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The Construction of Four Level Second Order Ratatable Designs

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SUMMARY

Nigam (1977) has given a method of construction of Second Order Rotatable Design for four levels which holds good for $r \geq 3\lambda$. It is pointed out in this paper that for $r \leq 3\lambda$ the result does not holds, since the positive solution of $t = \alpha^2/\beta^2$ does not exist for specific parameter of BIBD. Then an alternative method of construction of SORD with four levels is given which applies even for $r < 3\lambda$. The catalogue of designs obtainable from the modified method of construction is also given.

Key words : Second order rotatable designs, Balanced incomplete block design.

1. INTRODUCTION

Nigam (1977) has given a method for constructing four level SORD through BIBD. He has shown that four level SORD can always be constructed through BIBD with $r \geq 3\lambda$. When, however, $r < 3\lambda$, he has shown that four level SORD can be constructed by adding 2^p level combinations obtained through the set $(\pm\alpha, \pm\alpha, \ldots, \pm\alpha)$ and $b2^p$ initial level combinations obtained through the BIBD. This always gives a v-dimensional SORDs with four level in $N = (b + 1)\,2^p$ design points, with

$$(r - 3\lambda - 2)t^2 - 6(r - \lambda)t + (5r - 2b - 3\lambda) = 0 \quad \text{where } t = \alpha^2/\beta^2 \qquad (1)$$

In the paper it is mentioned that equation (1) yield a positive solution of t when $r < 3\lambda$. But it is not universally true. Following table shows some of the BIBD with $(r < 3\lambda)$, where a positive solution of equation (1) does not exist. Hence SORD can not be constructed through these BIBDs.

BIBD (v, b, r, k, λ)	
(6, 10, 5, 3, 2)	(7, 7, 4, 4, 2)
(8, 14, 7, 4, 3)	(9, 18, 8, 4, 3)
(9, 18, 10, 5, 5)	(10, 18, 9, 5, 4)
(11, 11, 6, 6, 3)	(11, 11, 5, 5, 2)
(12, 22, 11, 6, 5)	(13, 26, 12, 6, 5)
(15, 15, 7, 7, 3)	

The method proposed by Nigam thus does not generally give four level SORD, if $r < 3\lambda$. The purpose of this note is to give an alternative method by which four level SORD can be constructed if $r < 3\lambda$ and Nigam's method is not applicable.

2. METHOD OF CONSTRUCTION

Following the notation adopted by Nigam, $N(= n_{ij})$ be the b × v incidence matrix of this BIBD such that $n_{ij} = \alpha$ if the j[th] treatment occurs in i[th] block and $n_{ij} = \beta$ otherwise.

If 2^p points are associated with each row of N in v-dimensions where 2^p denotes a resolution V fractional replicate of 2^v in ± 1 levels. There are $b2^p$ design points from BIBD. Then four level SORD with levels $\pm\alpha$ and $\pm\beta$ can be constructed from the $b2^p$ design points of the BIBD.

If $r < 3\lambda$, then four level SORD can be constructed by replicating y times a set of $v2^p$ design points obtained through permutations of $(\pm\beta, \pm\alpha, \ldots, \pm\alpha)$. This gives v-dimensional SORDs in $N = (b + vy)2^p$ design points, with

$$[(r - 3\lambda) - (2v - 5)y]t^2 - 6[(r - \lambda) + y]t + (5r - 2b - 3\lambda + y) = 0 \quad \text{where } t = \alpha^2/\beta^2 \qquad (2)$$

where y be the positive integer. The value of y is chosen such that $y > -5r + 2b + 3\lambda$.

Proof. Since $r < 3\lambda$

If we want to construct SORD, all the following relations should hold

(A) $\sum_{u=1}^{N} x_{iu} = 0$, $\sum_{u=1}^{N} x_{iu}x_{ju} = 0$, $\sum_{u=1}^{N} x_{iu}^3 = 0$

$\sum_{u=1}^{N} x_{iu}x_{ju}x_{lu} = 0$, $\sum_{u=1}^{N} x_{iu}^2 x_{ju}x_{lu} = 0$

$\sum_{u=1}^{N} x_{iu}^3 x_{ju} = 0$, $\sum_{u=1}^{N} x_{iu} x_{ju}x_{lu}x_{mu} = 0$ etc.

$(i \neq j \neq l \neq m; i, j, l, m = 1, 2, ..., v)$

(B) (i) $\sum_{u=1}^{N} x_{iu}^2 = \lambda_2 N, (i = 1, 2, ..., v)$

(ii) $\sum_{u=1}^{N} x_{iu}^4 = c\lambda_4 N, (i = 1, 2, ..., v)$

(C) $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \lambda_4 N, (i \neq j; i, j = 1, 2, ..., v)$

Under these conditions and using normal equations, obtain a SORD with realtions

(D) $\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 \; (i \neq j = 1, 2, ..., v)$

(E) $\lambda_4/\lambda_2^2 > v/(v+2)$ (3)

Where λ_2 and λ_4 are constants and condition (E) is non-singularity condition.

For this design, which is generated from the BIBD and y replicate permutations of $(\pm\beta, \pm\alpha, ..., \pm\alpha)$ satisfy the condition (A) of equation (3) is true obviously. The conditions (B) and (C) of equation (3) are true as follows

(B) (i) $\sum_{u=1}^{N} x_{iu}^2 = r2^p\alpha^2 + (b-r)2^p\beta^2 + y2^p\beta^2$

$+ (v-1)\, y2^p\alpha^2 = \lambda_2 N, (i = 1, 2, ..., v)$

(ii) $\sum_{u=1}^{N} x_{iu}^4 = r2^p\alpha^4 + (b-r)2^p\beta^4 + y2^p\beta^4$

$+ (v-1)\, y2^p\alpha^4 = 3\lambda_4 N, (i = 1, 2, ..., v)$

(C) $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \lambda 2^p\alpha^4 + (b-2r+\lambda)2^p\beta^4$

$+ (r-\lambda)2^{p+1}\alpha^2\beta^2 + 2^{p+1}y\alpha^2\beta^2 + (v-2)y2^p\alpha^4$

$= \lambda_4 N, \; (i \neq j; i, j = 1, 2, ..., v)$ (4)

This design satisfies the second order rotatable arrangement if

$$\sum x_i^4 = 3\sum x_i^2 x_j^2$$

$\Rightarrow \; r2^p\alpha^4 + (b-r)2^p\beta^4 + 2^p y\beta^4 + (v-1)y2^p\alpha^4$

$= 3[\lambda 2^p\alpha^4 + (b-2r+\lambda)2^p\beta^4 + (r-\lambda)2^{p+1}\alpha^2\beta^2$

$+ 2^{p+1}y\alpha^2\beta^2 + (v-2)y2^p\alpha^4]$

$\Rightarrow \; r\alpha^4 + (b-r)\beta^4 + y\beta^4 + (v-1)y\alpha^4$

$= 3[\lambda\alpha^4 + (b-2r+\lambda)\beta^4 + 2(r-\lambda)\alpha^2\beta^2$

$+ 2y\alpha^2\beta^2 + (v-2)y\alpha^4]$

$\Rightarrow \; [(r-3\lambda)-(2v-5)y]t^2 - 6[(r-\lambda)+y]t$

$+ (5r-2b-3\lambda+y) = 0$ where $t = \alpha^2/\beta^2$

This equation has only one real positive solution if and only if $(5r-2b-3\lambda+y) > 0$.

For this, the value of y can be chosen such that $y > (-5r+2b+3\lambda)$. Putting the minimum value of y in Equation (2), the value of t is obtained. Now, there may be infinite number of solutions for the value of α and β. By taking $\lambda_2 = 1$, the value of β can be obtained. After it, the value of α can be obtained. This design contains N = (b+yv)2^p points. Each factor has 4 levels $-\beta, -\alpha, \alpha, \beta$. Hence, the design exists.

Note. This method can also be applied for construction of four level SORD when $r \geq 3\lambda$ but as the number of points increases the method given by Nigam (1977) is better than the proposed method for $r \geq 3\lambda$.

Example. Consider the BIB Design with v = 6, b = 10, r = 5, k = 3, $\lambda = 2$. The minimum value of y is 2. Now using Equation (2) the value of t is obtained as 0.0328. Since $t = \alpha^2/\beta^2 = 0.0328$. By taking $\lambda_2 = 1$ and using $\alpha^2 = 0.0328\beta^2$, the value of β^2 can be obtained as 2.9365. Substituting the value of β^2 in $\alpha^2 = 0.0328\beta^2$, The value of α^2 can be obtained as 0.0963. Hence, this design contains 704 design points. Each factor has 4 levels -1.7136, -0.3104, 0.3104 and 1.7136. This design also satisfies non-singularity condition. Hence, the design exists.

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Catalogue. Values of 'α' and 'β' for four level SORD using BIBD for $r < 3\lambda$ when $\lambda_2 = 1$

No. of factors	Parameters of Associated BIB Design	Set of combination	α	β	N
5	$(5, 10, 6, 3, 3) \times 2^4$	$(\pm\beta, \pm\alpha, ..., \pm\alpha) \times 2^4$	0.4576	1.6066	240
6(i)	$(6, 10, 5, 3, 2) \times 2^5$	$y(=2)(\pm\beta, \pm\alpha, ..., \pm\alpha) \times 2^5$	0.3103	1.7136	704
6(ii)	$(6, 15, 10, 4, 6) \times 2^5$	$(\pm\beta, \pm\alpha, ..., \pm\alpha) \times 2^5$	0.5193	1.6810	672
7	$(7, 7, 4, 4, 2) \times 2^6$	$(\pm\beta, \pm\alpha, ..., \pm\alpha) \times 2^6$	0.4074	1.7565	896
8	$(8, 14, 7, 4, 3) \times 2^6$	$y(=3)(\pm\beta, \pm\alpha, ..., \pm\alpha) \times 2^6$	0.2886	1.8886	2432
9(i)	$(9, 18, 8, 4, 3) \times 2^7$	$y(=6)(\pm\beta, \pm\alpha, ..., \pm\alpha) \times 2^7$	0.2523	2.0681	9216
9(ii)	$(9, 12, 8, 6, 5) \times 2^7$	$(\pm\beta, \pm\alpha, ..., \pm\alpha) \times 2^7$	0.5126	1.8328	2688
9(iii)	$(9, 18, 10, 5, 5) \times 2^7$	$y(=2)(\pm\beta, \pm\alpha, ... \pm\alpha) \times 2^7$	0.2818	1.8421	4608
10	$(10, 18, 9, 5, 4) \times 2^7$	$y(=4)(\pm\beta, \pm\alpha, ... \pm\alpha) \times 2^7$	0.2759	2.0489	7424
11(i)	$(11, 11, 6, 6, 3) \times 2^9$	$y(=2)(\pm\beta, \pm\alpha, ... \pm\alpha) \times 2^9$	0.3675	2.0525	16896
11(ii)	$(11, 11, 5, 5, 2) \times 2^9$	$y(=4)(\pm\beta, \pm\alpha, ... \pm\alpha) \times 2^9$	0.3382	2.2328	28160
12	$(12, 22, 11, 6, 5) \times 2^8$	$y(=5)(\pm\beta, \pm\alpha, ... \pm\alpha) \times 2^8$	0.2676	2.1977	20992
13	$(13, 26, 12, 6, 5) \times 2^8$	$y(=8)(\pm\beta, \pm\alpha, ... \pm\alpha) \times 2^8$	0.2471	2.3684	33280
15	$(15, 15, 7, 7, 3) \times 2^8$	$y(=5)(\pm\beta, \pm\alpha, ... \pm\alpha) \times 2^8$	0.3337	2.5027	23040