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Robustness of Optimal Block Designs for Triallel Cross Experiments Against Interchange of A Pair of Crosses

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SUMMARY

Robustness of some optimal block designs for triallel crossing plans against interchange of a pair of crosses has been dealt with using connectedness and efficiency criteria. The interchanged crosses may have no line in common, one line in common or two lines in common with the substituted cross. Each of these aspects has been investigated separately.

Key words : Triallel crosses, Interchange of crosses, Robustness, Nested block designs.

1. INTRODUCTION

In plant breeding experiments, triallel crosses form an important class of mating designs, which are used for studying the genetic properties of a set of inbred lines. When reciprocal crosses are excluded, p inbred lines give rise to $n_c = 3\ {}^pC_3$ possible crosses of the type $(i_1 \times i_2) \times i_3;\ i_1 \neq i_2 \neq i_3,\ (i_1, i_2, i_3, = 1, 2, \ldots, p)$ {see e.g. Hinkelmann (1965)}. The mating designs for triallel crosses introduced by Rawlings and Cockerham (1962a) are generally conducted in completely randomized designs (CRD), or in randomized complete block (RCB) designs as environmental designs involving n_c crosses. As p increases, the number of triallel crosses, n_c, increases manifold. As a consequence more resources are required for conducting experiments. Furthermore, accommodation of large number of crosses in a RCB design usually results into large intra block variances. A sample of the complete triallel crosses, i.e., partial triallel crosses (P.T.C.) introduced by Hinkelmann (1965) and subsequently studied by Arora and Aggarwal [(1984), (1989)], Ceranka *et al.* (1990) and Ponnuswamy and Srinivasan (1991), etc. may be used in such situations.

Recently Das and Gupta (1997), following the approach of Gupta and Kageyama (1994) and Dey and Midha (1996) in case of diallel experiments, constructed block designs starting with p lines rather than n_c crosses that are universally optimal in $D(p, b, k)$. $D(p, b, k)$ denotes the class of connected block designs for triallel crosses in p lines with b blocks each of size k; the total number of experimental units, n, being less than n_c. For construction of these designs, Das and Gupta (1997) used nested balanced block designs with sub-block size 3.

The optimal design theory developed in the above investigations assumes absence of disturbances like missing observations, outlying observations or inadequacy of assumed model, etc. These assumptions may, however, be violated in real life; thus rendering even an optimal design poor. Consequently, the design that is efficient for estimating various treatment contrasts may no longer remain efficient after it undergoes a disturbance. Interchange of a pair of treatments (crosses) is one such aberration that needs attention during the execution of an experiment. Interchange of a pair of crosses is said to have occurred, if two experimental units belonging to different blocks receive the crosses originally designated for the other. Such discrepancy may occur due to the following reasons

(i) due to interchange of tags or labels attached to the seed packets of different crosses that could not be detected before the application of crosses to the experimental units, and

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(ii) human errors in the preparation of field layout plan, where each of a pair of blocks accommodates a cross originally designated for the other.

These types of disturbances were first reported by Pearce (1948) in a randomized block design set up. These disturbances have been termed as mechanical errors by Gomez and Gomez (1976) whereas Pearce (1983) called these as errors in the application of the treatments. The properties of the original design may be affected in presence of such discrepancy. Therefore, there is a need to address the problem by knowing the designs that are insensitive to such types of disturbances. Batra *et al.* (1997) studied the robustness of block designs against interchange of a pair of treatments. Recently, Panda *et al.* (2001) investigated the robustness of optimal block designs for triallel cross experiments against exchange of a cross. Here, an attempt has been made to investigate the robustness of optimal block designs for triallel crosses against interchange of a pair of crosses.

In Section 2, preliminaries of block designs for triallel crosses have been discussed. The interchanged crosses may have no line in common, one line or two lines in common with the substituted cross. Each of these cases has been investigated separately in Sections 3, 4 and 5, respectively. In each situation, a relationship between the information matrix of the resulting design and that of the original design has been established. The eigenvalues of the information matrix of the resulting design have been obtained when the original design is variance balanced with respect to line effects. Robustness has been investigated using the connectedness criterion {see e.g., Ghosh (1979)} and the efficiency criterion {see e.g., John (1976)}.

2. EXPERIMENTAL SET-UP

Let d be a block design with b blocks each of size k for a triallel cross experiment in p inbred lines. Evidently, $n = bk$ is the total number of observations. In a triallel cross experiment the genotypic effect of the hybrid consists of single-line effects, two-line specific effects and three-line specific effects. We assume here that for a partial triallel cross experiment {in which every line appears as half parent an equal number of times, say r_H and every line appears as full parent an equal number of times, say r_F, in a single set of crosses and each of the crosses $(i_1 \times i_2) \times i_3$ appears at most once} the two- and three- line specific effects are of importance. The line effects, still, are of two types viz. effects as half parent and effects as full parent, i.e., the ordering of lines in a triallel cross is important. Some plant breeders argue that these ordering effects can also be averaged over line effects. Hence, in the present investigation, similar to Das and Gupta (1997), we consider the situations where the ordering of lines in a triallel cross is not of importance and consider the following linear additive fixed effect model for the observations

$$\mathbf{Y} = \mu \mathbf{1}_n + \Delta_1' \mathbf{g} + \Delta_2' \beta + \varepsilon \qquad (2.1)$$

where $\mathbf{Y}$ is $n \times 1$ vector of observed responses, μ is general mean effect, $\mathbf{g}$ and β are the column vectors of p line and b block effects, respectively. $\mathbf{1}_n$ is the vector of n unities, Δ_1' is the design matrix of observation vs line effects i.e. the $(\alpha, \beta)^{th}$ element in Δ_1' is 1, if α^{th} observation pertains to β^{th} line and is zero, otherwise. Δ_2' is the design matrix of observation vs block effects i.e., the $(\alpha, \beta)^{th}$ element in Δ_2' is 1, if α^{th} observation pertains to β^{th} block and is zero, otherwise. Also, ε is the vector of random errors which follows $\mathbf{N}_n(\mathbf{0}, \sigma^2 \mathbf{I}_n)$.

The information matrix for estimating linear functions of line effects, using d, under model (2.1) is

$$\mathbf{C}_d = \mathbf{G}_d - \mathbf{N}_d \mathbf{K}_d^{-1} \mathbf{N}_d' \qquad (2.2)$$

where $\mathbf{G}_d = \Delta_1 \Delta_1' = ((g_{dii'}))$ with $g_{dii} = s_{di}$, and for $i \neq i'$, $g_{dii'}$ being the number of crosses in d in which i and i' appear together and s_{di} is the number of replications for the i^{th} line. $\mathbf{N}_d = \Delta_1 \Delta_2' = ((n_{dij}))$. n_{dij} being the number of times the line i occurs in block j of d and $\mathbf{K}_d = \Delta_2 \Delta_2'$ is the diagonal matrix of block sizes. The design d will be connected if and only if $\text{rank}(\mathbf{C}_d) = p - 1$. Henceforth, we shall deal with connected designs only. Furthermore, Das and Gupta (1997) gave several results on the optimality of the designs considered in D (p, b, k). For better understanding, we state two of these theorems in Appendix.

Das and Gupta (1997) have obtained optimal designs for triallel crosses using nested balanced block designs with sub block size 3. Consider a nested balanced block design d_n with parameters $v = p, b_1, b_2, k_1, k_2 = 3$. If we now identify the treatments of d_n as lines of a triallel cross experiment and perform crosses among lines appearing in the same sub-block of d_n, we get a block design d^* for a triallel experiment involving p lines with

$n = b_2$ crosses arranged in $b = b_1$ blocks, each of size $k = k_1/3$. The information matrix of d^* is

$$\mathbf{C}_{d^*} = (p-1)^{-1} k^{-1} b \{3k(k-1-2x) + p\, x(x+1)\} [\mathbf{I}_p - (1/p)\mathbf{1}_p\mathbf{1}'_p] \quad (2.3)$$

Thus, from (2.3) and Theorem 2, $d^* \in D(p, b, k)$, constructed using a nested balanced block design with parameters p, $b_1 = b$, $b_2 = bk$, $k_1 = 3k$, $k_2 = 3$ is universally optimal in $D(p, b, k)$. Now, we discuss the robustness of optimal triallel crosses in the following sections.

3. ROBUSTNESS AGAINST INTERCHANGE OF DISTINCT CROSSES

Without loss of generality, we assume that the cross involving lines 1, 2 and 3 in Block-1 has been interchanged by the cross involving lines 4, 5 and 6 in Block-2 of the design d. We denote the resulting design after interchange of the crosses by d_I. The incidence matrix of d in the partition form can be written as

$$\mathbf{N}_d = \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{N}_p \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{N}_u \end{bmatrix} \quad (3.1)$$

where

- $\mathbf{n}_i$ is the 6×1 vector corresponding to affected lines vs j^{th} affected block, $(i = 1, 2)$
- $\mathbf{u}_i$ is the $(p-6) \times 1$ vector corresponding to unaffected lines vs i^{th} affected block
- $\mathbf{N}_p$ is the $6 \times (b-2)$ matrix corresponding to affected lines vs unaffected blocks
- $\mathbf{N}_u$ is the $(p-6) \times (b-2)$ matrix corresponding to unaffected lines vs unaffected blocks

After interchange of distinct crosses, the resulting incidence matrix $(\mathbf{N}_I)$ will be

$$\mathbf{N}_I = \begin{bmatrix} \mathbf{n}_{1I} & \mathbf{n}_{2I} & \mathbf{N}_p \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{N}_u \end{bmatrix} \quad (3.2)$$

where, $\mathbf{n}_{1I} = (\mathbf{n}_1 - \mathbf{e}_1 + \mathbf{e}_2)$, and $\mathbf{n}_{2I} = (\mathbf{n}_2 + \mathbf{e}_1 - \mathbf{e}_2)$ with

$$\mathbf{e}_1 = [1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0]'$$

and $$\mathbf{e}_2 = [0 \;\; 0 \;\; 0 \;\; 1 \;\; 1 \;\; 1]'$$

As the cross involving lines 1, 2 and 3 is interchanged with the cross involving lines 4, 5 and 6, their incidence is one in their respective blocks. However, the lines belonging to the substituted cross may either be present or absent in the blocks where substitution takes place. Thus, $\mathbf{n}_1$ and $\mathbf{n}_2$ can be expressed as

$$\mathbf{n}_1 = [1 \;\; 1 \;\; 1 \;\; x_1 \;\; x_2 \;\; x_3]', \text{ and}$$
$$\mathbf{n}_2 = [x_4 \;\; x_5 \;\; x_6 \;\; 1 \;\; 1 \;\; 1]' \quad (3.3)$$

where x_i, $(i = 1, 2, \ldots, 6)$ takes values 1 or 0 depending upon the presence or absence of the corresponding line. The resulting information matrix $(\mathbf{C}_I)$ of d_I can be written as (details are given in the Appendix)

$$\mathbf{C}_I = \mathbf{C}_d - \mathbf{A}_I \quad (3.4)$$

It can be easily seen that $\mathbf{A}_I$ is symmetric with row and column sum zero and $\mathbf{A}_I$ commutes with information matrix $\mathbf{C}_d$ of a variance balanced block design for triallel crosses. Therefore, the eigenvalues of resulting information matrix $(\mathbf{C}_I)$ can be obtained by subtracting the eigenvalues of $\mathbf{A}_I$ from that of $\mathbf{C}_d$. Thus, the eigenvalues of $\mathbf{C}_I$ are

(i) $\theta_0 = 0$ with multiplicity 1

(ii) $\theta_{1I} = \mu$ with multiplicity $(p-3)$

(iii) $\theta_{2I} = (\mu - \theta_1)$ and

(iv) $\theta_{3I} = (\mu - \theta_2)$ (3.5)

Here, $\mu = 3b(k-1)/(p-1)$ is the unique non-zero eigenvalue of C_d. For design d_I to be connected, rank$(\mathbf{C}_I)$ should be $p - 1$. In other words $\mu \neq \theta_i, i = 1, 2$. It is difficult to obtain the conditions for which $\mu \neq \theta_i$ in general. Hence, we shall study the particular designs. Furthermore, for a connected design, the efficiency of the resulting design (d_I) relative to the original design (d) is seen to be

$$E = \frac{(p-1)\mu^{-1}}{\left[(p-3)\theta_{1I}^{-1} + \theta_{2I}^{-1} + \theta_{3I}^{-1}\right]} \quad (3.6)$$

All the designs can be classified into 10 distinct cases depending upon the values of x'_i s, $(i = 1, \ldots, 6)$. Rest of the cases are identical with these 10 cases in the sense that the other cases reduce to these by simply renumbering the lines and the blocks. The ten distinct cases and the corresponding eigenvalues of $\mathbf{A}_I$ are given in Table 3.1.

Here, we have considered, the symmetric BIB designs for obtaining the optimal block designs for triallel

Table 3.1. Distinct cases and the eigenvalues of $\mathbf{A}_l$

Case No.	Values of x_1	x_2	x_3	x_4	x_5	x_6	Eigenvalues θ_1, θ_2
I	0	0	0	0	0	0	$\pm\sqrt{(36k-16-12\alpha)}/k$
II	0	0	0	0	0	1	$\left[1\pm\sqrt{(36k-16-12\alpha)}\right]/k$
III	0	0	0	0	1	1	$\left[2\pm\sqrt{(36k-12-12\alpha)}\right]/k$
IV	0	0	0	1	1	1	$\left[3\pm\sqrt{(36k-12\alpha)}\right]/k$
V	0	0	1	0	0	1	$\left[2\pm\sqrt{(36k-12\alpha-4)}\right]/k$
VI	0	0	1	0	1	1	$\left[3\pm\sqrt{(36k-12\alpha)}\right]/k$
VII	0	1	1	0	1	1	$\left[4\pm\sqrt{(36k+12-12\alpha)}\right]/k$
VIII	1	1	1	0	0	1	$\left[4\pm\sqrt{(36k-12\alpha-20)}\right]/k$
IX	1	1	1	0	1	1	$\left[5\pm\sqrt{(36k-12\alpha+19)}\right]/k$
X	1	1	1	1	1	1	$\left[6\pm\sqrt{(36k-12\alpha+36)}\right]/k$

crosses so that the value of α and thereby the non-zero eigenvalues of A_l can easily be obtained.

The design d_l becomes disconnected when $\theta_i, (i=1,2)$ equals μ, the non-zero eigenvalue of C_d. As an illustration, we consider the following.

Example 3.1: Consider an experimental situation in which 24 distinct triallel crosses involving 9 lines are arranged in 12 blocks each of size 2. The design with block contents is

Block-1	(1 × 2 × 3), (4 × 5 × 6)
Block-2	(1 × 2 × 3), (7 × 8 × 9)
Block-3	(4 × 5 × 6), (7 × 8 × 9)
Block-4	(1 × 4 × 7), (2 × 5 × 8)
Block-5	(1 × 4 × 7), (3 × 6 × 9)
Block-6	(2 × 5 × 8), (3 × 6 × 9)
Block-7	(1 × 6 × 8), (2 × 4 × 9)
Block-8	(1 × 6 × 8), (3 × 5 × 7)
Block-9	(2 × 4 × 9), (3 × 5 × 7)
Block-10	(1 × 5 × 9), (2 × 6 × 7)
Block-11	(1 × 5 × 9), (3 × 4 × 8)
Block-12	(2 × 6 × 7), (3 × 4 × 8)

Now, in this design, if the cross (1 × 2 × 3) in Block-1 gets interchanged with cross (7 × 8 × 9) in Block-2, it is the Case IV in Table 3.1. For this case,

$$\mathbf{n}_1 = [1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0]' \text{ and}$$

$$\mathbf{n}_2 = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]'$$

Further, it can be easily seen that the unique non-zero eigenvalues of information matrix $\mathbf{C}_d$ is 4.5 with multiplicity 8. The non-zero eigenvalues of $\mathbf{A}_l$ are 4.5 and −1.5 respectively. Using (3.5), it can be easily seen that two of the eigenvalues of $\mathbf{C}_l$ are zero and the design becomes disconnected.

Similarly, when the cross (4 × 5 × 6) in Block-1 gets interchanged with cross (7 × 8 × 9) in Block-2, then, it is the Case *I* given in Table 3.1. Similarly one can observe other cases.

The general condition for which d_l remains connected cannot be obtained due to uncertainty in the number of observations, n ($< n_c$). Therefore, the connectedness has been studied for particular designs obtainable from the methods of construction given by Das and Gupta (1997). The designs with $p \le 30$ are given in Table *A*1 (Appendix). Efficiencies (*E's*) of d_l relative to d for connected designs in different cases have been computed using (3.6). It is seen that all the designs except D_3, D_4 and D_{12} have $E \geq 0.9500$ and thus are robust from the connectedness and efficiency criterion. The design D_3 is disconnected in cases IV and VI. The value of E is 0.8346 in Case-X for design D_4 and 0.7906 in Case-X for design D_{12} indicating that these designs are not robust.

4. INTERCHANGE OF CROSSES WITH ONE LINE IN COMMON

Without loss of generality, we assume that the cross involving 1, 2 and 5 in Block-1 has been interchanged with the cross involving lines 3, 4 and 5 in Block-2, so that line 5 is common in both the crosses.

Here, like (3.1), the incidence matrix can be written as

$$\mathbf{N}_d = \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{N}_p \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{N}_u \end{bmatrix} \tag{4.1}$$

where

$\mathbf{n}_i$ is the 5 × 1 vector corresponding to affected lines vs i^{th} affected block (i = 1, 2)

$\mathbf{u}_i$ is the (p – 5) × 1 vector corresponding to unaffected lines vs i^{th} affected block

$\mathbf{N}_p$ is the 5 × (b – 2) matrix corresponding to affected lines vs unaffected blocks

$\mathbf{N}_u$ is the (p – 5) × (b – 2) matrix corresponding to unaffected lines vs unaffected blocks

After interchange of a pair of crosses having one line in common the resulting incidence matrix becomes

$$\mathbf{N}_l = \begin{bmatrix} \mathbf{n}_{1l} & \mathbf{n}_{2l} & \mathbf{N}_p \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{N}_u \end{bmatrix} \tag{4.2}$$

where $\mathbf{n}_{1l} = (\mathbf{n}_l - \mathbf{e}_1 + \mathbf{e}_2)$, and $\mathbf{n}_{2l} = (\mathbf{n}_2 - \mathbf{e}_2 + \mathbf{e}_1)$ with $\mathbf{e}_1 = [1 \quad 1 \quad 0 \quad 0 \quad 1]'$, and $\mathbf{e}_2 = [0 \quad 0 \quad 1 \quad 1 \quad 1]'$

Let $\mathbf{n}_1 = [1 \quad 1 \quad x_1 \quad x_2 \quad 1]'$

and $\mathbf{n}_2 = [x_3 \quad x_4 \quad 1 \quad 1 \quad 1]'$

where x_i, (i = 1, 2, 3, 4) takes values one or zero depending on the presence or absence of the line 3, 4 in Block-1 and lines 1, 2 in Block-2, respectively. Substitution of $\mathbf{n}_i$ and $\mathbf{n}_{il}$,(i = 1, 2) in (4.2) yields (A2.5) with values of **X, Y, Z** and **S** given at (A2.8) in the Appendix.

All the designs can now be classified into 6 distinct cases depending upon the values of x_i (0, 1), (i = 1 ,2, 3, 4). These cases along with the corresponding non-zero eigenvalues of A_l are listed in Table 4.1.

Example 4.1: As an illustration, we again consider Example 3.1 and suppose that the cross (1 × 2 × 3) in Block-1 gets interchanged with cross (1 × 4 × 7) in Block-4. It is the Case IV given in Table 4.1. For this case

$\mathbf{n}_1 = [1 \quad 1 \quad 1 \quad 0 \quad 1]'$ and

$\mathbf{n}_2 = [1 \quad 0 \quad 1 \quad 1 \quad 1]'$

It can be easily seen that the unique non-zero eigenvalues of information matrix $\mathbf{C}_d$ in this case is 4.5 with multiplicity 8. The non-zero eigenvalues of $\mathbf{A}_l$ are 3 and –1 respectively. Furthermore, using (3.6), it can be easily seen that the efficiency of this design is 0.8148. Similarly one can observe other cases.

Table 4.1. Distinct cases and the non-zero eigenvlues of $\mathbf{A}_l$

Case No.	Values of x_1	x_2	x_3	x_4	Eigenvalues θ_1, θ_2
I	0	0	0	0	$\pm\left[\sqrt{(24k-8\alpha+8)}\right]/k$
II	0	0	0	1	$\left[1\pm\sqrt{(24k-8\alpha-8)}\right]/k$
III	0	0	1	1	$\left[2\pm\sqrt{(24k-8\alpha)}\right]/k$
IV	1	0	1	0	$\left[2\pm\sqrt{(24k-8\alpha)}\right]/k$
V	1	1	1	0	$\left[3\pm\sqrt{(24k-8\alpha+8)}\right]/k$
VI	1	1	1	1	$\left[4\pm\sqrt{(24k-8\alpha+16)}\right]/k$

Here also connectedness and efficiencies of d_l relative to d have been computed for optimal block designs for triallel crosses that are symmetric with respect to the lines. It is seen that all the designs given in Table A1 except D_1, D_2, D_3, D_4, D_{11}, and D_{16} have $E \geq 0.9500$ and as such are robust. The design D_1 is disconnected in Case V. For the design D_2, E = 0.5000 in Case VI, for D_3, E = 0.2424 in Case-V, for D_4, E = 0.9223 in Case VI, for D_{11}, E = 0.7246 in Case IV, and for D_{16}, E = 0.7391 in Case IV implying that these designs are not robust.

5. INTERCHANGE OF CROSSES WITH TWO LINES IN COMMON

Without loss of generality, we assume that the cross involving lines 1, 3 and 4 in Block-1 has been interchanged with the cross involving lines 2, 3 and 4 in Block-2, so that the lines 3 and 4 are common in both the crosses. Here, $\mathbf{n}_i$, $\mathbf{u}_i$, (i = 1, 2), $\mathbf{N}_p$ and $\mathbf{N}_u$ of $\mathbf{N}_d$ in (3.1) are of order 4 × 1, (p – 4) × 1, 4 × (b – 2) and (p – 4) × (b – 2), respectively having same meaning.

After interchange of a pair of crosses having two lines in common $\mathbf{n}_{il}$,(i = 1, 2) will be

$\mathbf{n}_{1l} = \mathbf{n}_1 - \mathbf{e}_1 + \mathbf{e}_2$, and $\mathbf{n}_{2l} = \mathbf{n}_2 + \mathbf{e}_1 - \mathbf{e}_2$ with

$$\mathbf{e}_1 = [1 \quad 0 \quad 1 \quad 1]', \text{ and } \mathbf{e}_2 = [0 \quad 1 \quad 1 \quad 1]' \tag{5.1}$$

$$\text{Let } \mathbf{n}_1 = [1 \quad x_1 \quad 1 \quad 1]' \text{ and } \mathbf{n}_2 = [x_2 \quad 1 \quad 1 \quad 1]' \tag{5.2}$$

where x_i, (i = 1, 2) takes values 1 or 0 depending on the presence or absence of the line 2 and 1 in Block-1 and Block-2, respectively. Substitution of $\mathbf{n}_i$, and $\mathbf{n}_{il}$, (i = 1, 2) in (4.2) yields (A2.5) with values of **X, Y, Z** and **S** given at (A2.10) in the Appendix.

Table 5.1. Distinct cases and eigenvalues of $\mathbf{A}_l$

Case No.	Values of x_1	x_2	Eigenvalues θ_1, θ_2
I	0	0	$\pm\sqrt{(12k-4-4\alpha)/k}$
II	0	1	$\frac{1}{k} \pm \sqrt{\frac{2}{k^2} + \frac{(12k-2-4\alpha)}{k}}$
III	1	1	$\frac{2}{k} \pm \sqrt{\frac{4}{k^2} + \frac{(12k-4\alpha)}{k}}$

Here, all designs can be classified into 3 distinct cases depending upon the values x_i, (i = 1, 2). These cases along with the non-zero eigenvalues are given in Table 5.1.

Example 5.1: As an illustration, we consider an experimental situation in which 14 distinct triallel crosses involving 7 lines are arranged in 7 blocks each of size 2. The design with block contents is

Block-1	(2 × 3 × 5), (4 × 6 × 7)
Block-2	(3 × 4 × 6), (1 × 5 × 7)
Block-3	(4 × 5 × 7), (1 × 2 × 6)
Block-4	(1 × 5 × 6), (2 × 3 × 7)
Block-5	(2 × 6 × 7), (1 × 3 × 4)
Block-6	(1 × 3 × 7), (2 × 4 × 5)
Block-7	(1 × 2 × 4), (3 × 5 × 6)

Now, if in this design, the cross (4 × 6 × 7) in Block-1 gets interchanged with cross (2 × 6 × 7) in Block-5, then it is the Case III given in Table 5.1. For this case

$$\mathbf{n}_1 = [1 \quad 1 \quad 1 \quad 1]', \text{ and } \mathbf{n}_2 = [1 \quad 1 \quad 1 \quad 1]'$$

It can be easily seen that the unique non-zero eigenvalues of information matrix $\mathbf{C}_d$ is 3.5 with multiplicity 6. The non-zero eigenvalues of $\mathbf{A}_l$ are $1 \pm \sqrt{3}$. From (3.6), it can be easily seen that the efficiency of this design is 0.9091.

Connectedness and efficiencies of the designs given in Table A1 have been studied as in earlier sections. The minimum value of E for the designs D_1, D_{11} and D_{16} is attained at 0.9091, 0.6667 and 0.6393, respectively in Case III. Rest of the designs for which interchange is possible have $E \geq 0.9500$ and are, therefore, robust as judged from the connectedness and efficiency criterion.

6. CONCLUSIONS

It is clear from the above that all the designs given in Table A1 (Appendix) are robust against interchange of a cross except the designs D_1, D_2, D_3, D_4, D_{11}, D_{12} and D_{16}. The designs D_1, D_2, D_3 become disconnected in some cases and D_4, D_{11}, D_{12} and D_{16} have relative efficiencies less than 0.9500 in one or more cases.

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APPENDIX

A1. Some results on optimality of designs in D (p, b, k).

Theorem 1. {Theorem 2.1, Das and Gupta (1997)}

for $d \in D(p, b, k)$

$$\text{tr}(\mathbf{C}_d) \le k^{-1}b \;\{3k(k-1-2x) + px(x+1)\}$$

where $x = [3k/p]$, [.] being the integer valued function and tr(**A**) denotes the trace of the matrix **A**.

Corollary 1. {Corollary 2.1, Das and Gupta (1999)}. For $d \in D(p, b, k)$, if $3k/p \ge 1$ (i.e., $x \ge 1$) then

$$\text{tr}(\mathbf{C}_d) \le k^{-1}b\{3k(k-1-2x) + px(x+1)\}$$

$$\le 3bk(p-3)/p \le 3b(k-1), \text{ and}$$

if $3k/p \le 1$ (*i.e.* $x = 0$) then $\text{tr}(\mathbf{C}_d) \le 3kb(k-1)$

Again, if $n_{dij} = 0$ or 1 for all i, j, then tr $(\mathbf{C}_d) = 3b(k-1)$

Theorem 2. {Theorem 2.2, Das and Gupta (1997)}

Let $d^* \in D(p, b, k)$, be a block design for triallel crosses satisfying

(i) $\text{tr}(\mathbf{C}_{d^*}) = k^{-1}b\;\{3k(k-1-2x) + px(x+1)\}$, and

(ii) $\mathbf{C}_{d^*}$ is completely symmetric

Then d^* is universally optimal in a relevant class of competing designs in $D(p, b, k)$ and, in particular, is A-optimal.

A2. Expressing the information matrix, $\mathbf{C}_I$, of $\mathbf{d}_I$ as $\mathbf{C}_I = \mathbf{C}_d - \mathbf{A}_I$

A2 (i) When the interchanged crosses are distinct

In this case, the diagonal matrix of block sizes ($\mathbf{K}_d$) in a partition form can be written as

$$\mathbf{K}_d = \begin{bmatrix} k_1 & 0 & \mathbf{0}' \\ 0 & k_2 & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_u \end{bmatrix} \qquad (A2.1)$$

It can be easily seen that

$$\mathbf{N}_l\mathbf{K}_d^{-1}\mathbf{N}_l' = \begin{bmatrix} \dfrac{\mathbf{n}_{1l}\mathbf{n}_{1l}'}{k_1} + \dfrac{\mathbf{n}_{2l}\mathbf{n}_{2l}'}{k_2} + \mathbf{N}_p\mathbf{K}_u^{-1}\mathbf{N}_p' & \dfrac{\mathbf{n}_{1l}\mathbf{u}_{1l}'}{k_1} + \dfrac{\mathbf{n}_{2l}\mathbf{u}_{2l}'}{k_2} + \mathbf{N}_p\mathbf{K}_u^{-1}\mathbf{N}_u' \\ \dfrac{\mathbf{u}_1\mathbf{n}_{1l}'}{k_1} + \dfrac{\mathbf{u}_2\mathbf{n}_{2l}'}{k_2} + \mathbf{N}_u\mathbf{K}_u^{-1}\mathbf{N}_p' & \dfrac{\mathbf{u}_1\mathbf{u}_1'}{k_1} + \dfrac{\mathbf{u}_2\mathbf{u}_2'}{k_2} + \mathbf{N}_u\mathbf{K}_u^{-1}\mathbf{N}_u' \end{bmatrix} \quad \text{(A2.2)}$$

Substituting the values of $\mathbf{n}_{1l}$ and $\mathbf{n}_{2l}$ from (3.2) in Section 3, we get

$$\mathbf{n}_{1l}\mathbf{n}'_{1l} = \mathbf{n}_1\mathbf{n}'_1 - \mathbf{n}_1(\mathbf{e}'_1 - \mathbf{e}'_2) - (\mathbf{e}_1 - \mathbf{e}_2)\mathbf{n}'_1 + (\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{e}'_1 - \mathbf{e}'_2)$$
$$\mathbf{n}_{2l}\mathbf{n}'_{2l} = \mathbf{n}_2\mathbf{n}'_2 + \mathbf{n}_2(\mathbf{e}'_1 - \mathbf{e}'_2) + (\mathbf{e}_1 - \mathbf{e}_2)\mathbf{n}'_2 + (\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{e}'_1 - \mathbf{e}'_2)$$
$$\mathbf{u}_1\mathbf{n}'_{1l} = \mathbf{u}_1\mathbf{n}'_1 - \mathbf{u}_1(\mathbf{e}'_1 - \mathbf{e}'_2)$$
$$\mathbf{u}_2\mathbf{n}'_{2l} = \mathbf{u}_2\mathbf{n}'_2 + \mathbf{u}_2(\mathbf{e}'_1 - \mathbf{e}'_2) \quad \text{(A2.3)}$$

Further, substitution of (A2.3) in (A2.2) yields

$$\mathbf{N}_l\mathbf{K}_d^{-1}\mathbf{N}_l' = \mathbf{N}_d\mathbf{K}_d^{-1}\mathbf{N}_d' + \mathbf{A}_l$$

The resulting information matrix ($\mathbf{C}_l$) of d_l can be written as

$$\mathbf{C}_l = \mathbf{C}_d - \mathbf{A}_l \quad \text{(A2.4)}$$

Here, $\mathbf{A}_l$ can be expressed as

$$\mathbf{A}_l = k^{-1}\begin{bmatrix} \mathbf{X}-\mathbf{Y} & \mathbf{S}' \\ \mathbf{S} & \mathbf{Z} \end{bmatrix} \quad \text{(A2.5)}$$

where

$$\mathbf{X} = [\mathbf{n}_2(\mathbf{e}_1' - \mathbf{e}_2') + (\mathbf{e}_1 - \mathbf{e}_2)\mathbf{n}_{2l}']$$
$$\mathbf{Y} = [\mathbf{n}_1(\mathbf{e}_1' - \mathbf{e}_2') - (\mathbf{e}_1 - \mathbf{e}_2)\mathbf{n}_{1l}']$$
$$\mathbf{S} = (\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{e}_1' - \mathbf{e}_2')$$
$$\mathbf{Z} = \mathbf{0}_{(p-6)\times(p-6)}, \text{ the null matrix}$$

Substituting the values of $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_{1l}$ and $\mathbf{n}_{2l}$ from (3.2) and (3.3) in (A2.2), we get

$$\mathbf{X} = \begin{bmatrix} 2x_4+1 & x_4+x_5+1 & x_4+x_6+1 & -x_4 & -x_4 & -x_4 \\ x_4+x_5+1 & 2x_5+1 & x_5+x_6+1 & -x_5 & -x_5 & -x_5 \\ x_4+x_6+1 & x_5+x_6+1 & 2x_6+1 & -x_6 & -x_6 & -x_6 \\ -x_4 & -x_5 & -x_6 & -1 & -1 & -1 \\ -x_4 & -x_5 & -x_6 & -1 & -1 & -1 \\ -x_4 & -x_5 & -x_6 & -1 & -1 & -1 \end{bmatrix}$$

and

$$\mathbf{Y} = \begin{bmatrix} 1 & 1 & 1 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & x_1 & x_2 & x_3 \\ x_1 & x_1 & x_1 & -(2x_1+1) & -(x_1+x_2+1) & -(x_1+x_3+1) \\ x_2 & x_2 & x_2 & -(x_1+x_2+1) & -(2x_2+1) & -(x_2+x_3+1) \\ x_3 & x_3 & x_3 & -(x_1+x_3+1) & -(x_2+x_3+1) & -(2x_3+1) \end{bmatrix}$$

The eigenvalues of $\mathbf{A}_l$ have been obtained by solving the equation

$$|\mathbf{A}_l - \theta I| = 0 \quad \text{(A2.6)}$$

Solution of (A2.6) yields, the non-zero eigenvalues of $\mathbf{A}_l$ as

$$\theta_1, \theta_2 = k^{-1}\left[(x_1+x_2+x_3+x_4+x_5+x_6) \pm\sqrt{(x_1+x_2+x_3+x_4+x_5+x_6)^2 + c}\right] \quad \text{(A2.7)}$$

where

$$c = 6Q + (x_1+x_2+x_4+x_5+x_6-5x_3)(2x_1-x_2-x_3)$$
$$-6(x_1-x_2)(x_2-x_3) - 3(x_1-x_6)(x_4+x_5+x_6)$$
$$+ (x_1+x_2+x_3)(3x_1-2x_5-x_6)$$
$$- (x_5-x_6)(4x_4+4x_5+10x_6)$$
$$- (x_4-x_5)(x_1+x_2+x_3+x_5+x_6-5x_4), \text{ and}$$
$$Q = \mathbf{u}'_1\mathbf{u}_1 + \mathbf{u}'_2\mathbf{u}_2 - 2\mathbf{u}'_1\mathbf{u}_2 \text{ with}$$
$$\mathbf{u}'_1\mathbf{u}_1 = 3k-3-x_4-x_5-x_6, \quad \mathbf{u}'_2\mathbf{u}_2 = 3k-3-x_1-x_2-x_3$$
$$\mathbf{u}'_1\mathbf{u}_2 = \alpha - x_1 - x_2 - x_3 - x_4 - x_5 - x_6$$

α being the number of common lines between the affected blocks.

A2(ii). When the interchanged crosses have one line in common

$$\text{Here, } \mathbf{X} = \begin{bmatrix} 2x_3+1 & x_3+x_4+1 & -x_3 & -x_3 & 1 \\ x_3+x_4+1 & 2x_4+1 & -x_4 & -x_4 & 1 \\ -x_3 & -x_4 & -1 & -1 & -1 \\ -x_3 & -x_4 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

$$\text{and } \mathbf{Y} = \begin{bmatrix} 1 & 1 & x_1 & x_2 & 1 \\ 1 & 1 & x_1 & x_2 & 1 \\ x_1 & x_1 & -(2x_1+1) & -(x_1+x_2+1) & -1 \\ x_2 & x_2 & -(x_1+x_2+1) & -(2x_2+1) & -1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

$$\mathbf{S} = (\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{e}'_1 - \mathbf{e}'_2), \text{ and } \mathbf{Z} = \mathbf{0}_{(p-5)\times(p-5)} \qquad (A2.8)$$

Solution of $|\mathbf{A}_l - \theta \mathbf{I}| = 0$ gives the non-zero eigenvalues of $\mathbf{A}_l$ as

$$\theta_1, \theta_2 = \left[(x_1 + x_2 + x_3 + x_4) \pm \sqrt{(x_1 + x_2 + x_3 + x_4)^2 + c}\right] \Big/ k \qquad (A2.9)$$

with $c = 4Q + (x_1 - x_2)(x_1 + x_3 + x_4 - 3x_2) + 2(x_1 - x_1)(x_1 + x_2 - x_3 - x_4) - (x_3 - x_4)(x_1 + x_2 + x_4 - 3x_3)$, and $Q = \mathbf{u}'_1\mathbf{u}_1 + \mathbf{u}'_2\mathbf{u}_2 - 2\mathbf{u}'_1\mathbf{u}_2$

A2 (iii). When the interchanged crosses have two lines in common

Here, **X, Y, S** and **Z** of $\mathbf{A}_l$ in (A2.5) are

$$\mathbf{X} = \begin{bmatrix} 2x_2+1 & x_2 & 1 & 1 \\ x_2 & -1 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \text{ and}$$

$$\mathbf{Y} = \begin{bmatrix} 1 & x_1 & 1 & 1 \\ x_1 & -2x_1-1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{S} = (\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{e}'_1 - \mathbf{e}'_2), \text{ and } \mathbf{Z} = \mathbf{0}_{(p-4)\times(p-4)} \qquad (A2.10)$$

Solution of $|\mathbf{A}_l - \theta I| = 0$, gives the non-zero eigenvalues of $\mathbf{A}_l$ as

$$\theta_1, \theta_2 = \frac{(x_1 + x_2)}{k} \pm \sqrt{\frac{(2x_1^2 + 2x_2^2) + 2Q}{k^2}} \qquad (A2.11)$$

where $Q = (6k - 2 + x_1 + x_2 - 2\alpha)/k$

A3. Some variance balanced block designs for triallel crosses with p ≤ 30

Table A1. Variance balanced block designs for triallel crosses for $p \le 30$ obtainable from Das and Gupta (1997)

Design	Parameters			Source
	p	b	k	
D1	7	7	2	S1-F4
D2	9	4	3	S1-F1
D3	9	12	2	S1-F3
D4	10	10	3	S1-P5(i)
D5	10	40	3	S1-F2
D6	13	13	4	S1-F4
D7	13	52	3	S1-F2
D8	15	35	4	S1-F3
D9	15	7	5	S1-F1
D10	16	16	5	S1-P13
D11	16	112	5	S1-F2
D12	19	19	3	S1-P6
D13	19	19	6	S1-F4
D14	19	76	3	S1-F2
D15	21	70	3	S1-F3
D16	21	70	4	S1-F3
D17	21	70	6	S1-F3
D18	21	10	7	S1-F1
D19	25	100	3	S1-F2
D20	25	25	8	S1-F4
D21	27	117	8	S1-F3
D22	27	13	9	S1-F1

S1: Das and Gupta (1997); F# : Family # of Designs
P : Table 3 of Preece [1983]