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Robustness of Nested Balanced Incomplete Block Designs Against Missing Data

Lalmohan Bhar and Amitava Dey
Indian Agricultural Statistics Research Institute, New Delhi-110012
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SUMMARY

Robustness aspects of nested balanced incomplete block designs against missing data have been investigated using connectedness and efficiency criteria. Sufficient condition for robustness of a design has been obtained for the loss of any m observations, using connectedness criterion. Designs robust against the loss of any m observations belonging to one sub-block, loss of any two observations belong to two different sub-blocks in a block or in different blocks and loss of any t observations in a block have been identified.

Key words : Connectedness, Efficiency, Nested block designs.

1. Introduction

Blocking is the technique used to bring about homogeneity of experimental units within a block, so that the treatment contrasts are estimated, making use of the intra-block information, with higher efficiency. In many field and laboratory experiments the experimental units or conditions differ due to several factors which influence the response under study. It might not always be possible to remove such heterogeneity in response due to the factors other than treatments by blocking alone. There are the experimental situations in which one or more factors are nested within the blocking factor. In such situations nested block designs can be adopted. For example, consider the following experiment quoted by Preece [16]. Suppose the half-leaves of a plant form the experimental units, on which a number of treatments, say, inoculations with sap from tobacco plants infected with Tobacco Necrosis Virus, are to be applied. Suppose the number of treatments is more than the number of suitable half-leaves per plant. Now, there is one source of variation present due to the variability among plants. Further, leaves within a plant may exhibit variation between themselves due to their being located on the upper branch, middle branch or on the lower branch of the same plant. Thus leaves within plants form a nested factor, being nested within plants. The half-leaves being experimental units, we then have two systems of 'blocks', leaves (may be called sub-blocks), being nested within plants (may be called blocks).

Kleczkowski [11] reported an experiment for a biological experiment on the effect of inoculating plants with virus. This design is actually based on a resolved nested balanced incomplete block design derived later by Preece [16]. Further use of this design was reported in 1965 by Kassanis and Kleczkowski [10]. This experimental background led to Preece's [16] statistical paper where nested balanced incomplete block designs (NBIBD) were defined for the first time and an incomplete table of them was given. Another experimental situation in hilly areas is reported by Parsad *et al.* [15]. Satpati and Parsad [18] reported some experimental situations both for field and laboratory conditions, where nested block designs can be used. For an excellent review of the subject, one may refer to Morgan *et al.* [13]. The paper by Morgan *et al.* [13] contains a catalogue of NBIB designs.

Statistical procedures followed for these designs, for making inductive inferences are based on ideal conditions. However, aberrations may occur due to some causes during experimentation. Loss of observations is one such aberration. A nested block design may become disconnected due to loss of observation(s). Therefore, there is a need to look for the designs that remain connected even after the loss of some observations. Ghosh ([4], [5], [6] and [7]) introduced a robustness property of designs against unavailability of any t (a positive integer) observations. Following Ghosh, a block design is termed as robust against the loss of observation(s) if the resulting design obtained after loss of observation(s) remains connected. Here after we call this criterion as *Criterion 1*.

Even if the design remains connected, the resulting design may become inefficient in the sense that some of the contrasts may be estimated with less precision. Hence, there is a need to examine the efficiency of the resulting design relative to the original design. Ghosh [7] studied the information contained in each observation in a given design that is robust against the loss of a single observation as per *Criterion 1*. If in an experiment, a most informative observation is unavailable, the loss will be more compared to a situation where a least informative observation is unavailable. Another criterion to check the efficiency of the resulting design, is in terms of A-efficiency of the design. A connected design is said to be robust against loss of observation(s) if the A-efficiency of the resulting design as compared to the original design is not too small. We call this criterion as *Criterion 2*.

Robustness of designed experiments against missing data has been studied extensively in the literature. For a review of the subject references may be made to Dey [2], Kageyama [9] and Lal *et al.* [12]. Robustness of incomplete block designs against missing data has been investigated from different angles e.g. Baksalary and Tabis [1], Dey [2], Dey *et al.* [3], Ghosh ([4], [5], [6] and [7]), Hedayat and John [8], Lal *et al.* [12] and so on. However, no work on this aspect seems to be available for another important class of designs, nested designs. The present article attempts to obtain some results on robustness for Nested

Balanced Block (NBB) designs. In Section 2, we obtain the sufficient conditions for robustness in general for nested block designs and in Section 3 results are applied to Nested Balanced Incomplete Block (NBIB) designs reported in Morgan *et al.* [13].

Denote an n-component vector of ones by $\mathbf{1}_n$, an identity matrix of order n by $\mathbf{I}_n$, and an m × n matrix of ones by $\mathbf{J}_{m\times n}$, with $\mathbf{J}_{m\times m}$ simply denoted by $\mathbf{J}_m$. Further $\mathbf{A}'$, $\mathbf{A}^-$ and $\mathbf{A}^+$ respectively denote the transpose, a generalized inverse (g-inverse) and the Moore-Penrose inverse of a matrix **A**.

For studying robustness, the following two theorems given by Dey [2] are useful (see also Dey *et al.* [3]).

Theorem 1. Let **A** and **B** be a pair of symmetric, nonnegative definite matrices of order n and let

$$\mathbf{A} = \mathbf{B} + \mathbf{G}\mathbf{G}'$$

where **G** is an n × m matrix. Then Rank (**A**) = Rank (**B**) if and only if $\mathbf{I}_m - \mathbf{G}'\mathbf{A}^-\mathbf{G}$ is positive definite.

Theorem 2. Let **A, B, G** be as in Theorem 1 and suppose that

$$\mathbf{I}_m - \mathbf{G}'\mathbf{A}^-\mathbf{G} = \mathbf{I}_m - \mathbf{G}'\mathbf{A}^+\mathbf{G}$$

is positive definite. Then

$$\mathbf{B}^+ = \mathbf{A}^+ + \mathbf{A}^+\mathbf{G}\left(\mathbf{I} - \mathbf{G}'\mathbf{A}^+\mathbf{G}\right)^{-1}\mathbf{G}'\mathbf{A}^+$$

2. *Conditions for Robustness*

Consider an experiment involving v treatments in a nested block design (d, say) with b blocks, there being q_j mutually exclusive and exhaustive sub-blocks in the j^{th} block, j = 1, 2, …, b; so that $b_1 = \sum_{j=1}^{b} q_j$ is the overall total number of sub-blocks. Let $\mathbf{N} = (n_{ij})$ be the $v\times b$ treatments-blocks incidence matrix, where n_{ij} denotes the number of replications of the i^{th} treatment in the j^{th} block, $i = 1, 2, \ldots, v$. The row sums of **N** are denoted by $\mathbf{r} = (r_1, r_2, \ldots, r_v)'$ and the column sums by $\mathbf{k} = (k_1, k_2, \ldots, k_b)'$, where r_i and k_j denote respectively the replication number of the i^{th} treatment and the size of the j^{th} block. Also $r_1 + \ldots + r_v = k_1 + \ldots + k_b = n$, the total number of experimental units. Let $\mathbf{M} = (m_{ij'(j)})$ denote the $v\times b_1$ treatments sub-blocks incidence matrix, where $m_{ij'(j)}$ denotes the replication number of the i^{th} treatment in the j'^{th} sub-block nested within the j^{th} block, $j' = 1, 2, \ldots, q_j$. The row sums of $\mathbf{M}$ are the elements of $\mathbf{r}$ while its column sums are the elements of the $b_1 \times 1$ vector $\mathbf{h} = (\mathbf{h}'_{(1)}, \ldots, \mathbf{h}'_{(b)})'$, where $\mathbf{h}'_{(j)} = (h_{1(j)}, \ldots, h_{q_j(j)})$, $\mathbf{1}'\mathbf{h}_{(j)} = k_j$. Here $h_{j'(j)}$ denotes the size of the j'^{th} sub-block nested in the j^{th} block. Let $\mathbf{R}$, $\mathbf{H}_j$, and $\mathbf{H}$ denote respectively the diagonal matrices whose diagonal elements are the successive elements of $\mathbf{r}$, $\mathbf{h}_j$ and $\mathbf{h}$. Let $\mathbf{W}$ be the $b \times b_1$ blocks - sub-blocks incidence matrix.

The model for the data can be written as

$$\mathbf{y} = \mu \mathbf{1}_n + \boldsymbol{\Delta}'\boldsymbol{\tau} + \mathbf{D}'\boldsymbol{\beta} + \boldsymbol{\Phi}'\boldsymbol{\eta} + \boldsymbol{\varepsilon} \tag{2.1}$$

where, $\boldsymbol{\tau}$ is v-component vector of treatment effects, $\boldsymbol{\beta}$ is b-component vector of block effects, $\boldsymbol{\eta}$ is b_1- component vector of sub-block effects, $\boldsymbol{\Delta}'$ is the $n \times v$ design matrix for treatments, $\mathbf{D}'$ is $n \times b$ design matrix for blocks, $\boldsymbol{\Phi}'$ is $n \times b_1$ design matrix for sub-blocks and $\boldsymbol{\varepsilon}$ denotes the vector of independent random errors with zero expectation and constant variance σ^2.

The information matrix $\mathbf{C}$ for estimating the treatment effects after eliminating the effects of other nuisance parameters, can be written as

$$\mathbf{C} = \mathbf{R} - \mathbf{M}\mathbf{H}^{-1}\mathbf{M}' \tag{2.2}$$

Now we order the *n* observations, such that the first k_1 observations come from the first block, the 2^{nd} k_2 observations come from the 2^{nd} block and so on. Again out of k_j observations first $h_{1(j)}$ observations come from the 1^{st} sub-block of the j^{th} block, 2^{nd} $h_{2(j)}$ observations come from the 2^{nd} sub-block of the j^{th} block and so on, j = 1, 2, ..., b; then we have $\mathbf{D}' = [\mathbf{D}_1 \ \ \mathbf{D}_2 \ \ldots \ \mathbf{D}_b]'$, where $\mathbf{D}'_j$ is a $k_j \times b$ matrix with j^{th} column of all unities and others of zero, $\mathbf{D}'\mathbf{D} = \text{diag}(k_1, \ldots, k_b)$, $\boldsymbol{\Phi}' = [\boldsymbol{\Phi}_{1(1)}, \boldsymbol{\Phi}_{2(1)}, \ldots, \boldsymbol{\Phi}_{q_1(1)}, \boldsymbol{\Phi}_{1(2)}, \ldots, \boldsymbol{\Phi}_{q_b(b)}]'$ where $\boldsymbol{\Phi}_{j'(j)}$ is a $h_{j'(j)} \times b_1$ design matrix for sub-block effects for the j'^{th} sub-block nested in j^{th} block; $j' = 1, 2, \ldots, q_j$ and $\boldsymbol{\Phi}'\boldsymbol{\Phi} = \text{diag}\ (h_{1(1)}, h_{2(1)}, \ldots, h_{q_1(1)}, \ldots, h_{q_b(b)})$. Similarly we partitioned $\boldsymbol{\Delta}'$ as $\boldsymbol{\Delta}' = (\boldsymbol{\Delta}_{1(1)}, \boldsymbol{\Delta}_{2(1)}, \ldots, \boldsymbol{\Delta}_{q_1(1)}, \ldots, \boldsymbol{\Delta}_{q_b(b)})'$ where $\boldsymbol{\Delta}_{j'(j)}$ is a $h_{j'(j)} \times v$ (0 – 1) design matrix for treatment effects for j'^{th} sub block in the j^{th} block. We also use the following notations

$$\boldsymbol{\Gamma} = (\mathbf{I} - \boldsymbol{\Phi}'\mathbf{H}^{-1}\boldsymbol{\Phi}) = \text{diag}\ (\boldsymbol{\Gamma}_{1(1)}, \ldots, \boldsymbol{\Gamma}_{q_b(b)}) \tag{2.3}$$

where $\Gamma_{j'(j)} = \left(\mathbf{I}_{h_{j'(j)}} - \frac{1}{h_{j'(j)}} \mathbf{J}_{h_{j'(j)}} \right)$; $j = 1, \ldots, b,\ j' = 1, 2, \ldots, q_j$

Then **C**-matrix given in (2.2) can be written as

$$\mathbf{C} = \Delta\, \Gamma \Delta'$$

Suppose that any m observations are lost from the design. Without any loss of generality we assume that $m_{j'(j)}$ (≥ 0) of these lost observations arise from $j'^{\,th}$ sub-block nested within j^{th} block such that

$$\sum_{j=1}^{b} \sum_{j'=1}^{q_j} m_{j'(j)} = m$$

We now define a $h_{j'(j)} \times h_{j'(j)}$ matrix $\mathbf{U}_{j'(j)}$ such that its u^{th} diagonal element is 1 if u^{th} observation is lost from the $j'^{\,th}$ sub-block nested within the j^{th} block, the other elements of $\mathbf{U}_{j'(j)}$ are zero, $u = 1, 2, \ldots, h_{j'(j)}$; $j' = 1, \ldots, q_j$; $j = 1, \ldots, b$. We also define $\mathbf{U}$ = diag $(\mathbf{U}_{1(1)}, \mathbf{U}_{2(1)}, \ldots, \mathbf{U}_{q_1(1)}, \ldots, \mathbf{U}_{1(b)}, \mathbf{U}_{2(b)}, \ldots, \mathbf{U}_{q_b(b)})$ and $\mathbf{A} = \mathbf{I}_n - \mathbf{U}$. The matrix **A** is symmetric and idempotent. If we denote the information matrix of the residual design by $\mathbf{C}_*$, then

$$\mathbf{C}_* = \Delta \mathbf{A} \Delta' - \Delta \mathbf{A} \mathbf{X} (\mathbf{X}' \mathbf{A} \mathbf{X})^{-} \mathbf{X}' \mathbf{A} \Delta'$$

where, $\mathbf{X} = [\mathbf{1}_n \quad \mathbf{D}' \quad \Phi']$

On simplification we get

$$\mathbf{C}_* = \mathbf{C} - \mathbf{V}' \mathbf{C}_0^{-} \mathbf{V}$$

where $\mathbf{V} = \Delta \Gamma \mathbf{U}$ and $\mathbf{C}_0 = \mathbf{U} \Gamma \mathbf{U}$

Recently Lal *et al.* [12] arrive at the similar structure of $\mathbf{C}_*$ while dealing with missing observations in block designs.

Theorem 3. The design *d* is *Criterion 1* robust against the loss of any m observations if and only if the smallest positive eigenvalue of **C** is strictly greater than the largest eigen value of $\mathbf{V}' \mathbf{C}_0^{-} \mathbf{V}$.

Proof: The proof follows from Theorem 1 and noting that $\mathbf{C}_0^{-}$ admits a unique Gramian root (see also Lal *et al.* [12]).

Once it is known that a design is *Criterion 1* robust , then it is of interest to examine the efficiency of residual design relative to original design and to decide robustness on the basis of *Criterion 2*. The efficiency of residual design with respect to original design is given by (Mukerjee and Kageyama [14])

$$E = \frac{\text{Sum of reciprocals of non - zero eigen value of } \mathbf{C}}{\text{Sum of reciprocals of non - zero eigen value of } \mathbf{C}_*} = \frac{\mathrm{tr}\left(\mathbf{C}^+\right)}{\mathrm{tr}\left(\mathbf{C}_*^+\right)}$$

Now from Theorem 2 we have

$$\mathrm{tr}(\mathbf{C}_*^+) = \mathrm{tr}(\mathbf{C}^+) + \mathrm{tr}\left(\mathbf{C}^+\mathbf{H}\left(\mathbf{I}_n - \mathbf{H}'\mathbf{C}^+\mathbf{H}\right)^{-1}\mathbf{H}'\mathbf{C}^+\right)$$

$$= \mathrm{tr}(\mathbf{C}^+) + g \tag{2.5}$$

where

$$g = \mathrm{tr}\left(\left(\mathbf{I}_n - \mathbf{H}'\mathbf{C}^+\mathbf{H}\right)^{-1}\mathbf{H}'\mathbf{C}^+\mathbf{C}^+\mathbf{H}\right)$$

For a balanced design $\mathbf{C}^+ = \frac{1}{\theta}\mathbf{I}_v$

If λ_i, $i = 1, 2, \ldots, m_0\ (\le m)$ are the m_0 non-zero eigenvalues of $\mathbf{H}'\mathbf{H}$, then the efficiency is

$$E = \left(1 + \frac{g_0}{v-1}\right)^{-1} \tag{2.6}$$

where $g_0 = \sum_{i=1}^{m_0} \frac{\lambda_i}{(\theta - \lambda_i)}$

3. Applications

In this section we apply the conditions obtained under Section 2 to NBIBD. For a NBIBD, all block sizes are equal (say k), all sub-block sizes are equal (say h) and each block contains equal number of sub-blocks (say q). For completeness, we recall the definition of a nested balanced incomplete block design of Preece [16].

Definition. A nested balanced incomplete block design with parameters ($v, b, k, r^*, \lambda, b_1, h, \lambda_1, q$) is a design for v treatments, each replicated r^* times with two systems of blocks such that :

(a) the second system is nested within the first, with each block from the first system, called henceforth a 'block', containing exactly q blocks from the second system, called hereafter as 'sub-blocks'

(b) ignoring the second system leaves a balanced incomplete block design with the usual parameters v, b, k, r^*, λ

(c) ignoring the first system leaves a balanced incomplete block design with parameters v, b_1, h, r^*, λ_1

3.1 Robustness of NBIBD when any *m* Observations Belonging to One Sub-Block are Lost

Suppose that m observations belonging to first sub-block of the first block of the design d are lost. Without loss of generality, we assume that between these two sub-blocks first α treatments are common. Then it is easy to see that

$$\mathbf{C}_* = \mathbf{C} - \mathbf{\Delta}_1\mathbf{\Gamma}_0\mathbf{U}_1(\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{U}_1)^{-}\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{\Delta}_1'$$

where $\mathbf{\Gamma}_0 = \mathbf{I}_n - \frac{1}{n}\mathbf{J}_n$, obtained from (2.3) and $\mathbf{\Delta}_{1(1)}$ and $\mathbf{U}_{1(1)}$ are written as $\mathbf{\Delta}_1$ and $\mathbf{U}_1$ respectively.

Since $\mathbf{\Delta}_1'\mathbf{\Delta}_1 = \mathbf{I}_h$, we have

$\mathbf{C}_* = \mathbf{C} - \mathbf{G}\mathbf{G}'$, where $\mathbf{G} = \mathbf{\Delta}_1\mathbf{\Gamma}_0\mathbf{U}_1(\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{U}_1)^{-}\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{\Delta}_1'$

We then obtain the following sufficient condition for robustness of d

Theorem 4. A NBIB design is robust against the loss of any m $(1 \leq m \leq h)$ observations in a sub-block, if the smallest positive eigenvalue of $\mathbf{C}$ is strictly larger than 1.

Proof. From Theorem 1, it follows that design d is robust as per *Criterion 1* against the loss of any m observations belonging to the same sub-block, if and only if $\mathbf{I}_v - \mathbf{G}'\mathbf{C}^{-}\mathbf{G}$ is positive definite, or equivalently, if and only if all the eigenvalues of $\mathbf{G}'\mathbf{C}^{-}\mathbf{G}$ are strictly smaller than unity. Let $\lambda_{max}(\mathbf{A})$ denote the largest eigenvalue of a symmetric nonnegative definite matrix $\mathbf{A}$. Then, $\lambda_{max}(\mathbf{G}'\mathbf{C}^{-}\mathbf{G}) = \lambda_{max}(\mathbf{G}'\mathbf{C}^{+}\mathbf{G}) = \lambda_{max}(\mathbf{C}^{+}\mathbf{G}\mathbf{G}')$

It is known that for a pair of symmetric non-negative matrices **A** and **B**

$$\lambda_{max}(\mathbf{AB}) \leq \lambda_{max}(\mathbf{A})\lambda_{max}(\mathbf{B}) \tag{3.1}$$

Hence, $\lambda_{max}(\mathbf{G}'\mathbf{C}^{+}\mathbf{G}) \leq \lambda_{max}(\mathbf{C}^{+})\lambda_{max}(\mathbf{G}\mathbf{G}')$ (3.2)

Now $\mathbf{G}\mathbf{G}' = \mathbf{\Delta}_1\mathbf{\Gamma}_0\mathbf{U}_1(\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{U}_1)^{-}\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{\Delta}_1'$

Since for two matrices **A** and **B**, the eigenvalues of **AB** and **BA** are the same, we calculate the eigenvalues of $(\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{U}_1)^{-}\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{\Delta}_1'\mathbf{\Delta}_1\mathbf{\Gamma}_0\mathbf{U}_1$. On simplification

$$(\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{U}_1)^{-}\mathbf{U}_1\mathbf{\Gamma}_0\mathbf{\Delta}_1'\mathbf{\Delta}_1\mathbf{\Gamma}_0\mathbf{U}_1 = \begin{bmatrix}\mathbf{I}_m & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \text{ if } 1 \leq m \leq h-1$$

$$= \mathbf{\Gamma}_0 \qquad \text{if } m = h$$

Thus the non zero eigenvalues of $\mathbf{GG'}$ are 1 with multiplicity m, if $1 \le m \le h-1$ and 1 with multiplicity $m-1$ if $m = h$. Hence from (3.2) we have

$$\lambda_{max}(\mathbf{G'C^+G}) \le \lambda_{max}(\mathbf{C^+}) \quad (3.3)$$

Using (3.3) and remembering that $\lambda_{max}(\mathbf{C^+}) = \{\lambda_{min}(\mathbf{C})\}^{-1}$, we get the required result, where λ_{min} is the smallest positive eigenvalue of **C**.

From Theorem 4 and noting that $\theta > 1$ for all NBIB designs, we get the following result.

Corollary 1. All NBIB designs reported in Morgan *et al.* [13] satisfy the sufficient condition of Theorem 4 and are thus robust against the loss of any m ($1 \le m \le h$) observations in a sub-block.

Now we examine the efficiency of the residual design and decide robustness on the basis of *Criterion 2.*

From (2.6) and using the eigenvalues of $\mathbf{GG'}$, we get the efficiency of the residual design E, as a function of m, to be

$$E(m) = \left[1 + \frac{m}{(v-1)(\theta-1)}\right]^{-1} \quad \text{if } 1 \le m \le h-1 \quad (3.4)$$

$$= \left[1 + \frac{h-1}{(v-1)(\theta-1)}\right]^{-1} \quad \text{if } m = h \quad (3.5)$$

Note that E(m) is a decreasing function of m, we therefore calculate E(m) for m = h, i.e., efficiency for the loss of all observations in a sub-block. We have calculated the efficiencies for all NBIB designs reported in Morgan *et al.* [13], and found that the efficiency is always greater than 0.90 for all the designs except one corresponding to serial number 1, for which it is 0.85. Therefore, all these designs are fairly robust as per *Criterion 2* against the loss of all the observations in a sub-block.

3.2. Robustness of Designs when any Two Observations are Lost

When any two observations are lost, following cases may arise:

Case(i) Two observations belong to the same sub-block.

This is a particular case of Section 3.1 when $m = 2$.

Case(ii) Two observations belong to two different sub-blocks.

Under this case again we get different sub-cases.

Case(ii)(a) Two sub-blocks belong to the same block.

Without loss of generality we assume that the first observations in the first sub-block and the first observations in the 2nd sub-block nested within the 1st block are lost. Then on simplification

$$\mathbf{C}_* = \mathbf{C} - \mathbf{V}\mathbf{V}', \text{ where } \mathbf{V} = \sqrt{\frac{k}{k-1}}\left[\Delta_1\Gamma_0\mathbf{U}_1 \qquad \Delta_2\Gamma_0\mathbf{U}_2\right]$$

Here $\Delta_{1(1)}$, $\Delta_{2(1)}$, $\mathbf{U}_{1(1)}$ and $\mathbf{U}_{2(1)}$ are written as Δ_1, Δ_2, $\mathbf{U}_1$ and $\mathbf{U}_2$ respectively.

For robustness study we need the eigenvalues of $\mathbf{V}\mathbf{V}'$ or $\mathbf{V}'\mathbf{V}$. Since, the two sub-blocks are disjoint, on simplification we get

$$\mathbf{V}'\mathbf{V} = \mathbf{I}_2 \otimes \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The non-zero eigen values of $\mathbf{V}'\mathbf{V}$ are 1 with multiplicity 2. If we denote the efficiencies for the present case by $E_{sb}(2)$, then from (2.6) we get

$$E_{sb}(2) = \left[1 + \frac{2}{(v-1)(\theta-1)}\right]^{-1}$$

Case(ii)(b) Two sub-blocks nested in different blocks

Without loss of generality, we assume that the first observations in the first sub-block nested in the first block and the first observations in the first sub-block nested within the second block are lost. Further two missing observations pertain to first two treatments and between these two sub-blocks α treatments are common. Then on simplification, we get

$$\mathbf{V}'\mathbf{V} = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

where the value of c depends on the configuration of missing observations.

Now we give a sufficient condition for the design d to be *Criterion 1* robust.

Theorem 5. A NBIB design is robust as per *Criterion 1* against the loss of any 2 observations if the smallest positive eigenvalue of $\mathbf{C}$ is strictly greater than 2.

Proof: From (3.1), we get

$$\lambda_{\max}(\mathbf{V}'\mathbf{C}^+\mathbf{V}) \le \lambda_{\max}(\mathbf{C}^+)\lambda_{\max}(\mathbf{V}\mathbf{V}') < \lambda_{\max}(\mathbf{C}^+)\,\mathrm{tr}(\mathbf{V}\mathbf{V}')$$

Now, $\mathrm{tr}(\mathbf{V}'\mathbf{V}) = 2$, hence the result, by noting that in other cases the maximum eigenvalue of $\mathbf{V}'\mathbf{V}$ is 1.

Corollary 2. All designs reported in Morgan *et al.* [13] satisfy the sufficient condition of Theorem 5 and are thus robust against the loss of any two observations.

Now we consider *Criterion 2* robustness against the loss of any two observations. The value of c is calculated for all possible cases of missing observations and are presented in Table 1.

Table 1. The value of c

Case	c
(ii) (b) (i) Two missing observations pertain to the same treatment	$\frac{h^2-2h+\alpha}{h(h-1)}$
(ii) (b) (ii) Two missing observations pertain to two different treatments and among these two treatments x are common in both the sub-blocks; x = 0, 1, 2	$\frac{\alpha-xh}{h(h-1)}$

If we denote the efficiency for the present case by $E_{db}(2)$, then from (2.6) and using the eigenvalues we get

$$E_{db}(2) = \left\{1+\frac{1}{v-1}\left[\frac{1+c}{\theta-1-c}+\frac{1-c}{\theta-1+c}\right]\right\}^{-1}$$

$$= \left\{1+\frac{2}{v-1}\left[\frac{\theta(\theta-1)}{(\theta-1)^2-c^2}-1\right]\right\}^{-1} \qquad (3.6)$$

For $\theta > 1$, from (3.6) it can easily be seen that E is monotonically decreasing function of c. Again the value of c for the *Case(ii) (b)(i)* is the largest for a fixed value of α, and c is monotonically increasing function of α. Therefore efficiency for the *Case (ii)(b)(i)* is the minimum among all the cases . We calculate the efficiency for the *Case (ii)(b)(i)* only . If the design is robust for the *Case (ii)(b)(i)*, it will be robust against the loss of any two observations in the design. The efficiency for the *Case (ii)(b)(i)* has been calculated for all the designs reported in Morgan *et al.* [13]. The efficiency for the designs corresponding to serial numbers 5 to 68 is greater than 0.90, where as it is 0.81, 0.88 and 0.87 for the designs corresponding to serial numbers 2, 3 and 4 respectively. The efficiency is only 0.50 for the design corresponding to serial number 1. Thus all these designs, except one corresponding to serial number 1 are fairly robust against the loss of any two observations.

3.3. Robustness of Designs when any t Observations in a Block are Lost

Suppose t $(1 \le t \le k)$ observations belonging to 1^{st} block of the design d are lost. Without loss of generality we assume that $t_{j'}$ observations belong to the j'^{th} sub-block, $j'=1, \ldots, q$, such that $\sum_{j'=1}^{q} t_{j'} = t$. Then $\mathbf{C}_* = \mathbf{C} - \mathbf{V}\mathbf{C}_0^{-}\mathbf{V}'$

For the present case $\mathbf{V} = \left[\Delta_1\Gamma_0\mathbf{U}_1 \quad \Delta_2\Gamma_0\mathbf{U}_2 \quad \ldots \quad \Delta_q\Gamma_0\mathbf{U}_q\right]$ and

$$\mathbf{C}_0 = \text{diag}\left(\mathbf{U}_1\Gamma_0\mathbf{U}_1, \ \ldots \ , \mathbf{U}_q\Gamma_0\mathbf{U}_q\right)$$

Here also $\Delta_{j'(j)}$ and $\mathbf{U}_{j'(j)}$ are written as $\Delta_{j'}$ and $\mathbf{U}_{j'}$ respectively.

For robustness study we need the eigenvalues of $\mathbf{V}\mathbf{C}_0^{-}\mathbf{V}'$ or $\mathbf{C}_0^{-}\mathbf{V}\mathbf{V}'$. On simplification

$$\mathbf{C}_0^{-}\mathbf{V}\mathbf{V}' = \mathbf{I}_q \otimes \mathbf{S}$$

where
$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_{t_{j'}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ if } \quad 1 \le t_{j'} \le h-1$$
$$= \Gamma_0 \qquad \text{if} \quad t_{j'} = h$$

Thus the non-zero eigen values of $\mathbf{C}_0^{-}\mathbf{V}\mathbf{V}'$ are 1 with multiplicity $\Sigma\left(t_{j'} - \delta_{j'}\right)$

where $\delta_{j'} = 1$ if $t_{j'} = h$

$= 0$ if $1 \le t_{j'} \le h-1$

Now using Theorem 3, we get the following result.

Theorem 6. The design *d* is robust against the loss of any t observations belonging to the same block if the smallest positive eigenvalue of **C** is strictly larger than unity.

Corollary 3. All designs reported in Morgan *et al.* [13] satisfy the sufficient condition of Theorem 6.

The efficiency of the residual design can be obtained using the following result.

Theorem 7: The efficiency E of the residual design in comparison with the original design when any t observations from the same block are lost in a NBIB design, is given by

$$E^{-1} = 1 + \sum_{j'=1}^{q}\left(E_{j'}^{-1} - 1\right)$$

where $E_{j'}$ is the corresponding efficiency of the design if any $t_{j'}$ observations from the j'^{th} sub-block are lost. •

More specifically $E_{j'}$ can be obtained from (3.4) and (3.5) by substituting m by $t_{j'}$.

Proof: From (2.5), we get

$$\text{tr}(\mathbf{C}_*^+) = \text{tr}(\mathbf{C}^+) + \alpha$$

where $$\alpha = \text{tr}\ [(\mathbf{I} - \mathbf{C}_0^{-1/2}\mathbf{V}'\mathbf{C}^+\mathbf{V}\mathbf{C}_0^{-1/2})\mathbf{C}_0^{-1/2}\mathbf{V}'\mathbf{C}^+\mathbf{C}^+\mathbf{V}\mathbf{C}_0^{-1/2}]$$

$$= \frac{1}{\theta^2}[(\mathbf{I} - \frac{1}{\theta}\mathbf{C}_0^{-1/2}\mathbf{V}'\mathbf{V}\mathbf{C}_0^{-1/2})(\mathbf{C}_0^{-1/2}\mathbf{V}'\mathbf{V}\mathbf{C}_0^{-1/2})]$$

Now for two sub-blocks j' and j'', $\Delta_{j'}'\Delta_{j'} = \mathbf{I}_h \ \forall\ j' = j''$

$$= \mathbf{0}\ \forall\ j' \neq j''$$

Hence $\mathbf{V}'\mathbf{V} = \text{diag}\ \left((\mathbf{U}_1\boldsymbol{\Gamma}_0\mathbf{U}_1), (\mathbf{U}_2\boldsymbol{\Gamma}_0\mathbf{U}_2), \ldots, (\mathbf{U}_q\boldsymbol{\Gamma}_0\mathbf{U}_q), \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}\right)$

and $\mathbf{C}_*^{-1/2}\mathbf{V}'\mathbf{V}\mathbf{C}_*^{-1/2} = \text{diag}\ \left((\mathbf{U}_1\boldsymbol{\Gamma}_0\mathbf{U}_1)^{-1/2}(\mathbf{U}_1\boldsymbol{\Gamma}_0\mathbf{U}_1)(\mathbf{U}_1\boldsymbol{\Gamma}_0\mathbf{U}_1)^{-1/2},\ \ldots\ ,\mathbf{0}\ ,\mathbf{0}\right)$

Thus $\alpha = \sum_{j'=1}^{q} \alpha_{j'}$

where $$\alpha_{j'} = \frac{1}{\theta^2}\text{tr}\left\{\mathbf{I} - \frac{1}{\theta}(\mathbf{U}_{j'}\boldsymbol{\Gamma}_0\mathbf{U}_{j'})^{-1/2}(\mathbf{U}_{j'}\boldsymbol{\Gamma}_0\mathbf{U}_{j'})(\mathbf{U}_{j'}\boldsymbol{\Gamma}_0\mathbf{U}_{j'})^{-1/2}\right\}$$
$$\left\{(\mathbf{U}_{j'}\boldsymbol{\Gamma}_0\mathbf{U}_{j'})^{-1/2}(\mathbf{U}_{j'}\boldsymbol{\Gamma}_0\mathbf{U}_{j'})(\mathbf{U}_{j'}\boldsymbol{\Gamma}_0\mathbf{U}_{j'})^{-1/2}\right\}$$

Now if we denote the efficiency for the loss of any $t_{j'}$ observations in the j'^{th} sub-block by $E_{j'}$, then

$$E_{j'} = \frac{\text{tr}(\mathbf{C}^+)}{\text{tr}(\mathbf{C}^+) + \alpha_{j'}} \quad \text{or} \quad \alpha_{j'} = (E_{j'}^{-1} - 1)\ \text{tr}(\mathbf{C}^+) \tag{3.7}$$

and $$E = \frac{\text{tr}(\mathbf{C}^+)}{\text{tr}(\mathbf{C}^+) + \alpha} \quad \text{or} \quad \alpha = (E^{-1} - 1)\ \text{tr}(\mathbf{C}^+) \tag{3.8}$$

Thus from (3.7) and (3.8) we have

$$E = \left[1 + \sum (E_{j'}^{-1} - 1)\right]^{-1}$$

We have calculated the efficiency for the loss of all observations in a block, *i.e.,* $E_{j'}$ is calculated from (3.5) and it is constant for all sub-blocks. We found that the efficiency is greater than 0.90 for the designs corresponding to serial numbers 4 to 7, 10, 11 and 13 to 68, where as it is above 0.83 for the designs corresponding to serial numbers 2, 3, 8, 9 and 12. The efficiency of the remaining design, *i.e.*, corresponding to serial number 1 is 0.75. Thus all these designs are fairly robust as per *Criterion 2* against the loss of all observations in a block, except one design corresponding to serial number 1.

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REFERENCES

[1] Baksalary, J. K. and Tabis, Z. (1987). Conditions for robustness of block designs against unavailability of data. *J. Statist. Plann. Inf.*, **16**, 49-54.

[2] Dey, A. (1993). Robustness of block designs against missing data. *Statistica Sinica*, **3**, 219-231.

[3] Dey, A., Midha, C. K. and Buchathal, D. C. (1996). Efficiency of the residual designs under the loss of observations in a block. *Jour. Ind. Soc. Agril. Stat.*, **49**, 237-248.

[4] Ghosh, S. (1979). On robustness of designs against incomplete data. *Sankhya*, **B40**, 204-208.

[5] Ghosh, S. (1982). Robustness of designs against the unavailability of data. *Sankhya*, **B44**, 50-62.

[6] Ghosh, S. (1982). Robustness of BIB designs against non-availability of data. *J. Statist. Plann. Inf.*, **6**, 29-32.

[7] Ghosh, S. (1982). Information in an observation in robust designs. *Commu. Stat. – Theory and Methods*, **11**, 1173-1184.

[8] Hedayat, A. and John, P. W. M. (1974). Resistant and susceptible BIB designs. *Ann. Statist.*, **2**, 148-158.

[9] Kageyama, S. (1990). *Robustness of Block Designs. Probability, Statistics and Design of Experiments* (Ed. R. R. Bahadur), 425-438 (New Delhi, Wiley Eastern).

[10] Kassanis, B. and Kleczkowski, A. (1965). Inactivation of a strain of tobacco necrosis virus and of the RNA isolated from it, by ultraviolet radiation of different wave-lengths. *Photochem. Photobiol.*, **4**, 209-214.

[11] Kleczkowski, A. (1950). Interpreting relationships between the concentrations of plant viruses and numbers of local lesions. *J. Gen. Microbiol.*, **4**, 53-69.

[12] Lal, K., Gupta, V. K. and Bhar, L. (2001). Robustness of designed experiments against missing data. *Journal of Applied Statistics*, **28**, 63-79.

[13] Morgan, J. P., Preece, D. A. and Rees, D. H. (2001). Nested balanced incomplete block designs. *Discrete Mathematics*, **231**, 351-389.

[14] Mukerjee, R. and Kageyama, S. (1990). Robustness of group divisible designs. *Commu. Stat. – Theory and Methods*, **19**, 3189–3203.

[15] Parsad, R., Gupta, V.K. and Voss, D. (2001). Optimal nested row-column designs. *Jour. Ind. Soc. Agril. Stat.*, **54**, 244-257.

[16] Preece, D.A. (1967). Nested balanced incomplete block designs. *Biometrika*, **54**, 479-486.

[17] Satpati, S. K. (2001). Nested block designs and their applications. Unpublished M.Sc. Thesis, Indian Agricultural Research Institute, New Delhi.

[18] Satpati, S. K. and Parsad, R. (2002). Nested partially balanced incomplete block designs. To appear in *ARS Combinatoria*.