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A Revisit to Alternative Estimators for Randomized Response Technique

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SUMMARY

In this paper we have suggested various estimators of population proportion π and their merits over Warner's [4] estimator examined through numerical illustrations.

Key words: Sensitive characteristic, Population proportion, Mean square error, Randomized response technique.

1. Introduction

Let there be a population possessing a sensitive attribute A, say. Warner [4] proposed a method known as randomized response (RR) technique to procure trustworthy data for estimating the proportion π of the population possessing the attribute A. The technique comprises of using a device with outcomes A and not-A with known probabilities p and (1 – p) respectively. The respondent observes the device's outcome which remains unknown to the investigator so that the respondent's privacy is protected. The respondent reports 'yes' if he has the characteristic shown by the device's outcome and 'no' otherwise. The probability of a yes answer is given by

$$\theta = p\pi + (1-p)(1-\pi) \tag{1.1}$$

Let n respondent be selected by simple random sampling with replacement (SRSWR) and n_1 be the number of 'yes' answers out of n responses. For estimating π, Warner [4] suggested an unbiased estimator

$$\hat{\pi}_w = \frac{\{\hat{\theta} - (1-p)\}}{(2p-1)}, \; p \neq \frac{1}{2} \tag{1.2}$$

with the variance

$$V(\hat{\pi}_w) = \frac{\theta(1-\theta)}{n(2p-1)^2} \tag{1.3}$$

where $\hat{\theta} = \frac{n_1}{n}$ is the proportion of yes answers in the sample.

Following Searls [2], Singh and Singh [3] suggested a class of estimators for π as

$$\hat{\pi}_\lambda = \frac{\{\lambda\hat{\theta} - (1-p)\}}{(2p-1)} \tag{1.4}$$

where λ is a constant to be chosen suitably such that mean square error (MSE) of $\hat{\pi}_\lambda$ is minimum.

The minimum MSE of $\hat{\pi}_\lambda$ for optimum value of $\lambda = \frac{n\theta}{\{1+(n-1)\theta\}}$ is given by

$$\min.\, \text{MSE}(\hat{\pi}_\lambda) = \frac{n\theta}{\{1+(n-1)\theta\}} V(\hat{\pi}_w) \tag{1.5}$$

Sampath *et al.* [1] suggested another class of estimator for π as

$$\hat{\pi}_\delta = \delta\frac{\{\hat{\theta} - (1-p)\}}{(2p-1)} = \delta\hat{\pi}_w \tag{1.6}$$

where δ is a constant such that MSE of $\hat{\pi}_\delta$ is minimum. The minimum MSE of $\hat{\pi}_\delta$ for optimum value of δ

$$\delta = \frac{\pi^2}{\{\pi^2 + V(\hat{\pi}_w)\}} = \delta_{opt} \tag{1.7}$$

is given by

$$\min.\, \text{MSE}(\hat{\pi}_\delta) = \frac{\pi^2 V(\hat{\pi}_w)}{\pi^2 + V(\hat{\pi}_w)} \tag{1.8}$$

From (1.3), (1.5) and (1.8), it can be easily shown that

$$\min.\, \text{MSE}(\hat{\pi}_\delta) \le \min.\, \text{MSE}(\hat{\pi}_\lambda) \le V(\hat{\pi}_w) \tag{1.9}$$

which clearly indicates that the estimator $\hat{\pi}_\delta$ suggested by Sampath *et al.* [1] is more efficient than Singh and Singh [3] and Warner [4] estimators under optimum conditions. However, the estimators $\hat{\pi}_\lambda$ and $\hat{\pi}_\delta$ cannot be used in practice as they depend on unknown constants. In this paper we have suggested estimators based on estimated optimum values and their properties are studied.

2. *Estimators Based on Estimated Optimum*

Since the optimum value of δ in (1.7) is not known in practice, it is, therefore, advisable to replace δ with its estimated optimum value

$$\hat{\delta}_{opt}^{(1)} = \frac{\hat{\pi}_w^2}{\left\{\hat{\pi}_w^2 + \frac{\hat{\theta}(1-\hat{\theta})}{n(2p-1)^2}\right\}} \tag{2.1}$$

which is obtained on replacing π^2 and $V(\hat{\pi}_w)$ by their consistent estimators $\hat{\pi}_w^2$ and $\hat{V}_1(\hat{\pi}_w) = \frac{\hat{\theta}(1-\hat{\theta})}{n(2p-1)^2}$ in (1.7). Substitution of $\hat{\delta}_{opt}^{(1)}$ in (1.6) yields an estimator for π as

$$\hat{\pi}_\delta^{(1)} = \frac{\hat{\pi}_w^3}{\left\{\hat{\pi}_w^2 + \frac{\hat{\theta}(1-\hat{\theta})}{n(2p-1)^2}\right\}} \tag{2.2}$$

Replacing π^2 by $\hat{\pi}_w^2$ and $V(\hat{\pi}_w)$ by its unbiased estimator $\hat{V}_2(\hat{\pi}_w) = \frac{\hat{\theta}(1-\hat{\theta})}{(n-1)(2p-1)^2}$ in (1.7), we get an estimate of δ_{opt} as

$$\hat{\delta}_{opt}^{(2)} = \frac{\hat{\pi}_w^2}{\left\{\hat{\pi}_w^2 + \frac{\hat{\theta}(1-\hat{\theta})}{(n-1)(2p-1)^2}\right\}} \tag{2.3}$$

and hence the resulting estimator is

$$\hat{\pi}_\delta^{(2)} = \frac{\hat{\pi}_w^3}{\left\{\hat{\pi}_w^2 + \frac{\hat{\theta}(1-\hat{\theta})}{(n-1)(2p-1)^2}\right\}} \tag{2.4}$$

We make the estimators in (2.2) and (2.4) more flexible by introducing the constant $h\ (\geq 0)$

$$\hat{\pi}_\delta^{(h)} = \frac{\hat{\pi}_w^3}{\left\{\hat{\pi}_w^2 + \left(\frac{h}{n}\right)\frac{\hat{\theta}(1-\hat{\theta})}{(2p-1)^2}\right\}} \tag{2.5}$$

If we set $h = \frac{n}{(n-1)}$ in (2.5), then the estimator $\hat{\pi}_\delta^{(h)}$ reduces to $\hat{\pi}_\delta^{(2)}$, while for $h = 1$, it reduces to $\hat{\pi}_\delta^{(1)}$.

The MSE of an estimator T can now be calculated from

$$\text{MSE}(T) = \sum_{n_1=0}^{n} (T-\pi)^2 \, {}^{n}C_{n_1} \, \theta^{n_1} (1-\theta)^{n-n_1} \tag{2.6}$$

as $\hat{\theta}$ follows a binomial distribution with parameters n and θ.

The percent relative efficiency (PRE) of T with respect to usual unbiased estimator $\hat{\pi}_w$ is given by

$$\text{PRE}(T, \hat{\pi}_w) = \frac{V(\hat{\pi}_w)}{\text{MSE}(T)} \times 100$$

$$= \frac{\theta(1-\theta)}{n(2p-1)^2}\left[\sum_{n_1=0}^{n} (T-\pi)^2 \, {}^{n}C_{n_1} \, \theta^{n_1} (1-\theta)^{n-n_1}\right]^{-1} \times 100 \tag{2.7}$$

We have computed the percentage relative efficiency of $\hat{\pi}_{\delta}^{(i)}$; $i = 1, 2$ and $\hat{\pi}_{\delta}^{(h)}$ with respect to $\hat{\pi}_w$ for different values of n, p, h, π and displayed in Tables 2.1 (a), 2.1 (b) and 2.1 (c).

Further, to obtain the approximate MSE of $\hat{\pi}_{\delta}^{(h)}$, we write

$$\hat{\theta} = \theta(1+\varepsilon)$$

such that

$$E(\varepsilon) = 0 \text{ and } E(\varepsilon^2) = \frac{(1-\theta)}{n\theta}$$

Expressing $\hat{\pi}_{\delta}^{(h)}$ in terms of ε's, we have

$$\hat{\pi}_{\delta}^{(h)} = \pi\left(1 + \frac{\theta}{(2p-1)\pi}\varepsilon\right)$$

$$\left[1 + \frac{h}{n}\frac{\theta(1-\theta)}{(2p-1)^2\pi^2}(1+\varepsilon)\left(1 - \frac{\theta}{(1-\theta)}\varepsilon\right)\left\{1 + \frac{\theta}{(2p-1)\pi}\varepsilon\right\}^{-2}\right]^{-1}$$

or

$$\left(\hat{\pi}_{\delta}^{(h)} - \pi\right) \cong \frac{\theta}{(2p-1)}\varepsilon - \frac{h}{n}\frac{\theta(1-\theta)}{(2p-1)^2\pi}\left[1 + \frac{\theta}{(2p-1)\pi}\varepsilon\left\{\frac{(1-2\theta)(2p-1)\pi}{\theta(1-\theta)} - 1\right\}\right]$$

Squaring both sides of the above expression and then taking expectations, we get the MSE of $\hat{\pi}_{\delta}^{(h)}$ to terms of order n^{-2} as

Table 2.1 (a). The percent relative efficiency of $\hat{\pi}_\delta^{(1)}, \hat{\pi}_\delta^{(2)}$ and $\hat{\pi}_\delta^{(h)}$ with respect to Warner's estimator $\hat{\pi}_w$

p↓	$\pi = 0.10$					
	Sample size(n)					
	Estimator↓	3	5	10	20	50
0.2	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	128.31	140.97	163.14	158.17	129.11
	$\hat{\pi}_\delta^{(2)}$	136.27	146.92	168.30	160.45	129.49
	$\hat{\pi}_\delta^{(h=6)}$	161.89	193.19	272.06	263.72	151.10
	$\hat{\pi}_\delta^{(h=16)}$	166.67	209.75	300.99	286.83	136.99
0.3	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	124.92	153.38	177.06	180.99	164.81
	$\hat{\pi}_\delta^{(2)}$	131.30	160.65	183.92	184.56	165.84
	$\hat{\pi}_\delta^{(h=6)}$	145.16	220.83	398.36	429.26	293.35
	$\hat{\pi}_\delta^{(h=16)}$	146.47	246.43	572.30	620.83	323.27
0.4	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	128.21	160.74	184.52	195.31	195.82
	$\hat{\pi}_\delta^{(2)}$	131.55	170.63	192.37	199.77	197.57
	$\hat{\pi}_\delta^{(h=6)}$	136.22	253.94	498.90	617.22	577.18
	$\hat{\pi}_\delta^{(h=16)}$	136.42	287.75	991.94	1384.23	1005.72
0.6	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	128.21	160.74	184.52	195.31	195.82
	$\hat{\pi}_\delta^{(2)}$	131.55	170.63	192.37	199.77	197.57
	$\hat{\pi}_\delta^{(h=6)}$	136.22	253.94	498.90	617.22	577.18
	$\hat{\pi}_\delta^{(h=16)}$	136.42	287.75	991.94	1384.23	1005.72
0.7	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	124.92	153.38	177.06	180.99	164.81
	$\hat{\pi}_\delta^{(2)}$	131.30	160.65	183.92	184.56	165.84
	$\hat{\pi}_\delta^{(h=6)}$	145.16	220.83	398.36	429.26	293.35
	$\hat{\pi}_\delta^{(h=16)}$	146.47	246.43	572.30	620.83	323.27

Table 2.1 (b). The percent relative efficiency of $\hat{\pi}_\delta^{(1)}, \hat{\pi}_\delta^{(2)}$ and $\hat{\pi}_\delta^{(h)}$ with respect to Warner's estimator $\hat{\pi}_w$

p↓	$\pi = 0.20$					
	Sample size(n)					
	Estimator↓	3	5	10	20	50
0.2	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	116.67	129.53	132.76	113.78	88.87
	$\hat{\pi}_\delta^{(2)}$	122.12	133.51	135.16	114.22	88.93
	$\hat{\pi}_\delta^{(h=6)}$	135.73	161.11	167.41	115.48	64.24
	$\hat{\pi}_\delta^{(h=16)}$	134.31	165.82	161.88	100.31	47.71
0.3	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	121.10	147.22	158.29	146.28	114.45
	$\hat{\pi}_\delta^{(2)}$	126.25	153.77	163.10	148.02	114.62
	$\hat{\pi}_\delta^{(h=6)}$	134.68	204.90	278.75	214.80	115.83
	$\hat{\pi}_\delta^{(h=16)}$	133.56	221.80	329.33	219.87	100.00
0.4	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	127.98	158.76	178.84	182.64	167.53
	$\hat{\pi}_\delta^{(2)}$	130.79	168.34	186.00	186.31	168.61
	$\hat{\pi}_\delta^{(h=6)}$	133.77	249.47	438.99	450.77	310.71
	$\hat{\pi}_\delta^{(h=16)}$	133.37	280.28	749.11	685.72	353.10
0.6	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	127.98	158.76	178.84	182.64	167.53
	$\hat{\pi}_\delta^{(2)}$	130.79	168.34	186.00	186.31	168.61
	$\hat{\pi}_\delta^{(h=6)}$	133.77	249.47	438.99	450.77	310.71
	$\hat{\pi}_\delta^{(h=16)}$	133.37	280.28	749.11	685.72	353.10
0.7	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	121.10	147.22	158.29	146.28	114.45
	$\hat{\pi}_\delta^{(2)}$	126.25	153.77	163.10	148.02	114.62
	$\hat{\pi}_\delta^{(h=6)}$	134.68	204.90	278.75	214.80	115.83
	$\hat{\pi}_\delta^{(h=16)}$	133.56	221.80	329.33	219.87	100.00

Table 2.1 (c). The percent relative efficiency of $\hat{\pi}_\delta^{(1)}, \hat{\pi}_\delta^{(2)}$ and $\hat{\pi}_\delta^{(h)}$ with respect to Warner's estimator $\hat{\pi}_w$

p↓	$\pi = 0.30$					
	Sample size(n)					
	Estimator↓	3	5	10	20	50
0.2	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	106.81	116.52	108.91	91.82	81.80
	$\hat{\pi}_\delta^{(2)}$	109.96	118.44	109.47	91.55	81.53
	$\hat{\pi}_\delta^{(h=6)}$	113.66	127.76	107.65	71.11	46.75
	$\hat{\pi}_\delta^{(h=16)}$	108.32	123.27	94.39	55.12	29.59
0.3	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	116.74	137.71	137.05	117.55	90.47
	$\hat{\pi}_\delta^{(2)}$	120.45	142.85	139.79	118.11	90.35
	$\hat{\pi}_\delta^{(h=6)}$	123.39	178.15	187.47	125.78	68.68
	$\hat{\pi}_\delta^{(h=16)}$	120.23	183.95	191.93	112.29	52.48
0.4	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	127.29	155.49	170.23	165.70	138.78
	$\hat{\pi}_\delta^{(2)}$	129.49	164.43	176.35	168.39	139.31
	$\hat{\pi}_\delta^{(h=6)}$	130.67	238.60	363.31	313.74	182.44
	$\hat{\pi}_\delta^{(h=16)}$	129.67	263.29	524.10	376.66	174.89
0.6	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	127.29	155.49	170.23	165.70	138.78
	$\hat{\pi}_\delta^{(2)}$	129.49	164.43	176.35	168.39	139.31
	$\hat{\pi}_\delta^{(h=6)}$	130.67	238.60	363.31	313.74	182.44
	$\hat{\pi}_\delta^{(h=16)}$	129.67	263.29	524.10	376.66	174.89
0.7	$\hat{\pi}_\delta^{(1)} = \hat{\pi}_\delta^{(h=1)}$	116.74	137.71	137.05	117.55	90.47
	$\hat{\pi}_\delta^{(2)}$	120.45	142.85	139.79	118.11	90.35
	$\hat{\pi}_\delta^{(h=6)}$	123.39	178.15	187.47	125.78	68.68
	$\hat{\pi}_\delta^{(h=16)}$	120.23	183.95	191.93	112.29	52.48

$$MSE\left(\hat{\pi}_{\delta}^{(h)}\right) = V(\hat{\pi}_w) + \frac{h\{V(\hat{\pi}_w)\}^2}{\pi^2}\left[h - 2\left\{\frac{(1-2\theta)(2p-1)\pi}{\theta(1-\theta)} - 1\right\}\right] \quad (2.8)$$

which is smaller than that of $\hat{\pi}_w$ if

either $0 < h < 2\left\{\frac{(1-2\theta)(2p-1)\pi}{\theta(1-\theta)} - 1\right\}$

or $\quad 2\left\{\frac{(1-2\theta)(2p-1)\pi}{\theta(1-\theta)} - 1\right\} < h < 0$

It is observed from Table (2.1) (a), 2.1 (b) and 2.1 (c) that

(i) When $\pi = 0.10$, the performance of the suggested estimators $\hat{\pi}_{\delta}^{(1)}, \hat{\pi}_{\delta}^{(2)}$ and $\hat{\pi}_{\delta}^{(h)}$ {h = 6, 16} are better than Warner's estimator $\hat{\pi}_w$. The estimator $\hat{\pi}_{\delta}^{(2)}$ is more efficient than $\hat{\pi}_{\delta}^{(1)}$. The efficiency of the estimator $\hat{\pi}_{\delta}^{(h)}$ increases as h increases. Thus the scalar 'h' plays a good role in improving the precision of the estimator $\hat{\pi}_{\delta}^{(h)}$.

(ii) When $\pi = 0.20$, the suggested estimators $\hat{\pi}_{\delta}^{(1)}, \hat{\pi}_{\delta}^{(2)}$ and $\hat{\pi}_{\delta}^{(h)}$ {h = 6, 16} perform well than Warner's estimator $\hat{\pi}_w$ except for higher values of n, in particular $(n = 50, p = 0.2)$. The estimator $\hat{\pi}_{\delta}^{(2)}$ is more efficient than $\hat{\pi}_{\delta}^{(1)}$. It is noted in general, that the efficiency of the estimator $\hat{\pi}_{\delta}^{(h)}$ increases as h increases, when $0.4 < p < 0.6\ (p \neq 0.5)$.

(iii) When $\pi = 0.3$, the performances of the suggested estimators $\hat{\pi}_{\delta}^{(1)}, \hat{\pi}_{\delta}^{(2)}$ and $\hat{\pi}_{h}^{(h)}$ $\{h = 6, 16\}$ are more efficient than Warner's estimator $\hat{\pi}_w$ for smaller values of n. However, they perform well for all values of n when p moves in the neighbourhood of 0.5 $(p \neq 0.5)$. The estimator $\hat{\pi}_{\delta}^{(2)}$ is more efficient than $\hat{\pi}_{\delta}^{(1)}$, except $n = 50$.

(iv) The gain in efficiency decreases as the value of π increases.

Finally, we conclude that the constructed estimators $\hat{\pi}_{\delta}^{(1)}, \hat{\pi}_{\delta}^{(2)}$ and $\hat{\pi}_{\delta}^{(h)}$ are more precise than Warner's estimator $\hat{\pi}_w$ for smaller values of π (i.e. for the populations in which number of persons possessing sensitive attribute is small). It is further noted that the substantial gain in efficiency due to suggested estimators $\hat{\pi}_{\delta}^{(1)}, \hat{\pi}_{\delta}^{(2)}$ and $\hat{\pi}_{\delta}^{(h)}\ (h = 6, 16)$ over Warner's estimator $\hat{\pi}_w$ is observed when the probability p (proportion of the sensitive attribute represented in the randomize response device) moves in the vicinity of 0.5

$(p \neq 0.5)$ and the sample size n is small. In practice, such sample sizes are desirable when the survey procedure, like randomized response technique, is costly.

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