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Some Contributions to T_2 - PBIBD

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SUMMARY

In this paper, some construction-methods of 2 new series of 2-associate triangular (T_2-) PBIBD's as well as the conditions under which the designs become Balanced Incomplete Block Design, are proposed.

Key words : T_m-association scheme, 2-associate triangular (T_2-) PBIBD's, Balanced Incomplete Block Design.

1. INTRODUCTION

In the history of association scheme, John (1966) extended the triangular association scheme to three associate classes. Further, the m-associate triangular (T_m-) association scheme had been introduced by Saha (1973). Many series of PBIBD's based on such association scheme, simply known as Triangular design, have been reported by Sinha (1977, 1981, 1983), Liu and Zheng (1988), Sinha and Kageyama (1990). Also Meitei (1996) has constructed 2 series of Triangular designs by partitioning X = {1, 2,...., n} into 3 disjoint subsets S_1, S_2 and S_3 from which a block is constructed so that its contents are all possible 2-sets: one element coming from S_1 and another from S_2 and all possible 2-sets: both elements coming from S_3.

The 2-associate triangular association scheme is defined as below.

Definition 1.1. Let there be $\binom{n}{2}$ treatments represented by all the possible sets $\{\alpha_1, \alpha_2\}$, assuming that $\{\alpha_1, \alpha_2\}$ and $\{\alpha_2, \alpha_1\}$ are identical, where α_i's integers satisfying $1 \le \alpha_1, \alpha_2 \le n$. If two treatments have an integer in common then they are first associates, otherwise second associates.

The parameters of the association scheme are

$$v = \binom{n}{2}, \; n_1 = 2(n-2), n_2 = \binom{n-2}{2}$$

$$p_{11}^1 = n-2, \; p_{11}^2 = 4$$

Natation 1.1. For a given set S, $\|S\|$ represents the cardinality of S.

For the undefined terms and definitions, it is referred to Raghavarao (1971).

2. THE CONSTRUCTION

Consider a set X = {1, 2,...., n}; $n \geq 5$ and partition it into 2 disjoint subsets S_1, S_2, say, $\|S_i\| = s_i$, i=1,2; $s_1 \neq s_2$. Obviously, $n = s_1+s_2$. A block is constructed from S_1 and S_2 such that the contents of the block are all possible 2-sets: one element coming from S_1 and another coming from S_2. Thus there are $\binom{n}{s_1}$ partitions of X into s_1 - set and s_2 -set. Each partition gives one block of size $s_1 s_2$ of a design given in the following theorem.

Theorem 2.1. There exists a series of 2-associate triangular PBIBD's with parameters

$$v = \binom{n}{2}, b = \binom{n}{s_1}, r = 2\binom{n-2}{s_1-1}, k = s_1 s_2$$

$$\lambda_1 = \delta_1\binom{n-3}{s_1-1} + \delta_2\binom{n-3}{s_1-2}, \lambda_2 = 4\delta_3\binom{n-4}{s_1-2}$$

where $s_1 \neq s_2$

$\delta_1 = 0, \text{if } s_2 < 2$ $\quad\quad$ $\delta_2 = 0, \text{if } s_1 < 2$
$\quad = 1$, otherwise $\quad\quad\quad\quad = 1$, otherwise

$\delta_3 = 0, \text{if } s_1 \text{ or } s_2 < 2$
$\quad = 1$, otherwise

Proof. The values of v and b can be obtained straight forward.

Consider a treatment $\{\alpha_1,\alpha_2\}$ for finding its replication-number. From the construction-method of block, it is clear that $\{\alpha_1,\alpha_2\}$ occurs in a block if the corresponding partition of X is such that

Case I : $\alpha_1 \in S_1; \alpha_2 \in S_2$ or

Case II : $\alpha_2 \in S_1; \alpha_1 \in S_2$

The partition of X into 2 disjoint subsets S_1 and S_2 where $\|S_1\| = s_1$ and $\|S_2\| = s_2$ means the selection of s_1 elements from X for forming s_1 and (n-s_1) i.e. s_2 remaining elements for forming S_2. Under the Case I, there are n-2 elements apart from α_1,α_2 in X, out of which any s_1-1 elements should belong to S_1 and n-(s_1-1)-2 i.e. s_2-1 remaining elements should belong to S_2. Thus there are $\binom{n-2}{s_1-1}\binom{s_2-1}{s_2-1}$ i.e. $\binom{n-2}{s_1-1}$ partitions where $\alpha_1 \in S_1; \alpha_2 \in S_2$. Similarly, under the Case II, there are $\binom{n-2}{s_1-1}\binom{s_2-1}{s_2-1}$ i.e. $\binom{n-2}{s_1-1}$ where $\alpha_2 \in S_1; \alpha_1 \in S_2$. Now, there are $2\binom{n-2}{s_1-1}$ partitions of X into 2 disjoint subsets S_1 and S_2 each giving one replication. Thus, $\{\alpha_1,\alpha_2\}$ occurs in $2\binom{n-2}{s_1-1}$ blocks.

Consider two first associates $\{\alpha_1,\alpha_2\}$ and $\{\alpha_1,\alpha_3\}$ for counting the number of blocks where they occur together in. They occur together in a block if the corresponding partition of X is such that

Case (i) : $\alpha_1 \in S_1; \alpha_2, \alpha_3 \in S_2$; or

Case (ii) : $\alpha_2, \alpha_3 \in S_1$, $\alpha_1 \in S_2$

Under the Case (i), there are n-3 elements apart from $\alpha_1,\alpha_2,\alpha_3$ in X, out of which any s_1-1 elements should belong to S_1 and n-(s_1-1)-3 i.e. s_2-2 remaining elements should belong to S_2. Thus there are $\binom{n-3}{s_1-1}\binom{s_2-2}{s_2-2}$ i.e. $\binom{n-3}{s_1-1}$ partitions where $\alpha_1 \in S_1$; $\alpha_2,\alpha_3 \in S_2$. Similarly, under the Case (ii), there are $\binom{n-3}{s_1-2}\binom{s_2-1}{s_2-1}$ i.e. $\binom{n-3}{s_1-2}$ partitions where $\alpha_2,\alpha_3 \in S_1$; $\alpha_1 \in S_2$. The total number of such distinct partitions of X is $\binom{n-3}{s_1-1}+\binom{n-3}{s_1-2}$, which proves the λ_i's value.

Clearly, it is known that any two second associates $\{\alpha_1,\alpha_2\}$ and $\{\beta_1,\beta_2\}$, say, appear together in a block if the corresponding partition of X is such that

(i) $\alpha_1,\beta_1 \in S_1; \alpha_2,\beta_2 \in S_2$ or

(ii) $\alpha_2,\beta_2 \in S_1; \alpha_1,\beta_1 \in S_2z$ or

(iii) $\alpha_1,\beta_2 \in S_1; \alpha_2,\beta_1 \in S_2$ or

(iv) $\alpha_2,\beta_1 \in S_1; \alpha_1,\beta_2 \in S_2$

each of which cases is possible in $\binom{n-4}{s_1-2}$ different ways. Thus, by construction-method of block the number of partitions arising from these cases gives that $\lambda_2 = 4\binom{n-4}{s_1-2}$.

When $s_1 = s_2$ in the above construction-method of block, each block repeats twice. Collecting one copy of each block, a corollary follows.

Corollary 2.1. For n even and greater than 5, there exists a series of 2-associate triangular PBIBD's with parameters

$$v=\binom{n}{2}, b=\frac{1}{2}\binom{n}{s_1}, r=\binom{n-2}{s_1-1}$$

$$k=s_1^2, \lambda_1=\frac{1}{2}\left[\binom{n-3}{s_1-1}+\binom{n-3}{s_1-2}\right], \lambda_2=2\binom{n-4}{s_1-2}$$

Remark 2.1. Under the condition $n^2-(1+4s_1)n+(2+4s_1^2)=0$; s_1 and $s_2 \geq 2$, the series of PBIBD's given in Theorem 2.1, becomes Balanced Incomplete Block Design.

Remark 2.2. Under the condition $n^2-(1+4s_1)n+(2+4s_1^2)=0$; $s \geq 2$, the series of PBIBD's given in Corollary 2.1, becomes Balanced Incomplete Block Design.

Remark 2.3. In Corollary 2.1, n must be even as $n = s_1+s_2$ and $s_1 = s_2$.

A 2-associate triangular PBIBD is presented as an example of Theorem 2.1, at below. In the Example 2.1, it can be seen that the triangular PBIBD, T60, Clatworthy (1973) is a special case of Theorem 2.1 when $n = 5$, $s_1 = 2$, $s_2 = 3$.

Example 2.1. Letting $n = 5, s_1 = 2, s_2 = 3$, the given series of triangular designs yields a triangular PBIBD with the parameters $v = 10 = b, r = 6 = k, \lambda_1 = 3, \lambda_2 = 4$. The final design so obtained is as follows where ( ) and { } indicate a block and a unit respectively

({1,3}, {1,4}, {1,5}, {2,3}, {2,4}, {2,5})
({1,2}, {1,4}, {1,5}, {3,2}, {3,4}, {3,5})
({1,2}, {1,3}, {1,5}, {4,2}, {4,3}, {4,5})
({1,2}, {1,3}, {1,4}, {5,2}, {5,3}, {5,4})
({2,1}, {2,4}, {2,5}, {3,1}, {3,4}, {3,5})
({2,1}, {2,3}, {2,5}, {4,1}, {4,3}, {4,5})
({2,1}, {2,3}, {2,4}, {5,1}, {5,3}, {5,4})
({3,1}, {3,2}, {3,5}, {4,1}, {4,2}, {4,5})
({3,1}, {3,2}, {3,4}, {5,1}, {5,2}, {5,3})
({4,1}, {4,2}, {4,3}, {5,1}, {5,2}, {5,3})

Again consider X = {1, 2,...., n} and all possible s-subsets of it. Treating all possible 2-subsets of X as treatments of a design, a block is constructed from a s-subset, S, say, such that the contents of the block are all possible 2-sets: both elements coming from S. Clearly there are $\binom{n}{2}$ s-subsets, each giving one block of size $\binom{s}{2}$. Thus, a theorem follows.

Theorem 2.2. There exists a series of 2-series of 2-associate triangular PBIBD's with parameters

$$v = \binom{n}{2}, b = \binom{n}{s}, r = \binom{n-2}{s-2}$$

$$k = \binom{s}{2}, \lambda_1 = \binom{n-3}{s-3}, \lambda_2 = \delta\binom{n-4}{s-4}$$

where $s \geq 3$ and $\delta = 0$ if $s < 4$ and 1, otherwise.

Proof. The proposed construction-method of block ensures that a treatment $\{\alpha_1, \alpha_2\}$, say, occurs in a block arisen from S if α_1 and $\alpha_2 \in S$. The number of such s-subsets is $\binom{n-2}{s-2}$ which provides the replication-number.

Two first associates $\{\alpha_1, \alpha_2\}$ and $\{\alpha_1, \alpha_3\}$, say, occur together in a block arisen from S if α_1, α_2 and block $\alpha_3 \in S$. And two second associates $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$, say, occur together in a block arisen from S if $\alpha_1, \alpha_2, \beta_1$ and $\beta_2 \in S$. The number of such s-subsets are $\binom{n-3}{s-3}$ and $\binom{n-3}{s-3}$ respectively. Thus, the values of λ_1 and λ_2 are obtained.

Remark 2.4. When n = s, the given series of PBIBD's in Theorem 2.2 may yield Balanced Incomplete Block Design. As the condition of n = s never arises, all the designs in this series are not Balanced Incomplete Block Design under any condition.

An illustration of Theorem 2.2 is made here below.

Example 2.2. When n = 6, s = 3, the given series of triangular designs yields a triangular PBIBD (T14, Clatworthy (1973)) with the parameters v = 15, b = 20, r = 4, k = 3, $\lambda_1 = 1, \lambda_2 = 0$.

The final design so obtained is as follows, where ( ) and { } indicate same as earlier

({1,2}, {2,3}, {1,3}),({1,2}, {2,4}, {1,4})
({1,2}, {2,5}, {1,5}), ({1,2}, {2,6}, {1,6})
({1,3}, {3,4}, {1,4}), ({1,3}, {3,5}, {1,5})
({1,3}, {3,6}, {1,6}), ({1,4}, {4,5}, {1,5})
({1,4}, {4,6}, {1,6}), ({1,5}, {5,6}, {1,6})
({2,3}, {3,4}, {2,4}), ({2,3}, {3,5}, {2,5})
({2,3}, {3,6}, {2,6}), ({2,4}, {4,5}, {2,5})
({2,4}, {4,6}, {2,6}), ({2,5}, {5,6}, {2,6})
({3,4}, {4,5}, {3,5}), ({3,4}, {4,6}, {3,6})
({3,5}, {5,6}, {3,6}), ({4,5}, {5,6}, {4,6})

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