0.024847
8	0.021112	0.021217	0.021312	0.023159	0.021144	0.021105	0.132343	0.021122	0.021115	0.021113	0.021130	0.021113
9	0.018348	0.018416	0.018478	0.019851	0.018363	0.018344	0.133085	0.018353	0.018349	0.018348	0.018358	0.018348
10	0.016220	0.016266	0.016309	0.017369	0.016228	0.016217	0.133676	0.016223	0.016220	0.016220	0.016226	0.016220
11	0.014532	0.014565	0.014596	0.015440	0.014537	0.014531	0.134156	0.014534	0.014532	0.014532	0.014536	0.014532
12	0.013161	0.013186	0.013209	0.013896	0.013164	0.013160	0.134556	0.013163	0.013161	0.013161	0.013164	0.013161
13	0.012026	0.020990	0.012062	0.012632	0.012028	0.012025	0.134894	0.012027	0.012026	0.012026	0.012028	0.012026
14	0.011071	0.011085	0.011099	0.011580	0.011072	0.011070	0.135183	0.011071	0.011071	0.011071	0.011072	0.011071
15	0.010256	0.010267	0.010228	0.010689	0.010256	0.010255	0.135433	0.010256	0.010256	0.010256	0.010257	0.010256
16	0.009552	0.009561	0.009570	0.009925	0.009553	0.009552	0.135652	0.009552	0.009552	0.009552	0.009553	0.009552
17	0.008939	0.008946	0.008953	0.009264	0.008939	0.008938	0.135846	0.008939	0.008939	0.008939	0.008939	0.008939
18	0.008399	0.008405	0.008411	0.008685	0.008400	0.008399	0.136018	0.008399	0.008399	0.008399	0.008400	0.008399
19	0.007921	0.007926	0.007931	0.008174	0.007921	0.007921	0.136171	0.007921	0.007921	0.007921	0.007921	0.007921
20	0.007494	0.007499	0.007503	0.007720	0.007495	0.007494	0.136310	0.007494	0.007494	0.007494	0.007495	0.007494

For $\pi = 0.5$, $p = 0.2$, $\theta = 0.5$, $\alpha = 0.5$												
m	$V(\hat{\pi}_h)$	$V_1(\hat{\pi}_h)$	$V_2(\hat{\pi}_h)$	$V_3(\hat{\pi}_h)$	$V_4(\hat{\pi}_h)$	$V_5(\hat{\pi}_h)$	$V_6(\hat{\pi}_h)$	$V_7(\hat{\pi}_h)$	$V_8(\hat{\pi}_h)$	$V_9(\hat{\pi}_h)$	$V_{10}(\hat{\pi}_h)$	$V_{11}(\hat{\pi}_h)$
3	0.157925	0.163936	0.173611	0.347222	-	0.186076	0.298959	0.158550	0.173611	-	0.158829	0.162037
5	0.084368	0.085490	0.086806	0.115741	0.092593	0.084896	0.317063	0.084445	0.084728	0.086806	0.084513	0.084452
6	0.068152	0.068764	0.069444	0.086806	0.070271	0.068298	0.321678	0.068186	0.068255	0.068359	0.068224	0.068174
7	0.057107	0.057474	0.057870	0.069444	0.057870	0.057159	0.325036	0.057124	0.057144	0.057140	0.057146	0.057114
8	0.049116	0.049353	0.049603	0.057870	0.049453	0.049138	0.327598	0.049125	0.049132	0.049124	0.049139	0.049119
9	0.043074	0.043235	0.043403	0.049603	0.043244	0.043084	0.329621	0.043079	0.043081	0.043076	0.043088	0.043075
10	0.038348	0.038462	0.038580	0.043403	0.038442	0.038353	0.331261	0.038351	0.038352	0.038349	0.038357	0.038348
11	0.034552	0.034636	0.034722	0.038580	0.034609	0.034555	0.332618	0.034554	0.034554	0.034552	0.034558	0.034552
12	0.031437	0.031501	0.031566	0.034722	0.031473	0.031439	0.333761	0.031439	0.031439	0.031437	0.031442	0.031437
13	0.028836	0.028885	0.028935	0.031566	0.028860	0.028837	0.334737	0.028837	0.028837	0.028836	0.028839	0.028836
14	0.026631	0.026670	0.026709	0.028935	0.026648	0.026632	0.335580	0.026632	0.026632	0.026631	0.026634	0.026631
15	0.024739	0.024770	0.024802	0.026709	0.024751	0.024739	0.336315	0.024739	0.024739	0.024739	0.024741	0.024739
16	0.023097	0.023122	0.023148	0.024802	0.023106	0.023097	0.336963	0.023097	0.023097	0.023097	0.023099	0.023097
17	0.021659	0.021680	0.021701	0.023148	0.021666	0.021660	0.337538	0.103359	0.021660	0.021659	0.021660	0.021659
18	0.020390	0.020407	0.020425	0.021701	0.020395	0.020390	0.338052	0.020390	0.020390	0.020390	0.020391	0.020390
19	0.019261	0.019275	0.019290	0.020425	0.019264	0.019261	0.338513	0.019261	0.019261	0.019261	0.019261	0.019261
20	0.018250	0.018262	0.018275	0.019290	0.018253	0.018250	0.338931	0.018250	0.018250	0.018250	0.018250	0.018250

variance $V(\hat{\pi}_h)$ followed by $V_{11}(\hat{\pi}_h)$, while for m = 8, 9, the upper bound $V_{11}(\hat{\pi}_h)$ is closest to the exact variance $V(\hat{\pi}_h)$ followed by $V_9(\hat{\pi}_h)$. For m = 10, the upper bound $V_{11}(\hat{\pi}_h)$ gives the exact variance $V(\hat{\pi}_h)$ followed by $V_9(\hat{\pi}_h)$. For m = 11, 12, the values of upper bounds $V_9(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ are same and equal to the exact variance $V(\hat{\pi}_h)$ followed by $V_j(\hat{\pi}_h)$, (j = 5, 7, 8, 10). For m = 15, 16, the upper bounds $V_5(\hat{\pi}_h)$, $V_7(\hat{\pi}_h)$, $V_8(\hat{\pi}_h)$, $V_9(\hat{\pi}_h)$, $V_{11}(\hat{\pi}_h)$ give the same value equal to the exact variance $V(\hat{\pi}_h)$ followed by $V_{10}(\hat{\pi}_h)$. For $17 \leq m \leq 20$, the values of the upper bounds $V_5(\hat{\pi}_h)$, $V_7(\hat{\pi}_h)$, $V_8(\hat{\pi}_h)$, $V_9(\hat{\pi}_h)$, $V_{10}(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ are same and exact to the true value of variance $V(\hat{\pi}_h)$ followed by $V_4(\hat{\pi}_h)$.

Thus, we see that the value of $V_7(\hat{\pi}_h)$ is closest to the exact variance $V(\hat{\pi}_h)$ for smaller size for m (i.e. $m \leq 7$) and for large m (≥ 8), $V_{11}(\hat{\pi}_h)$ is best choice.

Finally, we conclude that, when the probability of yes answers greater than or equal to θ (i.e. $\theta \geq 0.5$), the upper bound $V_7(\hat{\pi}_h)$ is to be preferred for $3 \leq m \leq 7$; while for $m \geq 8$, the upper bound $V_{11}(\hat{\pi}_h)$ is to be preferred. When $\theta < 0.5$, $m = 3$, the upper bound $V_7(\hat{\pi}_h)$ is to be preferred, while for $3 < m \leq 5$, $V_{11}(\hat{\pi}_h)$ is to be chosen. However, $\theta \leq 0.5$, $m \geq 5$, $V_{11}(\hat{\pi}_h)$ is a good choice along with other upper bounds. In general, we note that the upper bounds $V_7(\hat{\pi}_h)$ and $V_{11}(\hat{\pi}_h)$ are to be preferred in practice. These upper bounds give values much closer (or equal to the exact variance $V(\hat{\pi}_h)$ for large values of m) to the exact variance $V(\hat{\pi}_h)$ in comparison to the upper bounds $V_1(\hat{\pi}_h)$ and $V_2(\hat{\pi}_h)$ suggested by Mangat and Singh (1991).

Remark 3.1. Following Sukhatme *et al.* (1984), we get an unbiased estimator of the variance $V(\hat{\pi}_h)$ as

$$\hat{V}(\hat{\pi}_h) = \frac{\hat{\theta}(1-\hat{\theta})}{(n-2)(2p-1)^2}$$

Remark 3.2. Similar studies can be carried out in the case where the respondents do not truthfully (or the respondents belonging to the sensitive class report with some probability)

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