Jour. Ind. Soc. Ag. Statistics
55(2), 2002 : 153-157

A Note on Linear Unbiased Invariant Estimator for Some Classes of Estimators

P.S. Shende and S.G. Prabhu Ajgaonkar
Marathwada University, Aurangabad
(Received : April, 1985)

SUMMARY

It is demonstrated that, unbiased linear invariant estimator does not exist for each of the T_1, T_2, T_3 and T_4 classes.

Key words: Classes of estimators, Unbiased linear invariant estimator.

1. Introduction

Let $U = (U_1, U_2, \ldots, U_N)$ denote a finite population consisting of N elements. Let Y_i represent the value of the characteristic under consideration for the i^{th} population element. The population total is denoted by $T = \sum_{i=1}^{N} Y_i$. A random sample of size n is drawn without replacement with arbitrary probabilities of selection at each draw.

For estimating the population total, Horvitz and Thompson [1] have formulated certain classes of linear estimators.

2. Classes of Linear Estimators

(a) Estimators belonging to the Horvitz and Thompson T_1 class are defined symbolically as

$$T_1 = \sum_{r=1}^{n} \alpha_r y_r \tag{1}$$

where α_r (r = 1, 2, ..., n) is a constant used as weight attached to the outcome at the r^{th} draw.

(b) T_2–class: T_2 class of linear estimators is defined as

$$T_2 = \sum_{r=1}^{n} \beta_i (y_r = Y_i) \tag{2}$$

where $\beta_i, i = 1, 2, \ldots N$ is a constant to be used as weight attached to the i^{th} population unit whenever it is included in the sample and y_r denotes the outcome at the r^{th} draw $(r = 1, 2, \ldots, n)$.

(c) T_3–class: T_3 class of linear estimators is defined as

$$T_3 = \beta^s \left(\sum_{r=1}^{n} y_r \right)_s \tag{3}$$

where β^s is a constant to be used as weight attached to the total of s^{th} sample, $s = 1, 2, \ldots, \binom{N}{n} n!$

(d) T_4–class: T_4 class is defined as

$$T_4 = \sum_{r=1}^{n} \beta_{ri} (y_r = Y_i) \tag{4}$$

where $\beta_{ri}, r = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, N$ is a constant used as weight attached to the i^{th} population unit, whenever, it is selected at the r^{th} draw. This class was defined by Koop [2] and Prabhu-Ajgaonkar [4] independently of each other.

3. Invariant Estimator

Roy and Chakravarti [3] defined a linear invariant estimator and demonstrated that in the class of linearly invariant unbiased estimators in general, there does not exist a best estimator.

A linear unbiased estimator $T = t(y)$ is called as invariant, if the transformation $y_i^* = ay_i + b, (i = 1, 2, \ldots, N)$ of the variate values transforms $t(y)$ to $t^*(y)$ where

$$t^*(y) = a\, t(y) + b$$

4. Results

(a) Prabhu-Ajgaonkar [4] demonstrated that, T_1 class is unbiased if and only if

$$\sum_{r=1}^{n} \alpha_r p_{ir} = 1, \quad i = 1, 2, \ldots, N \tag{5}$$

where p_{ir} is the probability that u_i gets selected at r^{th} draw.

The unbiased T_1 class is linearly invariant, if

$$T_1^* = \sum_{r=1}^{n} \alpha_r (ay_r + b)$$

$$= a\sum_{r=1}^{n} \alpha_r y_r + b\sum_{r=1}^{n} \alpha_r$$

$$= aT_1 + b$$

This is so if $\sum_{r=1}^{n} \alpha_r = 1$

In equation (5), summing over i, we get

$$\sum_{r=1}^{n} \alpha_r \sum_{i=1}^{N} p_{ir} = N$$

or $$\sum_{r=1}^{n} \alpha_r = N \text{ as } \sum_{i=1}^{N} p_{ir} = 1 \text{ which is a contradiction}$$

Thus, linearly invariant estimator does not exist in the unbiased T_1 class of linear estimators.

(b) Horvitz and Thompson have proved that a unique unbiased estimator exists in the T_2 class, if $\beta_i = \frac{1}{\pi_i}$. Where π_i is the probability of including the i^{th} population unit in the sample (i = 1, 2, ..., N).

Let T_2^* be the transformed estimator

Then $T_2^* = \sum_{r=1}^{n} \frac{y_r^*}{\pi_r}$

$$= a \sum_{r=1}^{n} \frac{y_r}{\pi_r} + b \sum_{r=1}^{n} \frac{1}{\pi_r}$$

This is invariant estimator, if

$$T_2^* = aT_2 + b$$

This is so, if $\sum_{r=1}^{n} \frac{1}{\pi_r} = 1$

But $\pi_r \leq 1$

or $\frac{1}{\pi_r} \geq 1$

$$\therefore \qquad \sum_{r=1}^{n} \frac{1}{\pi_r} \geq n$$

Hence, unbiased invariant estimator does not exist in the T_2 class.

(c) $T_3 = \beta^s \left(\sum_{r=1}^{n} y_r \right)_s$, T_3 will be unbiased estimator, if $\sum_{s \in i} \beta^s \, p(s) = 1$ where p(s) is the probability of getting the s^{th} sample, $s = 1, 2, \ldots \binom{N}{n} n!$ and $\sum_{s \in i}$ is the sum over all samples which contain the i^{th} unit.

Now $T_3^* = a\beta^s \left(\sum_{r=1}^{n} y_r \right)_s + \beta^s nb$

T_3^* is invariant, if

$\beta^s nb = b$ i.e. $\beta^s = \frac{1}{n}$

This value of β^s does not satisfy the conditions of unbiasedness. Therefore, an unbiased linear estimator (invariant) does not exists in the T_3 class.

(d) $T_4 = \sum_{r=1}^{n} \beta_{ri}\left(y_r = Y_i\right)$

The condition of unbiasedness is

$$\sum_{r=1}^{n} \beta_{ri} p_{ir} = 1, i = 1, 2, \ldots N$$

where p_{ir} is the probability of selecting the i^{th} unit at the r^{th} draw.

$$T_4^* = \sum_{r=1}^{n} \beta_{ri} y_i^*$$

T_4 is invariant, if $\sum_{r=1}^{n} \beta_{ri} = 1$

Subsequently, the T_4 class is unbiased linear invariant if, the following two conditions are satisfied.

$$\sum_{r=1}^{n} \beta_{ri} p_{ir} = 1 \tag{1}$$

and

$$\sum_{r=1}^{n} \beta_{ri} = 1 \tag{2}$$

Now, $\pi_i = \sum_{r=1}^{n} p_{ir} = \sum_{r=1}^{n} p_{ir} \sum_{r=1}^{n} \beta_{ri} \geq \sum_{r=1}^{n} \beta_{ri}\, p_{ir} = 1$ which is a contradiction as π_i is a probability.

Thus, an unbiased invariant estimator does not exist in the T_4 class.

REFERENCES

[1] Horvitz, D.G. and Thompson, D.J. (1952). A generalization of sampling without replacement from a finite universe. *J. Amer. Stat. Assoc.*, **47**, 663-685.

[2] Koop, J.C. (1963). On the axioms of sample formation and their bearing on the construction of linear estimators in sampling theory for finite universes. I, II and III. *Metrica*, **7**, 81-114-165-204.

[3] Roy, J. and Chakravarti, I.M. (1960). Estimating the mean of a finite population. *The Annals of Mathematical Statistics*, **31**, 392-398.

[4] Prabhu-Ajgaonkar, S.G. (1970). A contribution to the theory of unique estimators for Horvitz and Thompson's class of linear estimators. *Metrika*, **15**, 40-47.