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Optimal Designs for Diallel Crosses[1]

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I am thankful to the members of the Indian Society of Agricultural Statistics for having elected me as the Sessional President of the 55th Session of the Society. I feel greatly honoured by your gesture. The Secretary of the Society informed me that one of the "duties" of the Sessional President is to deliver a Technical Address during the conference. After some thought, I decided to choose this topic for my address, because it (i) is a topic of current interest, (ii) is likely to be of some value to agricultural research statisticians and, (iii) has sufficient technical content to qualify as a "Technical" address.

In what follows, a brief review of recent results on optimal designs for complete and partial diallel crosses is presented.

1. Introduction and Preliminaries

The diallel cross is a type of mating design used in plant breeding to study the genetic properties of a set of inbred lines. The purpose of such an experiment is to compare the lines with respect to their general combining abilities. Often, apart from inferring on general combining abilities, an experimenter is also interested in inferring on "cross effects" or, specific combining abilities. For genetic interpretation of these parameters, see Griffing [11] and Hinkelmann [14].

Experimental (or, environmental) design issues in the context of diallel and partial diallel crosses have been studied quite extensively in the literature; e.g., Curnow [15], Hinkelmann [14], Gupta *et al.* [21] and the references given therein. However, the problem of finding optimal designs in this context has received attention only recently.

A common type of diallel cross experiment involves $\frac{p(p-1)}{2}$ crosses of the type $(i \times j), i < j, i, j = 1, 2, \ldots, p,$ where p is the number of inbred lines under

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consideration. A diallel cross of the above type has been called in the literature a complete diallel cross. With the increase in the number of lines p, the number of crosses in the experiment increases rapidly; for example, with p = 5 lines, there are only 10 crosses while with p = 10 lines, the number of crosses rapidly increases to 45. In such a situation, assuming that heterogeneity (in the experimental material) exists in one direction, adoption of a randomized complete block design with crosses as treatments would result in a large error variance, even with a moderate number of lines.

In order to control the error, one would therefore look for an appropriate incomplete block design in preference to a complete block design. One possibility is to use available incomplete block designs, for instance, a balanced incomplete block design, for the experiment, identifying crosses with treatments. This approach has been advocated, e.g., by Das and Giri [8] and Ceranka and Mejza (1988). Another approach proposed is to start with an incomplete block design for the usual treatment-block structure, treat the treatments as lines and make all possible pairwise crosses among the lines within a block; see e.g., Ghosh and Divecha [10] and Sharma [5]. Both these approaches, however are not appropriate if one is interested in the statistical properties, like optimality, of the design. Even an optimal block design (for the usual treatment structure) may turn out to be of poor efficiency when used for a diallel cross experiment and examples can easily be found to demonstrate that this indeed is the case. It is therefore clear that special techniques are needed to obtain "good" designs for (complete) diallel cross experiments.

If the number of lines p is large, the number of crosses $\left(=\frac{p(p-1)}{2}\right)$ in a complete diallel cross experiment may become prohibitively large for an experimenter and in such cases, one might use a 'sample' of the crosses only, leading to what are known as partial diallel crosses. In the context of partial diallal crosses, there are two problems that need to be tackled: (i) firstly, one has to find a 'good' sample (or, a good partial diallel cross plan) in the unblocked situation and, (ii) arrange the crosses so chosen in a block design, so that the final design is optimal for the estimation of say, the general combining ability effects.

In the following sections, we present a brief survey on optimal designs for diallel crosses. Sections 2–4 deal with complete diallel crosses. In Section 2, a formulation of the contrasts representing the general and specific combining ability effects is given. In Section 3, we show how in general, nested balanced incomplete block designs with sub-block size two can be used to obtain universally optimal designs for diallel crosses. In Section 4, we show that triangular partially balanced incomplete block designs satisfying a certain parametric condition also lead to optimal designs for diallel crosses under two different models. Finally, in Section 5, some aspects of optimal partial diallel crosses are discussed.

2. *A Formulation of the General and Specific Combining Ability Effects*

In a recent paper, Chai and Mukerjee [3] gave a nice interpretation of the general and specific combining ability parameters. We describe this development here. Consider a diallel cross experiment with $p > 2$ lines. A typical cross is denoted by $(i \times j), 1 \le i < j \le p$. Thus, there are $v = \frac{p(p-1)}{2}$ crosses in all. These crosses are regarded as treatments, in the usual terminology of experimental designs. Let τ_{ij} be the fixed effect of the cross $(i \times j)$. Then, we can represent τ_{ij} as

$$\tau_{ij} = \bar{\tau} + g_i + g_j + s_{ij}, (1 \le i < j \le p) \tag{2.1}$$

where $\bar{\tau}$ is the mean effect of the treatments, the $\{g_i\}$ stand for the general combining abilities, the $\{s_{ij}\}$ denote the specific combining abilities, and

$$g_1 + \ldots + g_p = 0 \tag{2.2}$$

$$s_{1i} + \ldots + s_{(i-1)i} + s_{i(i+1)} + \ldots + s_{ip} = 0, (1 \le i \le p) \tag{2.3}$$

An explanation of the inevitability of the conditions (2.2) and (2.3) appears to be in order. First observe that when specific combining ability parameters are absent, (2.1) becomes $\tau_{ij} = \bar{\tau} + g_i + g_j, 1 \le i < j \le p$. If we average over i and j, then we see that (2.2) holds and, hence, it follows that

$$g_i = (p-2)^{-1}\{\tau_{1i} + \ldots + \tau_{(i-1)i} + \tau_{i(i+1)} + \ldots + \tau_{ip} - (p-1)\bar{\tau}\}, 1 \le i \le p \tag{2.4}$$

However, the definition of general combining ability parameters, as contrasts among the τ_{ij} does not (and, should not) depend on whether the model has specific combining ability effects or not. Even in the presence of specific combining ability effects, with $\{g_i\}$ given by (2.4), (2.2) holds and it is not difficult to see from (2.1) that $\{s_{ij}\}$ must satisfy (2.3).

Let us arrange the crosses in the order $(1 \times 2), \ldots, (1 \times p), (2 \times 3), \ldots, (2 \times p), \ldots, (\overline{p-1} \times p)$. Let $\mathbf{g} = (g_1, \ldots g_p)'$ and let $\boldsymbol{\tau}$ and $\mathbf{s}$ be the $v \times 1$ vectors with elements $\{\tau_{ij}\}$ and $\{s_{ij}\}$ respectively. We now attempt to express the general and specific combining ability effects, i.e., $\mathbf{g}$ and $\mathbf{s}$, in terms or $\boldsymbol{\tau}$ explicitly. Suppose P is a $p \times v$ matrix with rows indexed by $1, \ldots, p$ and columns by the pairs $(i, j), 1 \le i < j \le p$, where the $\{t, (i, j)\}^{th}$ element of P is unity if $t \in (i, j)$ and zero otherwise. Then, one can verify that

$$PP' = (p-2)I_p + J_p, (PP')^{-1} = (p-2)^{-1}\left(I_p - \frac{1}{2(p-1)}J_p\right) \tag{2.5}$$

$$\mathbf{P1}_v = (p-1)\mathbf{1}_p, \mathbf{P}'\mathbf{1}_p = 2\mathbf{1}_v \tag{2.6}$$

where for positive integers α, β, I_α is the α^{th} order indentity matrix, $\mathbf{1}_\alpha$ is the $\alpha \times 1$ vector of all ones and $J_{\alpha\beta} = \mathbf{1}_\alpha \mathbf{1}'_\beta$; $\mathbf{J}_{\alpha\alpha}$ is simply denoted by J_α. Then, (2.1) can be written as

$$\tau = \bar{\tau}\mathbf{1}_v + P'\mathbf{g} + \mathbf{s} \tag{2.7}$$

where $$\mathbf{1}_p\mathbf{g} = 0, \; P\mathbf{s} = 0 \tag{2.8}$$

It is now not hard to see that

$$\mathbf{g} = H_1\boldsymbol{\tau}, \mathbf{s} = \boldsymbol{\tau} - \bar{\boldsymbol{\tau}}\,\mathbf{1}_v - P'\mathbf{g} = H_2\boldsymbol{\tau} \tag{2.9}$$

where the matrices H_1 and H_2 are given by

$$H_1 = (p-2)^{-1}(P - 2p^{-1} J_{pv}) \tag{2.10}$$

and $$H_2 = I_v - (p-2)^{-1}(P'P - 2(p-1)^{-1}J_v) \tag{2.11}$$

One can check easily that

$$\text{Rank}(H_1) = p-1, \text{Rank}(H_2) = v-p \tag{2.12}$$

Thus, **g** and **s** are represented by contrasts among the $\{\tau_{ij}\}$ that carry p – 1 and v – p degrees of freedom respectively. Also, contrasts representing **g** are orthogonal to those representing **s**. Another interesting fact is that when p = 3, $H_2 = 0$ and thus, s = 0. Therefore, when p = 3, all of the information is contained in the $\{g_i\}$ and hence, the optimal designs for this case are the same as those under a model with no specific combining ability.

3. Nested Designs and Optimality

In this section, we describe some optimal block designs for complete diallel cross experiments under a model with no specific combining ability effect. Let d be a block design for a complete diallel cross experiment involving p inbred lines and b blocks each of size $k(\geq 2)$. This means that there are k crosses in each of the blocks of d. Further, let r_{di} denote the number of times the i^{th} cross appears in d, i = 1, 2, ..., p(p – 1)/2, and similarly, let w_{dj} denote the number of times the j^{th} line occurs in crosses in the whole design $d, j = 1, 2, \ldots, p$. It is then easy to see that $\sum_{i=1}^{p(p-1)/2} r_{di} = bk$ and $\sum_{j=1}^{p} w_{dj} = 2bk$. We also let $n = bk$ denote the number of observations generated by d. For the data obtained from the design d, we postulate the model

$$\mathbf{Y} = \mu\mathbf{1}_n + \Delta_1\mathbf{g} + \Delta_2\boldsymbol{\beta} + \varepsilon \tag{3.1}$$

where $\mathbf{Y}$ is the $n \times 1$ vector of observed responses, μ is a general mean effect, $\mathbf{g}$ and $\boldsymbol{\beta}$ are vectors of p general combining ability effects and b block effects respectively, Δ_1, Δ_2 are the corresponding design matrices, that is, the (s, t)th element of Δ_1 is one if the sth observation pertains to the tth line, and is zero, otherwise; similarly (s, t)th element of Δ_2 is 1 if the sth observation comes from the tth block, and is zero, otherwise; ε is the vector of random error components, these components being distributed with mean zero and constant variance σ^2. In (3.1), the elements of ε take care of specific combining ability effects as well as unassignable variation. Under the model (3.1), it can be shown that the coefficient matrix of the reduced normal equations for estimating linear functions of general combining ability effects using a design d is

$$C_d = G_d - N_d N_d' / k \tag{3.2}$$

where $G_d = (g_{dii'})$, $N_d = (n_{dij})$, $g_{dii} = w_{di}$, and for $i \neq i'$, $g_{dii'}$ is the number of times the cross $(i \times i')$ appears in d; n_{dij} is the number of times the line i occurs in block j of d. The matrix C_d is often referred to as the information matrix of d.

A design d is called connected if and only if all elementary comparisons among general combining ability effects are estimable using d. A necessary and sufficient condition for a design d to be connected is that Rank $(C_d) = p - 1$. We denote by $\mathcal{D}(p, b, k)$, the class of all such connected block designs $\{d\}$ with p lines, b blocks each of size k. We then have

Theorem 3.1. For any design $d \in \mathcal{D}(p, b, k)$

$$\mathrm{tr}(C_d) \leq k^{-1} b \{2k(k - 1 - 2x) + px(x + 1)\}$$

where $x = \left[\frac{2k}{p}\right]$, [.] is the greatest integer function and for a square matrix A, tr(A) stands for its trace. Equality holds if and only if $n_{dij} = x$ or $x + 1$ for all $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, b$.

For a proof, see Das *et al.* [6]. Note that if $2k < p$ then $x = 0$ and in that case we have

$$\mathrm{tr}(C_d) \leq 2b(k - 1), d \in \mathcal{D}(p, b, k) \tag{3.3}$$

We now attempt to find universally optimal designs for diallel cross experiments. For more on the universal optimality criterion, see Kiefer [14]. Recall that a universally optimal design is in particular also A-optimal, that is, such a design minimizes the average variance of the best linear unbiased estimators of all elementary contrasts among the parameters of interest (the general combining ability effects, in the present context). Making an appeal to a result of Kiefer [14] and to Theorem 3.1, we have the following result.

Theorem 3.2. Let $d^* \in \mathcal{D}(p, b, k)$ be a block design for diallel crosses, and suppose C_{d*} satisfies

(i) $tr(C_{d*}) = k^{-1}b\{2k(k-1-2x) + px(x+1)\}$ and

(ii) C_{d*} is completely symmetric

Then d^* is universally optimal in $\mathcal{D}(p, b, k)$, and in particular minimizes the average variance of the best linear unbiased estimators of all elementary contrasts among the general combining ability effects.

There is an interesting connection between nested balanced incomplete block designs of Preece [18] and optimal designs for diallel crosses. For completeness, we recall the definition of a nested balanced incomplete block design.

Definition 3.1. A nested balanced incomplete block design with parameters $(v^*, b_1, k_1, r^*, \lambda_1, b_2, k_2, \lambda_2, m)$ is a design for v^* treatments, each replicated r^* times with two systems of blocks such that

(a) the second system is nested within the first, with each block from the first system, called henceforth as 'block' containing exactly m blocks from the second system, called hereafter as 'sub-blocks.

(b) ignoring the sub-blocks leaves a balanced incomplete block design with usual parameters $v^*, b_1, k_1, r^*, \lambda_1$

(c) ignoring the blocks leaves a balanced incomplete block design with parameters $v^*, b_2, k_2, r^*, \lambda_2$

Consider now a nested balanced incomplete block design d with parameters $v^* = p, b_1, k_1, k_2 = 2, r^*$. If we identify the treatments of d as lines of a diallel experiment and perform crosses among the lines appearing in the same sub-block of d (which is of size two), we get a block design d^* for a complete diallel experiment involving p lines, each cross being replicated $r = \frac{2b_2}{\{p(p-1)\}}$ times, and $b = b_1$ blocks, each of size $k = \frac{k_1}{2}$. Such a design $d^* \in \mathcal{D}(p, b, k)$; also, for such a design, $n_{d*ij} = 0$ or 1 for $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, b$ and

$$C_{d*} = (p-1)^{-1}\, 2b(k-1)(I_p - p^{-1} J_p) \tag{3.4}$$

Clearly C_{d*} given by (3.4) is completely symmetric and $tr(C_{d*}) = 2b(k-1)$ which equals the upper bound for $tr(C_d)$ given by (3.3). Thus, one concludes that the design d^* is universally optimal in $\mathcal{D}(p, b, k)$. It is also easy to see that using d*, each elementary contrast among general combining ability effects is estimated with a variance

$$\frac{(p-1)\sigma^2}{\{b(k-1)\}} \tag{3.5}$$

Further, if the nested balanced incomplete block design with parameters $v^* = p, b_1, k_1, \ b_2 = \frac{b_1 k_1}{2}, k_2 = 2$ is such that $\lambda_2 = 1$ or, equivalently, if

$$b_1 k_1 = p(p-1) \tag{3.6}$$

then the optimal design d^* for diallel crosses derived from this design has each cross replicated just once and hence uses the minimal number of experimental units. Summarising therefore we have the following result due to Das *et al.* [6].

Theorem 3.3. The existence of a nested balanced incomplete block design d with parameters $v^* = p, b_1 = b, b_2 = bk, k_1 = 2k, k_2 = 2$ implies the existence of a universally optimal incomplete block design d^* for complete diallel crosses. Further, if the parameters of d satisfy (3.6), then d^* has the minimal number of experimental units.

Instead of the design $d^* \in \mathcal{D}(p, b, k)$ based on the nested balanced incomplete block design d, if one adopts a randomized complete block design with $r = \frac{2bk}{\{p(p-1)\}}$ blocks, each block having all the $\frac{p(p-1)}{2}$ crosses, the C-matrix can easily be shown to be

$$C_R = r(p-2)(I_p - p^{-1} J_p)$$

so that the variance of the best linear unbiased estimator of any elementary contrast among the general combining ability effects is $\frac{2\sigma_1^2}{\{r(p-2)\}}$, where σ_1^2 is the per observation variance in the case of randomized block experiment. Thus the efficiency factor of the design $d^* \in \mathcal{D}(p, b, k)$, relative to a randomized complete block design is given by

$$e = \frac{2b(k-1)}{r(p-2)(p-1)} = \frac{p(k-1)}{k(p-2)} \tag{3.7}$$

Next, we describe some specific families of nested balanced incomplete block designs, leading to optimal block designs for diallel crosses. Gupta and Kageyama [13] obtained two such families of nested balanced incomplete block designs. These families, in our notation have the following parameters

Series 1: $v^* = p = 2t + 1 = b_1, b_2 = t(2t+1), k_1 = 2t, k_2 = 2$

Series 2: $v^* = p = 2t, b_1 = 2t - 1, b_2 = t(2t-1), k_1 = 2t, k_2 = 2$

It is easy to verify that the designs in Series 1 and 2 above satisfy (3.6) and hence use the minimal number of experimental units. Several other families of nested balanced incomplete block designs satisfying the condition (3.6) exist and can therefore be used to derive optimal designs for diallel crosses with minimal number of experimental units. As before, we denote the parameters of the design for complete diallel crosses by p, b, k, r where p is the number of lines, b, the number of blocks, k, the number of crosses per block or the block size and r, the number of times each cross is replicated in the design.

Family 1. Let $p = 4t + 1, t \geq 1$ be a prime or a prime power and x be a primitive element of the Galois field of order p, G F (p). Consider the t initial blocks

$$\{(x^i, x^{i+2t})(x^{i+t}, x^{i+3t})\}, i = 0, 1, 2, 3, \ldots, t-1$$

As shown by Dey *et al.* [8], these initial blocks, when developed in the sense of Bose [2], give rise to a nested balanced incomplete block design with parameters $v^* = p = 4t + 1, k_1 = 4, b_1 = t(4t + 1), k_2 = 2$. Using this design, one can get an optimal design for diallel crosses with. minimal number of experimental units and parameters $p = 4t + 1$, $b = t(4t + 1)$, $k = 2, r = 1$. It is interesting to note that this family of designs has the smallest block size, k = 2.

Example 3.1. Let t = 2 in Family 1. Then a nested balanced incomplete block design with parameters $v^* = p = 9, b_1 = 18, k_1 = 4, k_2 = 2, \lambda_2 = 1$ can be constructed by developing the following initial blocks over $GF(3^2)$

$$\{(1, 2), (2x + 1, x + 2)\}, \{(x, 2x), (2x + 2, x + 1)\}$$

where x is a primitive element of $GF(3^2)$ and the elements of $GF(3^2)$ are 0, 1, 2, x, x + 1, x + 2, 2x, 2x + 1, 2x + 2. Adding successively the non-zero elements of $GF(3^2)$ to the contents of the initial blocks, the full nested design is obtained. The design for diallel crosses is exhibited below, where the lines have been relabelled 1 through 9, using the correspondence $0 \to 1, 1 \to 2, 2, \to 3, x \to 4,$ $x + 1 \to 5, x + 2 \to 6, 2x \to 7,\ 2x + 1 \to 8, 2x + 2 \to 9$

$[2 \times 3, 6 \times 8]$	$[1 \times 3, 4 \times 9]$	$[1 \times 2, 5 \times 7]$	$[5 \times 6, 2 \times 9]$
$[4 \times 6, 3 \times 7]$	$[4 \times 5, 1 \times 8]$	$[8 \times 9, 3 \times 5]$	$[7 \times 9, 1 \times 6]$
$[7 \times 8, 2 \times 4]$	$[4 \times 7, 5 \times 9]$	$[5 \times 8, 6 \times 7]$	$[6 \times 9, 4 \times 8]$
$[1 \times 7, 3 \times 8]$	$[2 \times 8, 1 \times 9]$	$[3 \times 9, 2 \times 7]$	$[1 \times 4, 2 \times 6]$
$[2 \times 5, 3 \times 4]$	$[3 \times 6, 1 \times 5]$		

This is a design for a complete diallel cross experiment for p = 9 lines in 18 blocks each of size two; each cross appears in the design just once. Two designs for p = 9 have been reported by Gupta and Kageyama [12]; both these designs have blocks of size larger than two. Further, no nested design listed by Preece [18] leads to an optimal design for diallel crosses with p = 9 lines in blocks of size two.

Family 2. Let $p = 6t + 1,\ t \geq 1$ be a prime or a prime power and x be a primitive element of GF(p). Consider the initial blocks

$$\left\{\left(x^{i}, x^{i+3t}\right)\left(x^{i+t}, x^{i+4t}\right)\left(x^{i+2t}, x^{i+5t}\right)\right\}, i = 0, 1, 2, \ldots, t-1$$

Dey *et al.* [8] show that these initial blocks, when developed give a solution of a nested balanced incomplete block design with parameters $v^{*} = p = 6t + 1, b_1 = t(6t + 1), k_1 = 6, k_2 = 2, \lambda_2 = 1$. Hence, using this series of nested balanced incomplete block designs, we get a solution for an optimal design for diallel crosses with parameters $p = 6t + 1, b = t(6t + 1), k = 3, r = 1$.

Example 3.2. Let t = 2 in Family 2. Then a nested balanced incomplete block design with parameters $v^{*} = p = 13, b_1 = 26, k_1 = 6, k_2 = 2, \lambda_2 = 1$ is obtained by developing over GF(13) the following two initial blocks

$$\{(1, 12), (4, 9), (3, 10)\} \text{and} \{(2, 11), (8, 5), (6, 7)\}$$

Using this nested design, an optimal design for diallel crosses with minimal number of experimental units and parameters p = 13, b = 26, k = 3 can be constructed.

Family 3. Let $12t + 7, t \geq 0$ be a prime or a prime power and suppose x = 3 is a primitive element of $GF(12t + 7)$. Then, as shown by Dey *et al.* [8], one can get a nested balanced incomplete block design with parameters $v^{*} = p = 12t + 8$, $b_1 = (3t + 2)(12t + 7)$, $k_1 = 4, k_2 = 2$ by developing the following $3t + 2$ initial blocks

$$\left\{(1, \infty), \left(x^{3t+2}, x^{6t+3}\right)\right\}$$

$$\left\{\left(x^{i}, x^{i+3t+1}\right)\left(x^{i+3t+2}, x^{i+6t+3}\right)\right\}, i = 1, 2, \ldots, 3t+1$$

here ∞ is an invariant variety. Using this family of nested designs, one can get a family of optimal designs for diallel crosses with minimal number of experimental units and parameters $p = 12t + 8, b = (3t + 2)(12t + 7), k = 2, r = 1$.

The next family of nested designs has $\lambda_2 = 2$ and hence in the design for diallel crosses derived from this family, each cross is replicated twice. However, this family of designs is of practical utility as the optimal designs for diallel crosses derived from this family of nested balanced incomplete block designs have a block size, k = 2.

Family 4. Let $p = 2t + 1$, $t \geq 1$ be a prime or a prime power and x be a primitive element of GF(2t + 1). Then as shown by Dey *et al.* [8], a nested balanced incomplete block design with parameters $v^* = 2t + 1, b_1 = t(2t + 1)$, $k_1 = 4$, $k_2 = 2$, $\lambda_2 = 2$ can be constructed by developing the following initial blocks over $GF(2t + 1)$

$$\{(0, x^{i-1}), (x^i, x^{i+1})\}, i = 1, 2, \ldots, t$$

Using this family of nested designs, a family of optimal designs for diallel crosses with parameters $p = 2t + 1, b = t(2t + 1), k = 2 = r$ can be constructed.

In particular, for t = 3, 5 we get optimal designs for diallel crosses with parameters

$$p = 7, b = 21, k = 2 = r, \text{and } p = 11, b = 55, k = 2 = r$$

For these values of p, no designs with block size two are available in Gupta and Kageyama [13].

Example 3.3. Let t = 3 in Family 4. Then a nested balanced incomplete block design with parameters $v^* = p = 7$, $b_1 = 21, k_1 = 4$, $k_2 = 2$, $\lambda_2 = 2$ is obtained by developing over GF(7) the following three initial blocks

$$\{(0,1), (3,2)\}; \{(0,3), (2,6)\}; \{(0,2), (6,4)\}$$

Using this nested design, an optimal design for diallel crosses with parameters p = 7, b = 21, k = 2 = r can be constructed and is shown below

$[0 \times 1, 2 \times 3]$	$[0 \times 3, 2 \times 6]$	$[0 \times 2, 4 \times 6]$
$[1 \times 2, 3 \times 4]$	$[1 \times 4, 0 \times 3]$	$[1 \times 3, 0 \times 5]$
$[2 \times 3, 4 \times 5]$	$[2 \times 5, 1 \times 4]$	$[2 \times 4, 1 \times 6]$
$[3 \times 4, 5 \times 6]$	$[3 \times 6, 2 \times 5]$	$[3 \times 5, 0 \times 2]$
$[4 \times 5, 0 \times 6]$	$[0 \times 4, 3 \times 6]$	$[4 \times 6, 1 \times 3]$
$[5 \times 6, 0 \times 1]$	$[1 \times 5, 0 \times 4]$	$[0 \times 5, 2 \times 4]$
$[0 \times 6, 1 \times 2]$	$[2 \times 6, 1 \times 5]$	$[1 \times 6, 3 \times 5]$

Here the lines are numbered 0 through 6.

4. Optimal Designs Based on Triangular PBIB Designs

In this section, we show how triangular incomplete block designs (which are a type of partially balanced incomplete block designs with two associate classes) can be used to derive optimal incomplete block designs for diallel crosses. To begin with, we consider the same model as in the previous section, namely, (3.1). These results are due to Dey and Midha [9] and Das *et al.* [19]. Let us recall the definition of a triangular design.

Definition 4.1. A binary block design with $v = \frac{p(p-1)}{2}$ treatments and b blocks, each of size k is called a triangular design if

(i) each treatment is replicated r times

(ii) the treatments can be indexed by a set of two labels (i, j), i < j, i, j = 1, 2, ..., p; two treatments, say (α, β) and (γ, δ) occur together in λ_1 blocks if either $\alpha = \gamma, \beta \neq \delta$, or $\alpha \neq \gamma, \beta = \delta$ or $\alpha = \delta$, $\beta \neq \gamma$, or $\alpha \neq \delta, \beta = \gamma$; otherwise, they occur together in λ_2 blocks.

Now, we derive a block design $d \in \mathcal{D}(p, b, k)$ for diallel crosses from a triangular design d_1 with parameters $v = \frac{p(p-1)}{2}$, b, r, k, λ_1, λ_2, by replacing a treatment (i, j) in d_1 with the cross $(i \times j), i < j; i, j = 1, 2, \ldots, p$. Then, we have the following lemma due to Dey and Midha [9]. For notational convenience, we write n_{ij} for n_{dij} in Lemma 4.1.

Lemma 4.1. For the design *d*, the following are true

(i) $\sum_{l=1}^{b} n_{il} = r(p-1), \sum_{i=1}^{p} n_{il} = 2k$

(ii) $\sum_{l=1}^{b} n_{il}^2 = r(p-1) + (p-1)(p-2)$

(iii) $\sum_{l=1}^{b} n_{il} n_{i'l} = r + (p-2)\{3\lambda_1 + (p-3)\lambda_2\}, i \neq i'; i, i' = 1, 2, \ldots, p$

Using Lemma 4.1, it can be shown that

$$C_d = \theta(I_p - p^{-1} J_p) \tag{4.1}$$

where $\theta = pk^{-1}\{r(k-1) - (p-2)\lambda_1\}$. Therefore, using the design d, any elementary comparison among general combining ability effects is estimated with a variance $\frac{2\sigma^2}{\theta}$, and the efficiency factor of the design relative to a randomized complete block design is $\frac{\theta}{\{r(p-2)\}}$. Further, from (4.1), it follows that

$$tr(C_d) = k^{-1}p(p-1)\{r(k-1) - (p-2)\lambda_1\} \tag{4.2}$$

Also, as shown in Theorem 3.1, for any design in $\mathcal{D}(p, b, k)$, the trace of the C-matrix is bounded above by

$$k^{-1}b\{2k(k-1-2x)+px(x+1)\} \tag{4.3}$$

where $x = \left[\frac{2k}{p}\right]$. Equating (4.2) and (4.3), we have the following result.

Theorem 4.1. A block design for complete diallel crosses derived from a triangular design with parameters $v = \frac{p(p-1)}{2}, b, r, k, \lambda_1, \lambda_2$ is universally optimal over $\mathcal{D}(p, b, k)$ if

$$p(p-1)(p-2)\lambda_1 = bx\{4k - p(x+1)\} \tag{4.4}$$

where $x = \left[\frac{2k}{p}\right]$. Further, when the condition in (4.4) holds, the efficiency factor is given by

$$e = p\frac{\{2k(k-1-2x)+px(x+1)\}}{\{2k^2(p-2)\}} \tag{4.5}$$

We now have the following result due to Das *et al.* [5].

Lemma 4.2. For a triangular design with parameters $v = \frac{p(p-1)}{2}, b, r, k, \lambda_1 = 0, \lambda_2$ the inequality $2k \le p$ holds.

It follows from Lemma 4.2 that for a triangular design with $\lambda_1 = 0$, $x = \left[\frac{2k}{p}\right] = 0$ if $2k < p$ and $4k - p(x+1) = 0$ if $2k = p$. Hence for triangular designs with $\lambda_1 = 0$, the condition in (4.4) is always satisfied. Thus we have the following corollary to Theorem 4.1, obtained by Dey and Midha [9].

Corollary 4.1. A triangular design with parameters $v = \frac{p(p-1)}{2}$, b, r, k, $\lambda_1 = 0, \lambda_2$ leads to a universally optimal design for diallel crosses.

Several other designs, not satisfying the condition of the above corollary are also optimal, as these satisfy the condition (4.4). Table 1 in Das *et al.* [5] lists all such optimal designs, derivable from triangular designs with at most 10 replications; see also Table 1 in Dey and Midha [9] in this connection.

The results of this section thus far are derived under a model with no specific combining abilities. Now, suppose that the model includes specific combining ability effects as well. The interest of the experimenter may still be in estimating optimally contrasts among general combining ability effects, but in the presence of specific combining ability effects, Chai and Mukerjee [3] have shown that diallel cross designs derived from triangular designs satisfying (4.4) remain optimal for the general combining ability effects even when the specific combining ability effects are included in the model. Thus, the findings in Das *et al.* [5] and Dey and Midha [9], on triangular designs satisfying (4.4) remain robust under a model including specific combining ability effects. Turning to the issue of optimality for the specific combining ability effects themselves, we have the following result due to Chai and Mukerjee [3].

Theorem 4.2. Let d_0 be a triangular design with usual parameters $v = \binom{p}{2}$, b r, k, $\lambda_1 > 0, \lambda_2 = 0$. Then d_0 is connected and is universally optimal in $\mathcal{D}(p, b, k)$ for any complete set of orthonormal contrasts representing the specific combining ability effects.

5. *Optimal Partial Diallel Crosses*

With a large number of lines p, a complete diallel cross may become prohibitively large and the use of a partial diallel cross is necessitated. The choice of a partial diallel cross plan has received considerable attention in the literature (see e.g., Arya [1], Hinkelmann and Kempthorne [15], Singh and Hinkelmann [20]), However, even in the unblocked situation, the issue of finding an optimal partial diallel cross plan has received relatively less attention. Having chosen an optimal partial diallel cross plan, further blocking of the crosses might be necessary to control the error and the literature on this aspect is also scanty. In this section, some aspects of finding optimal partial diallel cross plans and their blocking are discussed.

Throughout this section, the model considered is the one in (3.1). To begin with, we consider an unblocked situation and thus, the block parameters in (3.1) are assumed to be absent. Subsequently, results under a model with block effects are discussed.

Suppose p = mn where $m \geq 2, n \geq 3$ are integers. Let us partition the set {1, ..., p} into m mutually exclusive and exhaustive subsets $S_1, \ldots, S_m$ each having n elements. Let

$$d^* = \{(i \times j) : 1 \leq i < j \leq p \text{ and } i, j \in S_u \text{ for some } u\} \tag{5.1}$$

If $\mathcal{D}(N, p)$ denotes the class of all N-observation partial diallel cross plans with $N < \binom{p}{2}$, then clearly, $d^* \in \mathcal{D}(N, p)$, where $N = \frac{1}{2}mn(n-1)$. Mukerjee [17] proved the following result.

Theorem 5.1. For each $m \geq 2$ and $n \geq 3$, the plan d^* is uniquely (up to isomorphism) E-optimal in $\mathcal{D}(N, p)$ where $N = \frac{1}{2} mn(n-1)$. Furthermore, the plan d^* is uniquely D- and A-optimal in $\mathcal{D}(N, p)$ for $n = 3$.

Recall that an E-optimal plan is one which, over a relevant class of competing plans, minimizes the maximum possible variance of a normalized contrast among the general combining ability effects. Similarly, a D- or, A-optimal plan can be interpreted in the present context. Mukerjee [17] also showed that for $n \geq 4$, the D- and A-efficiencies of d^* are at least as large as e_D and e_A where

$$e_D = \frac{\left[\{2(n-1)\}^{m-1}(n-2)^{m(n-1)}\right]^{\frac{1}{v-1}}}{\lambda_0}$$

$$e_A = \frac{\lambda_0^{-1}(v-1)}{\frac{1}{2}(n-1)^{-1}(m-1)+(n-2)^{-1}m(n-1)}$$

and $\lambda_0 = \frac{(n-1)(p-2)}{(p-1)}$. Even when judged by these conservative bounds, d^* is seen to perform satisfactorily under the D- and A-criteria in a large number of cases. Thus, for instance, among the 29 pairs (m, n) satisfying $m \geq 2, n \geq 4, p = mn \leq 30$, there are 24 cases for which e_D exceeds 0.90 and 15 for which $e_D \geq 0.95$; similarly in 19 cases, $e_A \geq 0.90$ and in 10 cases, $e_A \geq 0.95$. From these numerical results, one might like to conjecture that d^* is indeed D- and A-optimal in $\mathcal{D}(N, p)$ for all $n \geq 3$. However, this conjecture remains to be proved and the proof of its truth (or, its falsity) appears to be highly non-trivial.

We now turn to the question of optimal and efficient blocking of partial diallel cross plans. Let $\overline{C}_d$ be the information matrix of a plan d under a model that does not include block effects and C_d be as in (3.2). Furthermore, let $\mathcal{D}(b, k, p)$ denote the class of partial diallel cross plans involving p parental lines and bk experimental units, arranged in b blocks of size k each. Then, we have the following result due to Gupta *et al.* [12].

Lemma 5.1. For any $d \in \mathcal{D}(b, k, p)$, $\overline{C}_d - C_d$ is nonnegative definite and, $\overline{C}_d = C_d$ if and only if the condition

$$N_d = b^{-1} w_d 1_b' \tag{5.2}$$

holds, where $w_d = (w_{d1}, \ldots, w_{dp})'$.

This lemma has useful implications in the construction of optimal or, efficient block designs for partial cross plans. Mukerjee [17] gives two classes of E-optimal block designs for the following two cases : (i) $n \geq 5$ is odd and (ii) $n \geq 4$ is even. We do not provide details here and refer the interested reader to the original source. For some more results on partial diallel cross designs, see Singh and Hinkelmann [21] and Das *et al.* [6].

REFERENCES

[1] Arya, A.S. (1983). Circulant plans for partial diallel crosses. *Biometrics*, **39**, 43-52.

[2] Bose, R.C. (1939). On the construction of balanced incomplete block designs. *Ann. Eugen.*, **9**, 353-399.

[3] Chai, F.S. and Mukerjee, R. (1999). Optimal designs for diallel crosses with specific combining abilities. *Biometrika*, **86**, 453-458.

[4] Curnow, R.N. (1963). Sampling the diallel cross. *Biometrics*, **19**, 287-306.

[5] Das, A., Dean, A.M. and Gupta, S. (1998). On optimality of some partial diallel cross designs. *Sankhya*, **B60**, 511-524.

[6] Das, A., Dey, A. and Dean, A.M. (1998). Optimal designs for diallel cross experiments. *Statist. Probab. Lett.*, **36**, 427-436.

[7] Das, M.N. and Giri, N.C. (1986). *Design and Analysis of Experiments*. Wiley Eastern, New Delhi.

[8] Dey, A., Das, U.S. and Banerjee, A.K. (1986). Construction of nested balanced incomplete block designs. *Calcutta Statist. Assoc. Bull.*, **35**, 161-167.

[9] Dey, A. and Midha, C.K. (1996). Optimal block designs for diallel crosses. *Biometrika*, **83**, 484-489.

[10] Ghosh, D.K. and Divecha, J. (1997). Two-associate class partially balanced incomplete block designs and partial diallel crosses. *Biometrika*, **84**, 245-248.

[11] Griffing, B. (1956). Concepts of general and specific combining ability in relation to diallel crossing systems. *Austral. J. Biol. Sci.*, **9**, 463-493.

[12] Gupta, S., Das, A. and Kageyama, S. (1994). Single replicate orthogonal block designs for circulant partial diallel crosses. *Commun. Statist.*, **A24**, 2601-2607.

[13] Gupta, S. and Kageyama, S. (1994). Optimal complete diallel crosses. *Biometrika*, **81**, 420-424.

[14] Hinkelmann, K. (1975). Design of genetical experiments. In: *A Survey of Statistical Design and Linear Models* (ed. J.N. Srivastava), 243-269, North Holland, Amsterdam.

[15] Hinkelmann, K. and Kempthorne, O. (1963). Two classes of group divisible partial diallel crosses. *Biometrika*, **50**, 281-291.

[16] Kiefer, J. (1975). Construction and optimality of generalized Youden designs. In : *A survey of Statistical Design and Linear Models* (ed. J.N. Srivastava), 333-353, North Holland, Amsterdam.

[17] Mukerjee, R. (1997). Optimal partial diallel crosses. *Biometrika*, **84**, 939-948.

[18] Preece, D.A. (1967). Nested balanced incomplete block designs. *Biometrika*, **54**, 479-486.

[19] Sharma, M.K. (1998). Partial diallel crosses through circular designs. *Jour. Ind. Soc. Agril. Stat.*, **51(1)**, 17-27.

[20] Singh, M. and Hinkelmann, K. (1990). On generation of efficient partial diallel crosses. *Biom. J.*, **32**, 177-187.

[21] Singh, M. and Hinkelmann, K. (1995). Partial diallel crosses in incomplete blocks. *Biometrics*, **51**, 1302-1314.