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Robust Second Order Rotatable Designs : Part III (RSORD)

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SUMMARY

In the context of response surface methodology, Box and Hunter [1] introduced the concept of rotatability assuming that the errors in the linear model are uncorrelated and homoscedastic. Recently Panda and Das [5], Das ([3], [4]) studied rotatability assuming that the errors in the linear model are correlated. This article examines the conditions of rotatability in a second order model when errors have the following variance-covariance structures: intra-class, inter-class, compound symmetry, and tri-diagonal structure. The robustness of usual Second Order Rotatable Designs (SORD) is investigated and construction of some *robust* SORD under the above correlation structures is given.

Key words : Response surface, Rotatability, Intra-class, Inter-class, Compound symmetry, Tri-diagonal structure.

1. Introduction

The concept of rotatability in the context of response surface designs was introduced by Box and Wilson [2] and then formally developed by Box and Hunter [1], assuming the errors to be uncorrelated and homoscedastic. A study of first order rotatable designs (FORDs) assuming that model has correlated errors was initiated by Panda and Das [5]. Das [3], studied SORDs assuming that the model has correlated errors. Starting with usual SORD, a method of construction of second order rotatable designs with correlated errors (SORDWCE) was introduced by Das [4].

The present article studies the second order rotatability conditions under intra-class, inter-class, compound symmetry structure and tri-diagonal structure of errors in the model. For a detailed literature and illustrations where such correlation structures arise in the observations, the reader may see Morrison [7], Press [8], Seber [9] and Lindsey [6]. The robustness of usual SORDs is investigated and it is observed that a SORD under the usual model with uncorrelated errors preserves rotatability property under the intra-class structure of errors and vice-versa. This holds irrespective of the value of the intra-class correlation coefficients. It is further seen that variance of estimated response of a SORDWCE under the intra-class correlation model is (under some mild conditions) more or less than that of a SORD under uncorrelated model, according as the intra-class correlation coefficient is positive or negative.

Again, if the design points are divided into some disjoint sets and if the points of each set satisfy the second order rotatability conditions in the usual sense, then the overall design is second order rotatable under the inter-class covariance structure of errors but the converse is not always true. The result of inter-class covariance structure also holds good in case of compound symmetry structure.

Finally, we undertake a detailed study of second order rotatability for tri-diagonal structure of errors. In the process we develop two methods of construction of second order rotatable designs under tri-diagonal structure of errors.

2. *Robust Rotatable Designs in Correlated Error Structures*

Das [3] obtained rotatability conditions when errors in observations are correlated. In the sequel we study the special cases of the correlated error structure mentioned in Section 1. The notations, rotatability conditions, non-singularity conditions, and variance function used here are as given in Das [3].

A natural and easily attainable property in case of regression designs is that of *rotatability*, which requires that the variance of a predicted value remains constant at points that are equidistant from the design center.

Definition 2.1. Robust Second Order Rotatable Design (RSORD): A design D on k factors which remains second order rotatable under the model (2.1) in Das [3] for all the variance-covariance matrices belonging to a well defined class W_0 = {W positive definite : $W_{N \times N}$ defined by a particular correlation structure neatly specified} is called a Robust Second Order Rotatable Design, under the correlation structure W_0.

3. *Robust Second Order Rotatable Designs Under Intra-class, Inter-class and Compound Symmetry Structure*

3.1 Intra-class Correlation Structure

We will now study RSORD under the intra-class variance covariance structure of errors given by

$$W_0 = \{D(e) = \sigma^2 [(1-\rho) I_N + \rho E_{N \times N}] = W_{N \times N}(\rho), \text{say}\} \quad (3.1)$$

$$W^{-1}_{N \times N}(\rho) = (\sigma^2)^{-1}[(\delta_1 - \gamma_1) I_N + \gamma_1 E_{N \times N}]$$

where $$\delta_1 = \frac{1+(N-2)\rho}{\{1+(N-1)\rho\}(1-\rho)}, \gamma_1 = -\frac{\rho}{\{1+(N-1)\rho\}(1-\rho)}$$

and $$\rho > -(N-1)^{-1} \tag{3.2}$$

Following general conditions of rotatability for SORD in the correlated case as in Das [3], it is clear that the only situation where any SORD under usual conditions also happens to be necessarily RSORD is intra-class correlation structure (3.1) for errors.

3.1.1 Conditions for Rotatability, Non-singularity and Variance Function

Following (3.3) and (3.5) in Das [3], it turns out that the necessary and sufficient conditions for second order rotatability and non-singularity under the structure (3.1) are exactly the *same* as those under independently and identically distributed error structure, whatever be the value of ρ in (3.1). However, the variance function depends on ρ. We have thus deduced the following theorem.

Theorem 3.1. A design is second order rotatable under intra-class structure of errors iff it is a SORD in the independent and identically distributed errors model, whatever be the value of intra-class correlation coefficient ρ.

Following (3.5) in Das [3], the underlying variance function is given by

$$V(\hat{y}_x/W_{N\times N}(\rho)) = A(N,k,\rho) + B(N,k,\rho)r^2 + C(N,k,\rho)r^4 = V_\rho, \text{say} \tag{3.3}$$

where $$v^{00} = \frac{\sigma^2\left[(k+2)\{1+(N-1)\rho\}\lambda_4 - kN\rho\lambda_2^2\right]}{T} = A(N,k,\rho); \text{say}$$

$$B(N,k,\rho) = (2a_1+e) = \frac{\sigma^2 (1-\rho)(k+2)\left[\lambda_4 - \lambda_2^2\right]}{\lambda_2 T}$$

$$C(N,k,\rho) = (c/2+d_1) = \frac{\sigma^2(1-\rho)[(k+1)\lambda_4 - \lambda_2^2(k-1)]}{2\lambda_4 T}$$

$$T = N\{(k+2)\lambda_4 - k\lambda_2^2\}, r^2 = \sum_{i=1}^{k} x_i^2 \tag{3.4}$$

and a_0, e, 1/c, d, a_1, d_1, v^{00} are given in Das [3].

The special case of SORD leading to V_0, follows upon substituting $\rho = 0$.

An interesting feature of the comparison between V_ρ and V_0 is that, $V_0 \lesseqgtr V_\rho$ according as $\rho \gtreqless 0$ provided

$$N[(k+2)\lambda_4 - k\lambda_2^2] > \left[\frac{(k+2)}{\lambda_2}\{\lambda_4(\lambda_2 + r^2) - \lambda_2^2 r^2\} + \frac{\{(k+1)\lambda_4 - (k-1)\lambda_2^2\} r^4}{2\lambda_4}\right]$$

3.2 Inter-class Correlation Structure

Consider a situation where the correlation structure (3.1) is altered to

$$W_0 = \{D(e) = I_m \otimes W_{n \times n}(\rho)\} \tag{3.5}$$

where $\otimes$ denotes Kronecker product and $W_{n \times n}(\rho)$ as in (3.1).

Structure (3.5) is known as Inter-class correlation structure.

3.2.1 Conditions for Rotatability and Non-singularity

Following (3.5) in Das [3], the necessary and sufficient conditions for second order rotatability under the correlation structure (3.5) simplify to

(I) (i) $v_{0.j} = \sum_{u=1}^{N} x_{ju} = 0$

$$v_{0.jl} = \sum_{u=1}^{N} x_{ju} x_{lu} = 0; \quad 1 \le j \le l \le k$$

(ii) $$v_{i.j} = \alpha \sum_{u=1}^{N} x_{iu} x_{ju} - \beta \left\{ \left(\sum_{u=1}^{n} x_{iu} \right)\left(\sum_{u=1}^{n} x_{ju} \right) + \ldots + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu} \right)\left(\sum_{u=n(m-1)+1}^{mn} x_{ju} \right) \right\} = 0, \quad 1 \le i \ne j \le k$$

(iii) (1) $$v_{ii.j} = \alpha \sum_{u=1}^{N} x_{iu}^2 x_{ju} - \beta \left\{ \left(\sum_{u=1}^{n} x_{iu}^2 \right)\left(\sum_{u=1}^{n} x_{ju} \right) + \ldots + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu}^2 \right)\left(\sum_{u=n(m-1)+1}^{mn} x_{ju} \right) \right\} = 0, \quad 1 \le i, j \le k$$

$$(2)\ \ v_{ij.l} = \alpha \sum_{u=1}^{N} x_{iu}x_{ju}x_{lu} - \beta\left\{\left(\sum_{u=1}^{n} x_{iu}x_{ju}\right)\left(\sum_{u=1}^{n} x_{lu}\right) + \ldots \right.$$

$$\left. + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu}x_{ju}\right)\left(\sum_{u=n(m-1)+1}^{mn} x_{lu}\right)\right\} = 0;\ \ 1 \le i < j, l \le k$$

$$(3)\ \ v_{ii.jl} = \alpha \sum_{u=1}^{N} x_{iu}^2 x_{ju}x_{lu} - \beta\left\{\left(\sum_{u=1}^{n} x_{iu}^2\right)\left(\sum_{u=1}^{n} x_{ju}x_{lu}\right) + \ldots \right.$$

$$\left. + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu}^2\right)\left(\sum_{u=n(m-1)+1}^{mn} x_{ju}x_{lu}\right)\right\} = 0$$

$$1 \le i,\ j < l \le k;\ \ (j, l) \ne (i, i)$$

$$(4)\ \ v_{ij.lt} = \alpha \sum_{u=1}^{N} x_{iu}x_{ju}x_{lu}x_{tu} - \beta\left\{\left(\sum_{u=1}^{n} x_{iu}x_{ju}\right)\left(\sum_{u=1}^{n} x_{lu}x_{tu}\right) + \ldots \right.$$

$$\left. + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu}x_{ju}\right)\left(\sum_{u=n(m-1)+1}^{mn} x_{lu}x_{tu}\right)\right\} = 0$$

$$1 \le i, l < j, t \le k;\ \ (i, j) \ne (l, t)$$

$$\text{(II) (i)}\quad v_{0.jj} = \{\sigma^2 (1 + (n-1)\rho)\}^{-1} \sum_{u=1}^{N} x_{ju}^2 = a_0;\ 1 \le j \le k$$

$$\text{(ii)}\quad v_{i.i} = \{\sigma^2\}^{-1}\left[\alpha \sum_{u=1}^{N} x_{iu}^2 - \beta\left\{\left(\sum_{u=1}^{n} x_{iu}\right)^2 + \ldots \right.\right.$$

$$\left.\left. + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu}\right)^2\right\}\right] = \frac{1}{e};\ 1 \le i \le k$$

$$\text{(iii)}\quad v_{ii.ii} = \{\sigma^2\}^{-1}\left[\alpha \sum_{u=1}^{N} x_{iu}^4 - \beta\left\{\left(\sum_{u=1}^{n} x_{iu}^2\right)^2 + \ldots + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu}^2\right)^2\right\}\right] = \left(\frac{2}{c} + d\right); 1 \le i \le k$$

$$\text{(III) (i)}\quad v_{ii.jj} = \{\sigma^2\}^{-1}\left[\alpha \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 - \beta\left\{\left(\sum_{u=1}^{n} x_{iu}^2\right)\left(\sum_{u=1}^{n} x_{ju}^2\right) + \ldots + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu}^2\right)\left(\sum_{u=n(m-1)+1}^{mn} x_{ju}^2\right)\right\}\right] = d;\ 1 \le i \ne j \le k$$

$$\text{(ii)}\quad v_{ij.ij} = \{\sigma^2\}^{-1}\left[\alpha \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 - \beta\left\{\left(\sum_{u=1}^{n} x_{iu} x_{ju}\right)^2 + \ldots + \left(\sum_{u=n(m-1)+1}^{mn} x_{iu} x_{ju}\right)^2\right\}\right] = \frac{1}{c};\ 1 \le i < j \le k$$

$$\text{(IV)}\ v_{ii.ii} = 2v_{ij.ij} + v_{ii.jj};\ 1 \le i < j \le k \tag{3.6}$$

where $v_{ii.ii}$, $v_{ii.jj}$ and $v_{ij.ij}$ are as in (II) (iii) and (III) (i), (ii) and

$$\alpha = \frac{1}{(1-\rho)},\ \beta = \frac{\rho}{(1-\rho)\{1+(n-1)\rho\}}$$

Non - singularity condition is given by

$$\text{(V)}\ \frac{2}{c} + k\left(d - \frac{a_0^2}{v_{00}}\right) > 0 \tag{3.7}$$

where $v_{00} = \dfrac{N}{1+(n-1)\rho}$ and $\dfrac{1}{c}$, d, a_0 are as in (3.6).

From (3.6) it is readily verified that if the design points are divided into 'm' sets of 'n' each having a SORD in the usual sense (i.e. when errors are uncorrelated) within each set, then the overall designs is SORDWCE under the inter-class structure (3.5), whatever be the value of intra-class correlation coefficient ρ.

Remark 3.1. It is not essential that the design points in each set should be SORD in the usual sense in order that the overall design become RSORD for all ρ under the inter-class structure (3.5) of errors.

It is now possible to construct designs satisfying (I) through (V) as in (3.6) and (3.7) but violating the conditions analogous to independently and identically distributed error structure relating to the individual group of observations. We provide an example below to highlight this point.

Example 3.1. Consider an experimental design consisting of three factors, four independent groups, each of eleven observations i.e. $k = 3, m = 4, n = 11$ and $D(e) = I_4 \otimes W_{11\times 11}(\rho)$, $N = mn = 44$.

Group I	1	2	3	4	5	6	7	8	9	10	11
x_1	1	–1	1	–1	1	–1	1	–1	2	0	0
x_2	1	–1	–1	1	1	–1	–1	1	0	2	0
x_3	1	–1	–1	–1	–1	1	1	1	0	0	2
Group II	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
x_1	1	–1	1	–1	1	–1	1	–1	2	0	0
x_2	1	–1	–1	1	1	–1	–1	1	0	–2	0
x_3	1	–1	–1	–1	–1	1	1	1	0	0	–2
Group III	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
x_1	1	–1	1	–1	1	–1	1	–1	–2	0	0
x_2	1	–1	–1	1	1	–1	–1	1	0	2	0
x_3	1	–1	–1	–1	–1	1	1	1	0	0	–2
Group IV	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
x_1	1	–1	1	–1	1	–1	1	–1	–2	0	0
x_2	1	–1	–1	1	1	–1	–1	1	0	–2	0
x_3	1	–1	–1	–1	–1	1	1	1	0	0	2

Group totals for x_i's are in the following table.

Group	I	II	III	IV
x_1	2	2	–2	–2
x_2	2	–2	2	–2
x_3	2	–2	–2	2

So the design points within a group do not satisfy the SORD conditions.

The condition (I) is readily satisfied.

In case of above example, the expressions (i), (ii) and (iii) in (II) simplify to

$$\frac{48}{(1+10\rho)}=a_0,\ \frac{48}{(1-\rho)}-\frac{16\rho}{(1-\rho)(1+10\rho)}=\frac{1}{e}$$

and $\frac{96}{(1-\rho)}-\frac{576\rho}{(1-\rho)(1+10\rho)}=\frac{2}{c}+d$ respectively and they are the same for $i=1,2,3$. (3.8)

Again, the conditions (i) and (ii) in (III) reduce to $\frac{32}{(1-\rho)}-\frac{576\rho}{(1-\rho)(1+10\rho)}=d$ and $\frac{32}{(1-\rho)}=\frac{1}{c}$ respectively and they are the same for $1\le i<j\le 3$. (3.9)

Therefore, (II) and (III) are satisfied.

Following (3.8) and (3.9), it is readily seen that (IV) is also satisfied.

Finally, non-singularity condition is verified by checking (3.5) in Das [3].

For the above example, the expression $\frac{2}{c}+k\left(d-\frac{a_0^2}{v_{00}}\right)$ simplifies to

$$\frac{64}{(1-\rho)}+3\left\{\frac{32}{(1-\rho)}-\frac{576\rho}{(1-\rho)(1+10\rho)}-\frac{(48)^2}{44(1+10\rho)}\right\}$$

$$=\frac{160}{(1-\rho)}-\frac{\rho(12)^3}{(1-\rho)(1+10\rho)}-\frac{1728}{11(1+10\rho)}=\frac{32}{11(1-\rho)}$$

which is positive.

Hence the non-singularity condition is also satisfied. Thus the overall design satisfies SORD conditions for all ρ.

3.3 Compound Symmetry Structure

In this section we turn our idea to a covariance structure of errors which is an extention of inter-class structure. It is termed as compound symmetry structure and is given by

$$W_0=\{D(e)=\sigma^2\,[I_m\otimes(A-B)+E_{m\times m}\otimes B]=W_{N\times N}\,(\rho,\rho_1),\text{ say}\} \quad (3.10)$$

where $A=(1-\rho)I_n+\rho E_{n\times n}$, $B=\rho_1 E_{n\times n}$, $N=mn$ and $\otimes$ denotes Kronecker product.

$$W_{N\times N}^{-1}\,(\rho,\rho_1)=(\sigma^2)^{-1}\,[I_m\otimes(A_1-B_1)+E_{m\times m}\otimes B_1]$$

where

$$A_1 = (\delta_2 - \gamma_2)I_n + \gamma_2 E_{n\times n}, B_1 = \delta_3 E_{n\times n}, \delta_2 = \gamma_2 + \frac{1}{(1-\rho)}$$

$$\gamma_2 = \frac{[(m-1)n\rho_1^2 - (m-2)n\rho\rho_1 - \{1+(n-1)\rho\}\rho]}{R}$$

$$\delta_3 = \frac{\{1-\delta_2 - (n-1)\rho\gamma_2\}}{(m-1)n\rho}$$

and $R = (1-\rho)\left[(m-2)n\rho_1\{1+(n-1)\rho\} + \{1+(n-1)\rho\}^2 - (m-1)n^2\rho_1^2\right]$ (3.11)

As with the inter-class correlation structure, it follows along the same line of arguments that even for a compound symmetry correlation structure (3.10), usual (i.e. errors are uncorrelated) second order rotatability within each set implies overall rotatability for all ρ and ρ_1. However, the converse is not necessarily true.

4. Tri-diagonal Correlation Structure

It is a covariance structure of errors which is a relaxation of intra-class structure or log-model co-variance structure of errors and is given by

$$W_0 = \left\{D(e) = \sigma^2\left[\begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix}\times\frac{1+\rho}{2} + \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix}\times\frac{1-\rho}{2}\right] = W_{2n\times 2n}(\rho), \text{say}\right\}$$

(4.1)

$$W_{2n\times 2n}^{-1}(\rho) = (\sigma^2)^{-1}\left[\begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix}\times\frac{1}{2(1+\rho)} + \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix}\times\frac{1}{2(1-\rho)}\right] \quad (4.2)$$

4.1 Conditions for Exact Rotatability

Following (3.3) in Das [3], necessary and sufficient conditions for second order rotatability under (4.1) simplify to

(I) (i) $v_{0.j} = \sum_{u=1}^{N} x_{ju} = 0$

$$v_{0.jl} = \sum_{u=1}^{N} x_{ju}x_{lu} = 0; 1 \le j < l \le k$$

$$\text{(ii)} \quad v_{i\,.\,j} = \sum_{u=1}^{N} x_{iu} x_{ju} - \rho \left\{ \sum_{u=1}^{n} x_{i(n+u)} x_{ju} + \sum_{u=1}^{n} x_{iu} x_{j(n+u)} \right\} = 0$$

$$1 \le i \ne j \le k$$

$$\text{(iii)} \quad (1) \quad v_{ii\,.\,j} = \sum_{u=1}^{N} x_{iu}^2 x_{ju} - \rho \left\{ \sum_{u=1}^{n} x_{i(n+u)}^2 x_{ju} + \sum_{u=1}^{n} x_{iu}^2 x_{j(n+u)} = 0 \right\}$$

$$1 \le i, j \le k$$

$$(2) \quad v_{ij\,.\,l} = \sum_{u=1}^{N} x_{iu} x_{ju} x_{lu} - \rho \left(\sum_{u=1}^{n} x_{i(n+u)} x_{j(n+u)} x_{lu} + \sum_{u=1}^{n} x_{iu} x_{ju} x_{l(n+u)} \right) = 0$$

$$1 \le i < j, l \le k$$

$$(3) \quad v_{ii\,.\,jl} = \sum_{u=1}^{N} x_{iu}^2 x_{ju} x_{lu} - \rho \left(\sum_{u=1}^{n} x_{i(n+u)}^2 x_{ju} x_{lu} + \sum_{u=1}^{n} x_{iu}^2 x_{j(n+u)} x_{l(n+u)} \right) = 0$$

$$1 \le i, j < l \le k; (j, l) \ne (i, i)$$

$$(4) \quad v_{ij\,.\,lt} = \sum_{u=1}^{N} x_{iu} x_{ju} x_{lu} x_{tu} - \rho \left(\sum_{u=1}^{n} x_{i(n+u)} x_{j(n+u)} x_{lu} x_{tu} + \sum_{u=1}^{n} x_{iu} x_{ju} x_{l(n+u)} x_{t(n+u)} \right) = 0$$

$$1 \le i, l < j, t \le k; (i, j) \ne (l, t)$$

$$\text{(II)} \quad \text{(i)} \quad v_{0\,.\,jj} = (1-\rho)\{\sigma^2(1-\rho^2)\}^{-1} \sum_{u=1}^{N} x_{ju}^2 = a_0; 1 \le j \le k$$

$$\text{(ii)} \quad v_{i\,.\,i} = \{\sigma^2(1-\rho^2)\}^{-1} \left[\sum_{u=1}^{N} x_{iu}^2 - 2\rho \sum_{u=1}^{n} x_{iu} x_{i(n+u)} \right] = \frac{1}{e}; 1 \le i \le k$$

$$\text{(iii)}\quad v_{ii.ii} = \{\sigma^2(1-\rho^2)\}^{-1}\left[\sum_{u=1}^{N} x_{iu}^4 - 2\rho\sum_{u=1}^{n} x_{iu}^2 x_{i(n+u)}^2\right]$$

$$= \left(\frac{2}{c} + d\right); 1 \le i \le k$$

$$\text{(III) (i)}\quad v_{ii.jj} = \{\sigma^2(1-\rho^2)\}^{-1}\left[\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 - \rho\left(\sum_{u=1}^{n} x_{i(n+u)}^2 x_{ju}^2 + \sum_{u=1}^{u} x_{iu}^2 x_{j(n+u)}^2\right)\right] = d;\ 1 \le i \ne j \le k$$

$$\text{(ii)}\quad v_{ij.ij} = \{\sigma^2(1-\rho^2)\}^{-1}\left[\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 - 2\rho\sum_{u=1}^{n} x_{iu} x_{ju} x_{i(n+u)} x_{j(n+u)}\right] = \frac{1}{c};\ 1 \le i < j \le k$$

$$\text{(IV)}\qquad v_{ii.ii} = 2v_{ij.ij} + v_{ii.jj}; 1 \le i < j \le k \tag{4.3}$$

where $N = 2n$, $v_{ii.ii}$, $v_{ii.jj}$ and $v_{ij.ij}$ are as in (II) (iii) and (III) (i), (ii).

Non - singularity condition is given by

$$\text{(V)}\ \frac{2}{c} + k\left(d - \frac{a_0^2}{v_{00}}\right) > 0 \tag{4.4}$$

where $v_{00} = \dfrac{N}{\sigma^2(1+\rho)}$, $N = 2n$ and $\dfrac{1}{c}$, d, a_0 are as in (4.3).

4.2 Method of Construction of Robust SORD

In this sub section we discuss two methods of construction of robust SORD under the tri-diagonal structure of the errors. The designs which are obtained by these methods satisfy the moment conditions given in (4.3).

Method 1 : We start with a usual SORD (i.e., those with uncorrelated errors) having 'n' design points involving k-factors. The set of 'n' design points can be extended to '2n' points by incorporating 'n' central points together just below or above the 'n' design points of the original design with which we started.

Features of the Method

(a) It can be readily verified that (4.3) holds for this design.

(b) To ensure non-singularity, we first examine the original design and the newly constructed design. For the original design with which we started, the following are the moment relations

$$\sum_{u=1}^{n} x_{iu}^2 = n\lambda_2;\ \sum_{u=1}^{n} x_{iu}^4 = 3n\lambda_4;\ \sum_{u=1}^{n} x_{iu}^2 x_{ju}^2 = n\lambda_4;\ 1 \le i \ne j \le k \qquad (4.5)$$

Using (4.5), for the newly constructed design, we get the following

$$\sum_{u=1}^{2n} x_{iu}^2 = n\lambda_2;\ \sum_{u=1}^{2n} x_{iu}^4 = 3n\lambda_4;\ \sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2 = n\lambda_4;\ 1 \le i \ne j \le k \qquad (4.6)$$

Using (4.3) and (4.6), the design parameters of the newly constructed designs are the following

$$a_0 = \frac{(1-\rho)n\lambda_2}{\sigma^2(1-\rho^2)},\ d = \frac{n\lambda_4}{\sigma^2(1-\rho^2)},\ \frac{1}{e} = \frac{n\lambda_2}{\sigma^2(1-\rho^2)},\ \frac{1}{c} = \frac{n\lambda_4}{\sigma^2(1-\rho^2)} \qquad (4.7)$$

Using (4.7) and noting $v_{00} = \frac{2n(1-\rho)}{\sigma^2(1-\rho^2)}$, the expression $\frac{2}{c} + k\left(d - \frac{a_0^2}{v_{00}}\right)$ simplifies to $\frac{n}{\sigma^2(1-\rho^2)}\left[(k+2)\lambda_4 - \frac{k\lambda_2^2(1-\rho)}{2}\right]$. Hence the non-singularity condition of the above design is

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}\frac{(1-\rho)}{2} \qquad (4.8)$$

It is readily seen that $0 \le \frac{(1-\rho)}{2} \le 1, -1 \le \rho \le 1$. The design, to start with, is a SORD in the usual sense so that $\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$. This condition indeed does satisfy the revised condition (4.8) derived here for all values of ρ. Thus the designs described in Method I satisfies non-singularity condition for all values of ρ. Therefore, the designs obtained by following Method I are robust SORD, under the error structure (4.1).

Method II : We start with a usual SORD i.e those with uncorrelated errors, having 'n' design points involving k factors. Let $(x_1, x_2, \ldots, x_k)$ be a usual

SORD of k factors with 'n' design points, where x_i is i-th explanatory variable or factor and x_i is a vector of order $n \times 1, 1 \le i \le k$. The set of 'n' design points can be extended to '2n' points by repeating i-th factor column just below itself, i.e., $(x_i', x_i')' = y_i$, say; $1 \le i \le k$. Now y_i is a vector of order $2n \times 1$ and $(y_1, y_2, \ldots, y_k)$ form a robust SORD under tri-diagonal structure of errors.

Features of the Method

(a) It can be readily verified that (4.3) holds for these designs.

(b) To ensure non-singularity condition, we first examine the original design and the newly constructed design. For the original design with which we started, the following are the moment relations

$$\sum_{u=1}^{n} x_{iu}^2 = n\lambda_2;\ \sum_{u=1}^{n} x_{iu}^4 = 3n\lambda_4;\ \sum_{u=1}^{n} x_{iu}^2 x_{ju}^2 = n\lambda_4; 1 \le i \ne j \le k \qquad (4.9)$$

Using (4.9), for the newly constructed design, we get the following

$$\sum_{u=1}^{2n} y_{iu}^2 = 2n\lambda_2;\ \sum_{u=1}^{2n} y_{iu}^4 = 6n\lambda_4;\ \sum_{u=1}^{2n} y_{iu}^2 y_{ju}^2 = 2n\lambda_4;\ 1 \le i \ne j \le k \qquad (4.10)$$

Using (4.3) and (4.10), the design parameters of the newly constructed designs are the following

$$a_0 = \frac{(1-\rho)2n\lambda_2}{\sigma^2(1-\rho^2)},\ d = \frac{2n\lambda_4}{\sigma^2(1-\rho^2)},\ \frac{1}{e} = \frac{2n\lambda_2}{\sigma^2(1-\rho^2)},\ \frac{1}{c} = \frac{2n\lambda_4}{\sigma^2(1-\rho^2)} \qquad (4.11)$$

Using (4.11) and noting $v_{00} = \dfrac{2n(1-\rho)}{\sigma^2(1-\rho^2)}$, the expression $\dfrac{2}{c} + k\left(d - \dfrac{a_0^2}{v_{00}}\right)$ simplifies to $\dfrac{2n}{\sigma^2(1-\rho^2)}\left[(k+2)\lambda_4 - k\lambda_2^2\right]$.

Hence the non-singularity condition of the above design is

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2} \qquad (4.12)$$

which is the non-singularity condition of the usual SORD to start with.

Hence the designs described in Method II satisfies non-singularity condition for all values of ρ. Therefore, the designs obtained by following Method II are robust SORD, under the error structure (4.1).

Note that variance function of robust SORD, obtained by following Method I and Method II can be easily obtained by following (3.7) in Das [3] and noting their corresponding design parameters.

Remark 4.1. The RSORD under tri-diagonal structure thus constructed above are not invariant under some permutations of the design points with respect to exact robust rotatability.

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