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Structural Time-Series Models for Describing Cyclical Fluctuations

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SUMMARY

Advantages of "Structural time-series modelling" approach over well-known "Auto Regressive Integrated Moving Average" (ARIMA) methodology are highlighted. In the former, peculiar features exhibited by the data dictate the particular type of model from the family to be employed. Various types of models which are capable of explaining "cyclical fluctuations" are discussed. As an illustration, all-India lac production data, which has prominent cycles, is modelled. Results are compared with corresponding analogue from ARIMA family. Finally, forecasting of all-India lac production is carried out.

Key words : ARIMA methodology, Cyclical fluctuations, Lac production data, Structural time-series models.

1. Introduction

Statistical modelling of time-series data in Agriculture is usually carried out by employing ARIMA methodology (Brockwell and Davis [2]). One disadvantage of this methodology is that the data series under consideration is assumed to be either stationary, or can be made so by differencing; however this is not always possible. A quite promising, mechanistic approach, which does not suffer from this drawback, is "Structural time-series modelling (STSM)" (Harvey [3]). The distinguishing feature of this methodology is that observations are regarded as made up of distinct components such as trend and cyclical fluctuations and each of which is modelled separately. The techniques that emerge from this approach are extremely flexible and are capable of handling a much wider range of problems than is possible through ARIMA approach. Purpose of this paper is to discuss STSM approach when there are prominent cyclical fluctuations. Finally, as an illustration, this methodology is applied to all-India lac production data for the period 1930-31 to 1998-99.

2. *Structural Time-Series Models*

A structural time-series model is set up in terms of various components of interest, viz. trend (μ_t), cyclical fluctuations (ψ_t), seasonal variations (S_t) and irregular (error) term (ε_t)

$$Y_t = \mu_t + \psi_t + S_t + \varepsilon_t \tag{1}$$

where Y_t is the observed time-series at time "t". In the absence of seasonal components, eq. (1) reduces to

$$Y_t = \mu_t + \psi_t + \varepsilon_t \tag{2}$$

For modelling cyclical fluctuations, the following three models are discussed

(i) Cycle Plus Noise Model (CNM)

Here the trend μ_t is assumed to be constant. Thus, the functional form of CNM (Harvey [3]) is

$$Y_t = \mu + \psi_t + \varepsilon_t \,,\ t = 1, 2, ..., T \tag{3}$$

The cyclical fluctuations ψ_t can be expressed as a mixture of sine and cosine terms

$$\psi_t = \alpha \cos(\lambda_c t) + \beta \sin(\lambda_c t) \tag{4}$$

where λ_c, $(\alpha^2 + \beta^2)^{1/2}$, $\tan^{-1}(\beta/\alpha)$ represent respectively the frequency, amplitude, and phase. The cycles in eq. (4) need to be made stochastic by allowing the parameters α and β to evolve over time. Following Harvey [3], the final form of eq. (4) can be written as

$$\begin{bmatrix}\psi_t\\ \psi_t^*\end{bmatrix} = \rho\begin{bmatrix}\cos\lambda_c & \sin\lambda_c\\ -\sin\lambda_c & \cos\lambda_c\end{bmatrix}\begin{bmatrix}\psi_{t-1}\\ \psi_{t-1}^*\end{bmatrix} + \begin{bmatrix}k_t\\ k_t^*\end{bmatrix} \tag{5}$$

where the correlation coefficient $\rho \in [0, 1]$ is a damping factor, and k_t and k_t^* are uncorrelated white-noise disturbance terms. Further, $\psi_0 = \alpha, \psi_0^* = \beta$. The new parameters are ψ_t, the current value of the cycle, and ψ_t^*, which appears by construction in order to form ψ_t. Eq. (5) is a vector AR (1) process.

Let L denote the lag operator, i.e.

$$L\,\psi_t = \psi_{t-1} \tag{6}$$

Then eq. (5) can be written as

$$\begin{bmatrix}1 - L\rho\cos\lambda_c & -L\rho\sin\lambda_c\\ L\rho\sin\lambda_c & 1 - L\rho\cos\lambda_c\end{bmatrix}\begin{bmatrix}\psi_t\\ \psi_t^*\end{bmatrix} = \begin{bmatrix}k_t\\ k_t^*\end{bmatrix} \tag{7}$$

Substituting for ψ_t in eq. (3) yields

$$Y_t = \mu + \frac{(1 - L\rho\cos\lambda_c)\,k_t + (L\rho\sin\lambda_c)k_t^*}{(1 - 2L\rho\cos\lambda_c + L^2\rho^2)} + \varepsilon_t,\ t = 1, 2, \ldots, T \qquad (8)$$

where ε_t is assumed to be uncorrelated with k_t and k_t^*.

For estimation of parameters, eq. (8) has to be put in state space form (Meinhold and Singpurwalla [6]) and then Kalman filter, prediction and smoothing (Koopman *et al.* [4]) are applied. In eq. (8), unobservable state, conditional on variances S_2^2 and σ_k^2, is done recursively using Kalman filter and smoother. In general, these are unknown and are treated as hyperparameters. Likelihood function can be evaluated by Kalman filter via prediction error decomposition (Shumway and Stoffer [8]). Maximizing likelihood function with respect to hyperparameters, using quasi-Newton optimization procedure, is referred to as "hyperparameter estimation". After estimation of parameters, prediction and smoothing are performed. The reduced form from ARIMA family corresponding to CNM is "Constant + ARIMA (2,2)".

(ii) Trend Plus Cycle Model (TCM)

As described by Harvey [3], TCM is given by the following equations

$$Y_t = \mu_t + \psi_t + \varepsilon_t,\ t = 1, 2, \ldots, T \qquad (9)$$

$$\mu_t = \mu_{t-1} + \beta_{t-1} + \eta_t \qquad (10)$$

$$\beta_t = \beta_{t-1} + \xi_t,\ t = \ldots, -1, 0, 1, \ldots \qquad (11)$$

In this, ε_t, η_t and ξ_t are the disturbance terms which follow Gaussian distributions with means 0 and variances $\sigma_\varepsilon^2, \sigma_\eta^2$ and σ_ξ^2 respectively and are called hyperparameters of the model. As mentioned in CNM model, estimation of state vector and hyperparameters is carried out by putting the model in state space form and subsequently Kalman filter is applied with proper initial values. The state space formulation of TCM is as follows

The measurement equation is

$$Y_t = [1 \quad 0 \quad 1 \quad 0]\alpha_t + \varepsilon_t,\ t = 1, 2, \ldots, T \qquad (12)$$

whereas the transition equation is

$$\alpha_t = \begin{bmatrix} \mu_t \\ \beta_t \\ \ldots \\ \psi_t \\ \psi_t^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & . & & \\ 0 & 1 & . & \multicolumn{2}{c}{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & . & \rho\cos\lambda_c & \rho\sin\lambda_c \\ \multicolumn{2}{c}{0} & . & -\rho\sin\lambda_c & \rho\cos\lambda_c \end{bmatrix} \begin{bmatrix} \mu_{t-1} \\ \beta_{t-1} \\ \ldots \\ \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \eta_t \\ \xi_t \\ \ldots \\ k_t \\ k_t^* \end{bmatrix} \qquad (13)$$

The covariance matrix of the vector of disturbances in eq. (13) is a diagonal matrix with diagonal elements $\{\sigma_\eta^2, \sigma_\xi^2, \sigma_k^2, \sigma_{k^*}^2\}$. Once the parameters are estimated using prediction error decomposition, Kalman filter, prediction and smoothing can be applied.

(iii) Cyclical Trend Model (CTM)

Here the cycle is actually incorporated within trend. Thus the model is given by the equations (Harvey [3])

$$Y_t = \mu_t + \varepsilon_t,\ t = 1, 2, \ldots, T \tag{14}$$

$$\mu_t = \mu_{t-1} + \psi_{t-1} + \beta_{t-1} + \eta_t \tag{15}$$

$$\beta_t = \beta_{t-1} + \xi_t \tag{16}$$

Estimation of parameters and hyperparameters is carried out by putting the model in state space form and applying Kalman filter. The essential difference between TCM and CTM is that, in the former, the observation Y_t depends on cyclical fluctuations ψ_t explicitly whereas, in the latter, it does so on ψ_{t-1} implicitly through the trend μ_t. As far as analogue form of CTM is concerned, there is no difference with TCM, i.e. ARIMA (2, 2, 4) model is the corresponding analogue of both TCM and CTM.

Goodness of Fit

Goodness of fit of models having equal number of hyperparameters is assessed using Prediction Error Variances (PEV). If the number of hyperparameters is different, one can use Akaike Information Criterion (AIC) which is given as

$$\text{AIC} = -2 \log_e L + 2n \tag{17}$$

or Schwartz - Bayes Information Criterion (SBC) given as

$$\text{SBC} = -2 \log_e L + n \log_e T \tag{18}$$

where L is the likelihood function, n denotes number of hyperparameters, and T is the sample size. Lower the values of these statistics, better is the fitted model.

3. An Illustration

As an illustration, data on all-India annual lac production for the period 1930-31 to 1998-99, obtained from the annual reports of Shellac Export Promotion Council, Kolkata, is utilized. Data up to the period 1990-91 is used for fitting purposes while subsequent data is used for examining goodness of fit of the models by comparing forecast values with the actual values. As the data set pertains to annual figures of lac production, only those models which do not incorporate "seasonal fluctuations" are considered. To the best of our knowledge, only STAMP software package (Koopman *et al.* [5]) has the capability to estimate parameters of "Structural time-series models". Version 6.0 of this software package downloaded from the world wide web site at http://www.econ.vu.nl/STAMP.htm is utilized for data analysis.

In the first instance, graphical display of data depicted in Fig.1 indicates presence of prominent cyclical fluctuations. Accordingly, all the three models capable of exhibiting cyclical fluctuations, viz. CNM, TCM, and CTM from the family of "Structural time-series models" are applied to the data set and the results are reported in Table 1. A perusal indicates that TCM performs the best as is evident from the values of three goodness of fit criteria. The maximum values for CNM imply that this model is not appropriate for describing the present data set. Thus, apart from cyclical fluctuations and noise, there is definitely presence of a trend component. The estimate of slope (β_t) for TCM, i.e. $\hat{\beta} = -0.39$, being negative, indicates a declining trend. The frequency estimate $\hat{\lambda} = 1.26$ corresponds to a period of $2\pi/\hat{\lambda}$, i.e. approximately 5 years. Further, $\hat{\rho} = 0.96$, being strictly less than unity, indicates that forecast function is a damped sine, or cosine wave. The situation is analogous to that of the AR(1) forecast function and indeed the condition that $\hat{\rho}$ be strictly less than unity is the one which is needed for stationarity. For comparing the performance of TCM with its corresponding analogue, ARIMA (2, 2, 4) model is fitted to the given data using SAS [7] statistical software package. Estimation of parameters is carried out using Melard's algorithm (Box *et al.* [1]). Parameter estimates of ARIMA (2, 2, 4) model are

Constant = – 0.41, AR 1 = 0.34, AR 2 = 0.30

MA 1 = –.09, MA 2 = 1.37, MA 3 = 0.18, MA 4 = – 0.46

Fig.2 displays the graph of autocorrelations on first to sixth lags for fitted ARIMA (2, 2, 4) model in respect of original as well as second order differenced series. It may be noted that for both the series, there is a spike at lag 5, which gives an indication of a possible five-year cycle [7]. Further, goodness of fit statistics for this model are

AIC = 317.23, SBC = 331.77, SE = 3.37

Thus, TCM performs much better than ARIMA (2, 2, 4) model. A comparison of forecasts with actual values for the next 8 years from 1991-92 onwards on the basis of ARIMA (2, 2, 4) model and TCM, presented in Table 1, also reflect the superiority of TCM over ARIMA (2, 2, 4) model. Thus TCM is the best model for describing cyclical fluctuations in all-India lac production data. To get visual insight, the graphs of fitted TCM and residuals respectively are exhibited in Figs.1 and 3.

Table 1. Summary statistics for structural time–series models

(i) Estimation of parameters :			
Parameter	CNM	TCM	CTM
μ_t (level)	18.86	15.41	14.76
β_t (slope)	2.03	–0.39	0.88
ψ_t (cycle)	3.89	2.15	3.06
σ^2_ε	52.76	33.54	45.73
σ^2_η	–	8.15	7.95
σ^2_ξ	1.25	2.86	3.48
Cycle Variance	10.77	10.63	8.29
Cycle Frequency	1.25	1.26	1.26
Cycle Amplitude	4.80	4.35	3.64
ρ	0.97	0.96	0.97
(ii) Goodness of fit statistics :			
AIC	270.02	246.87	261.75
SBC	282.39	262.37	273.75
SE	5.83	3.24	4.08
(iii) Forecast values (in thousand tonnes) for TCM :			
Year	Actual values	Forecast by TCM	Forecast by ARIMA (2, 2, 4)
1991-92	10.81	14.11	14.25
1992-93	11.68	12.40	13.97
1993-94	20.52	17.26	13.32
1994-95	22.46	16.14	12.81
1995-96	20.05	17.79	12.23
1996-97	19.76	17.30	11.67
1997-98	17.55	13.15	11.09
1998-99	10.36	10.29	10.52

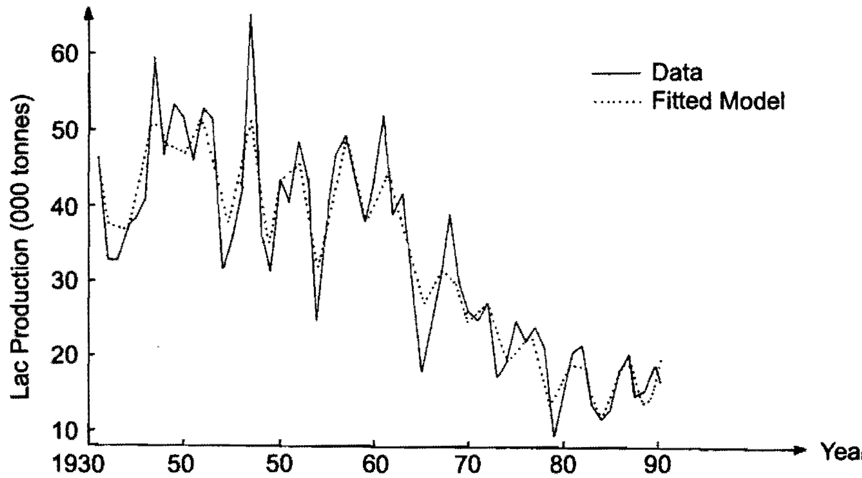

Fig. 1. Graph of trend plus cycle model along with all-India lac production data

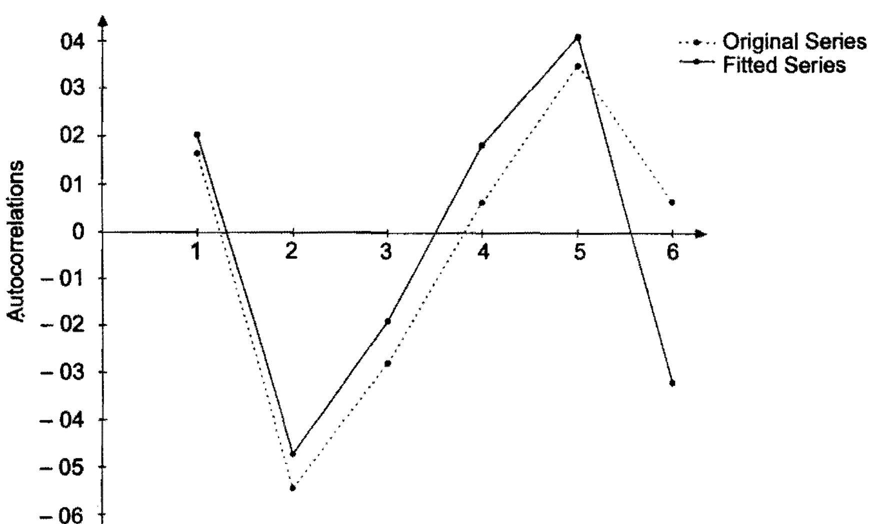

Fig. 2. Autocorrelations on first to six lags for fitted ARIMA (2, 2, 4) model

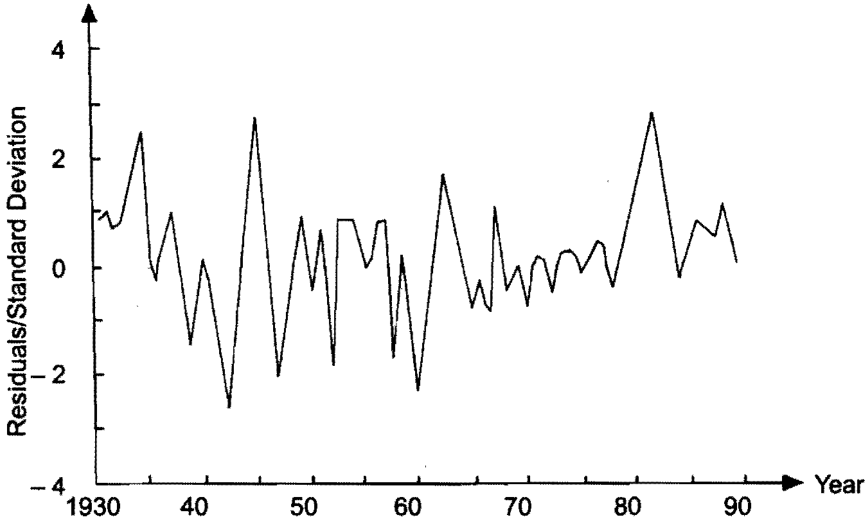

Fig. 3. Graph of residuals of trend plus cycle model over time

4. Concluding Remarks

The methodology is discussed for describing time-series data with marked cyclical fluctuations. As an illustration, the methodology is applied to all-India lac production data. Some descriptive studies do exist in the literature concerning lac population growth but these have dealt with only one aspect, i.e. either cyclic fluctuations or declining trend. This is probably for the first time that we have analytically demonstrated using sophisticated statistical tools that, apart from a declining tend, there are prominent five-year cycles in all-India lac production. One plausible reason for such cycles is the existence of cyclical fluctuations in various factors, like temperature, humidity, and rainfall; all these have a marked influence on lac production. In reality, the lac production system is a three-species interacting system comprising trees (i.e. food for the lac insect), lac insect, and predators of lac insect. Thus there is a need to study the multivariate extension of 'Structural time-series models'. Appropriate estimation procedures along with relevant computer programs to handle such situations also have to be developed. Simultaneously, detailed time-series data concerning above-mentioned species is also required. Then only it may be possible to assign a biological explanation for the cyclical behaviour of lac production. However, it is hoped that, in due course of time, research workers in other disciplines would also start applying 'Structural time-series models' in their data analysis.

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