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Estimation of Error Variance for a Three Way Layout in ANOVA Model-II using Preliminary Test of Significance

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SUMMARY

In the present paper, an estimator of the error variance for a three-way layout in random effects model incorporating two PTS has been proposed. Expressions for bias and MSE of the proposed estimator have been derived and partial checks have been made. Some theoretical results have been established. It has been observed that the proposed estimator dominates unbiased estimator of error variance in certain range of nuisance parameters. Further a comparison of its performance in the bias and MSE with the estimator proposed by Singh and Gupta [5] reveals that, the proposed estimator is better than the earlier one. Recommendations regarding its applications have been attempted.

Key words : Preliminary test of significance, Error variance, Synthesis of mean squares, Bias, Mean square error, Relative efficiency, Nuisance parameter.

1. Introduction

Suppose that a agricultural equipment(s) producing concern is producing some small parts to be used in the equipments say sprayer, etc. The parts are being produced, using a large number of machines of same make and model. The concern may be interested in getting an answer to the question : 'Is there any difference between the machines?' Since the total number of machines in use is very large, it is not possible to make such a study by taking samples of output of all machines. Therefore, keeping this and other related problems in mind, the following experiment is performed :

A random sample of I machines from the lot of machines and J workers from the totality of workers has been selected independently. Each worker is assigned to work on a machine for one day. A random sample of K batches of materials produced by each worker on a machine from the total output is selected. Since for any machine or worker or batch of material, there may be considerable variation, we will treat the output as though it is a continuous random variable.

Let Y_{ijkl} denotes the 1^{th} observation in the k^{th} batch of material produced by j^{th} worker if he uses i^{th} machine. The sample observations can well be represented by a complete three-way layout, designating machines as factor A, workers as factor B and batches as factor C. Thus, we can assume that

$$Y_{ijkl} = \mu + a_i^A + a_j^B + a_k^C + a_{ij}^{AB} + a_{jk}^{BC} + a_{ik}^{AC} + a_{ijk}^{ABC} + e_{ijkl} \qquad (1.1)$$

$$i = 1, ..., I; j = 1, ..., J; k = 1, ... K; l = 1, ..., L$$

The random variables a_i^A are uncorrelated and have N $(0, \sigma_A^2)$ distribution. Similarly a_j^B have N $(0, \sigma_B^2)$, ... a_{ijk}^{ABC} have N $(0, \sigma_{ABC}^2)$ distributions. The errors e_{ijkl} are independently and identically distributed with mean zero and variance σ_e^2. We are interested in testing the main hypothesis $H_A : \sigma_A^2 = 0$ against the alternative $H'_A : \sigma_A^2 > 0$, i.e. we are interested in examining whether there is any significant difference between the machines from which these I machines have been drawn at random beyond their variation from 1^{th} batch to another or in their use by different workers. We note that to test H_A no exact test is available unless we assume that either of the two, two factor interactions are zero. Singh and Gupta [5] have proposed an estimate of error variance assuming that the interaction AC is zero and AB may or may not be zero. Singh, Singh and Ali [6] have addressed to the problem of estimation of error variance in a mixed ANOVA model using two preliminary tests of significance.

We have proposed an estimate of error variance when the interactions AC and AB may or may not be present. Then (1.1) can be written as :

$$Y_{ijkl} = \mu + a_i^A + a_j^B + a_k^C + a_{ij}^{AB} + a_{jk}^{BC} + a_{ik}^{AC} + a_{ijk}^{ABC} + e_{ijkl}$$

$$\text{if } \sigma_{AC}^2 > 0 \text{ and } \sigma_{AB}^2 > 0$$

$$Y_{ijkl} = \mu + a_i^A + a_j^B + a_k^C + + a_{jk}^{BC} + a_{ik}^{AC} + a_{ijk}^{ABC} + e_{ijkl}$$

$$\text{if } \sigma_{AC}^2 > 0 \text{ and } \sigma_{AB}^2 = 0 \qquad (1.2)$$

$$Y_{ijkl} = \mu + a_i^A + a_j^B + a_k^C + a_{ij}^{AB} + a_{jk}^{BC} + + a_{ijk}^{ABC} + e_{ijkl}$$

$$\text{if } \sigma_{AC}^2 = 0 \text{ and } \sigma_{AB}^2 > 0$$

$$Y_{ijkl} = \mu + a_i^A + a_j^B + a_k^C + a_{jk}^{BC} + + a_{ijk}^{ABC} + e_{ijkl}$$

$$\text{if } \sigma_{AC}^2 = 0 \text{ and } \sigma_{AB}^2 = 0$$

A portion of the analysis of variance resulting from the above model is given in Table 1.1.

Table 1.1. Analysis of variance for a three-way layout in ANOVA Model-II

Source of Variation	Degrees of Freedom	Mean Squares	Expected Mean Squares
A	$n_4 = (I-1)$	V_4	$\sigma_4^2 = \sigma_e^2 + L\,\sigma_{ABC}^2 + KL\sigma_{AB}^2 + JL\sigma_{AC}^2 + JKL\sigma_A^2$
AB	$n_3 = (I-1)(J-1)$	V_3	$\sigma_3^2 = \sigma_e^2 + L\,\sigma_{ABC}^2 + KL\sigma_{AB}^2$
AC	$n_2 = (I-1)(K-1)$	V_2	$\sigma_2^2 = \sigma_e^2 + L\,\sigma_{ABC}^2 + JL\sigma_{AC}^2$
ABC	$n_1 = (I-1)(J-1)(K-1)$	V_1	$\sigma_1^2 = \sigma_e^2 + L\,\sigma_{ABC}^2$

The mean squares V_i's ($i = 1, 2, 3$, and 4) are independently distributed as $\sigma_i^2\,\chi_i^2/n_i$, where χ_i^2 is a central chi-square statistic with n_i degrees of freedom.

If we can assume that either $\sigma_{AC}^2 = 0$ or $\sigma_{AB}^2 = 0$, then an estimate of error variance is V_2 or V_3 respectively. However, if $\sigma_{AC}^2 \geq 0$ and $\sigma_{AB}^2 \geq 0$, then for an estimate of the error variance use :

$$V = \begin{cases} V_{123} \text{ if } \dfrac{V_2}{V_1} < F(n_2, n_1, \alpha_1) \text{ and } \dfrac{V_3}{V_{12}} < F(n_3, n_{12}, \alpha_2) \\ V_2 \text{ if } \dfrac{V_2}{V_1} \geq F(n_2, n_1, \alpha_1) \text{ and } \dfrac{V_3}{V_1} < F(n_3, n_1, \alpha_3) \\ V_3 \text{ if } \dfrac{V_2}{V_1} < F(n_2, n_1, \alpha_1) \text{ and } \dfrac{V_3}{V_{12}} \geq F(n_3, n_{12}, \alpha_2) \\ V_A \text{ if } \dfrac{V_2}{V_1} \geq F(n_2, n_1, \alpha_1) \text{ and } \dfrac{V_3}{V_1} \geq F(n_3, n_1, \alpha_3) \end{cases} \tag{1.3}$$

where

$$V_{123} = \frac{n_1V_1 + n_2V_2 + n_3V_3}{n_1 + n_2 + n_3} \quad .$$

$$V_{12} = \frac{n_1V_1 + n_2V_2}{n_1 + n_2}$$

$$V_A = V_3 + V_2 - V_1,\ n_{ijk} = n_i + n_j + n_k$$

2. *Bias of V*

Bias is defined as $B = E(V) - (\sigma_3^2 + \sigma_2^2 - \sigma_1^2)$. Since the sum of squares $n_i V_i / \sigma_i^2$, $i = 1, 2, 3$ are independently distributed as central χ_i^2 statistics with n_i degrees of freedom, the joint probability density function of V_1, V_2, V_3 is given by

$$f(V_1, V_2, V_3) = K_1 V_1^{\frac{n_1}{2}-1} V_2^{\frac{n_2}{2}-1} V_3^{\frac{n_3}{2}-1} \exp\left[-\frac{1}{2}\left\{\frac{n_1 V_1}{\sigma_1^2} + \frac{n_2 V_2}{\sigma_2^2} + \frac{n_3 V_3}{\sigma_3^2}\right\}\right] dV_1\, dV_2\, dV_3 \quad (2.1)$$

where

$$K_1 = \frac{\left(\frac{n_1}{\sigma_1^2}\right)^{\frac{n_1}{2}} \left(\frac{n_2}{\sigma_2^2}\right)^{\frac{n_2}{2}} \left(\frac{n_3}{\sigma_3^2}\right)^{\frac{n_3}{2}}}{2^{\frac{n_{123}}{2}} \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right) \Gamma\left(\frac{n_3}{2}\right)}$$

making the following transformations in (2.1)

$$u_1 = \frac{n_1 V_1}{\sigma_1^2},\ u_2 = \frac{n_2 V_2}{n_1 V_1}\theta_{12},\ u_3 = \frac{n_3 V_3}{n_1 V_1}\theta_{13} \quad (2.2)$$

where $\theta_{13} = \frac{\sigma_1^2}{\sigma_3^2}$, $\theta_{12} = \frac{\sigma_1^2}{\sigma_2^2}$

The joint probability density function of u_1, u_2 and u_3 is given by

$$g(u_1, u_2, u_3) = K_2 u_1^{\frac{n_{123}}{2}-1} u_2^{\frac{n_2}{2}-1} u_3^{\frac{n_3}{2}-1} \exp\left[-\frac{1}{2}u_1 (1 + u_2 + u_3)\right] du_1\, du_2\, du_3 \quad (2.3)$$

where

$$K_2 = \left[2^{\frac{n_{123}}{2}} \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right) \Gamma\left(\frac{n_3}{2}\right)\right]^{-1}$$

The expected value of V is given by

$$E(V) = E_1 P(R_1 \,\&\, R'_1) + E_2 P(R_2 \,\&\, R) + E_3 P(R_1 \,\&\, R'_2) + E_4 P(R_2 \,\&\, R') \quad (2.4)$$

$$E(V) = E_1 [V = V_{123}/R_1 \& R'_1] + E_2 [V = V_2/R_2 \& R] + E_3 [V = V_3/R_1 \& R'_2] + E_4 [V = V_A / R_4 \& R'] \quad (2.5)$$

where

$P_1 = P(R_1 \& R'_1)$ $P_2 = P(R_2 \& R)$ $P_3 = P(R_1 \& R'_2)$ $P_4 = P(R_2 \& R')$

$R_1 : V_2 / V_1 < \beta'$, $R'_1 : V_3 / V_{12} < \delta'$, $R_2 : V_2 / V_1 \geq \beta'$, $R'_2 : V_3 / V_{12} \geq \delta'$

$R : V_3 / V_1 < \delta'_1$, $R' : V_3 / V_1 \geq \delta'_1$, $\beta' = \frac{n_2}{n_1} \theta_{12} \beta$, $\delta' = \frac{n_3}{n_{12}} \theta_{23} \delta$

$\delta'_1 = \frac{n_3}{n_1} \theta_{13} \delta_1$, $\beta = F(n_2, n_1; \alpha_1)$, $\delta = F(n_3, n_{12}; \alpha_2)$, $\delta_1 = F(n_3, n_1; \alpha_3)$

are the $F(m, n; \alpha)$ 100% points of F- distribution with (m, n) degrees of freedom, at α% level of significance.

$$E_1 = \frac{K_2}{P_1} \int_{u_1 = 0}^{\infty} \int_{u_2 = 0}^{\beta'} \int_{u_3 = 0}^{\delta'(\theta_{12} + u_2)} \left(\frac{n_1 V_1 + n_2 V_2 + n_3 V_3}{n_{123}} \right) g(u_1, u_2, u_3)\, du_1\, du_2\, du_3 \quad (2.6)$$

or $\quad E_1 = A_1 + A_2 + A_3$

Integrating (2.6) and using standard results of integration we have :

$$A_1 = \frac{K_2}{P_1} \frac{\sigma_1^2}{n_{123}} \int_{u_1 = 0}^{\infty} \int_{u_2 = 0}^{\beta'} \int_{u_3 = 0}^{\delta'(\theta_{12} + u_2)} u_1^{\frac{n_{123}}{2}} u_2^{\frac{n_2}{2} - 1} u_3^{\frac{n_3}{2} - 1} \exp\left\{ -\frac{u_1}{2} (1 + u_2 + u_3) \right\} du_1\, du_2\, du_3$$

which integrates out as

$$A_1 = \frac{K_2}{P_1} \frac{\sigma_1^2}{n_{123}} 2^{\frac{n_{123}}{2} + 1} \Gamma\left(\frac{n_{123}}{2} + 1 \right) \sum_{i = 0}^{\frac{n_3}{2} - 1} \frac{(-1)^i \binom{\frac{n_3}{2} - 1}{i}}{\left(\frac{n_{12}}{2} + i + 1 \right)} \left[B_{X_0}\left(\frac{n_2}{2}, \frac{n_1}{2} + 1 \right) - \sum_{j = 0}^{i} \binom{i}{j} \frac{B_{Y_0}(q, p + 1)}{(1 + \delta' \theta_{12})^{p + 1} (1 + \delta')^q} \right]$$

where $p = \frac{n_1}{2} + i - j,\ q = \frac{n_2}{2} + j$

Similarly A_2 and A_3 can be integrated. Again, integrating out the other expectations in (2.5), finally we get, E(V) expressed as a fraction of σ_1^2 as :

$$\frac{E(V)}{\sigma_1^2} = \frac{n_1}{n_{123}} I_{X_0}\left(\frac{n_2}{2}, \frac{n_1}{2}+1\right) + \frac{1}{\theta_{12}}\left[1 - \frac{n_{13}}{n_{123}} I_{X_0}\left(\frac{n_2}{2}+1, \frac{n_1}{2}\right)\right]$$

$$+ \frac{1}{\theta_{13}} \frac{n_3}{n_{123}} I_{X_0}\left(\frac{n_2}{2}, \frac{n_1}{2}\right) - \frac{1}{B\left(\frac{n_2}{2}, \frac{n_1}{2}+1\right) B\left(\frac{n_3}{2}, \frac{n_{12}}{2}+1\right)} \sum_{i=0}^{\frac{n_3}{2}-1} \frac{(-1)^i \binom{\frac{n_3}{2}-1}{i}}{\left(\frac{n_{12}}{2}+i+1\right)}$$

$$\sum_{j=0}^{i} \binom{i}{j} \left[\frac{n_1}{n_{123}} \frac{B_{Y_0}(q, p+1)}{(1+\delta'\theta_{12})^{p+1}(1+\delta')^q} + \frac{\{B(q, p+1) - B_{M_0}(q, p+1)\}}{(1+\delta'_1)^{p+1}}\right]$$

$$- \frac{1}{\theta_{12}} \frac{1}{B\left(\frac{n_2}{2}+1, \frac{n_1}{2}\right) B\left(\frac{n_3}{2}, \frac{n_{12}}{2}+1\right)} \frac{n_2}{n_{123}} \sum_{i=0}^{\frac{n_3}{2}-1} \frac{(-1)^i \binom{\frac{n_3}{2}-1}{i}}{\left(\frac{n_{12}}{2}+i+1\right)}$$

$$\sum_{j=0}^{i} \binom{i}{j} \left[\frac{B_{Y_0}(q+1, p)}{(1+\delta'\theta_{12})^{p}(1+\delta')^{q+1}}\right] + \frac{1}{\theta_{13}} \frac{1}{B\left(\frac{n_2}{2}, \frac{n_1}{2}\right) B\left(\frac{n_3}{2}+1, \frac{n_{12}}{2}\right)}$$

$$\sum_{i=0}^{\frac{n_3}{2}} \frac{(-1)^i \binom{\frac{n_3}{2}}{i}}{\left(\frac{n_{12}}{2}+i\right)} \sum_{j=0}^{i} \binom{i}{j}$$

$$\left[\frac{n_{12}}{n_{123}} \frac{B_{Y_0}(q, p)}{(1+\delta'\theta_{12})^{p}(1+\delta')^q} + \frac{\{B(q, p) - B_{M_0}(q, p)\}}{(1+\delta'_1)^{p}}\right]$$

where

$$X_0 = \frac{\beta'}{1+\beta'},\quad Y_0 = \frac{(1+\delta')\beta'}{(1+\delta'\theta_{12})(1+\delta')\beta'},\quad M_0 = \frac{\beta'}{1+\delta'_1+\beta'}$$

Numerical values of bias for the data set considered in Section 6 have been assembled in Tables 1-4.

Special Cases : As a partial check on this result we let $\beta = 0$ and $\delta_1 = 0$ i.e. we always reject both the hypotheses $H_1 : \sigma^2_{AC} > 0$ and $H'_1 : \sigma^2_{AB} > 0$ and we use V_A as an estimator of σ^2. In this case $X_0 = 0$ and $Y_0 = 0$, hence E(V) reduces to : $\sigma_3^2 + \sigma_2^2 - \sigma_1^2$.

Again if we take the limits $B \to \infty$ and $\delta \to \infty$, i.e., we never reject both the hypotheses then from E(V) we observe that

$$E(V) = E(V_{123}) = \frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2}{n_{123}},$$ as it should be, because V_{123} is an unbiased estimator of σ^2 in this case.

3. Mean Square Error of V

The mean square error of the variance estimator V is given by the following relation :

$$MSE(V) = E(V^2) - 2E(V)(\sigma_3^2 + \sigma_2^2 - \sigma_1^2) + (\sigma_3^3 + \sigma_2^2 - \sigma_1^2)^2$$

in order to evaluate the mean square error of V we are required to find out $E(V^2)$ for this purpose we follow the same method which was used for deriving E(V). The final expression for the MSE is not included to save space.

4. Mathematical Results

Result 4.1 : For a given set of degrees of freedom and $\theta_{13} = 1$ i.e., $\sigma_1^2 = \sigma_3^2$ and $\theta_{12} = 1$ i.e., $\sigma_1^2 = \sigma_2^2$ the mean value of V expressed as a fraction of σ_1^2 lies between $(1 - \alpha_1)(1 - \alpha_2)$ and $1 + (1 - \alpha_1) + (1 - \alpha_3) + (1 - \alpha_1)(1 - \alpha_2)$.

Corollary : For a given set of degrees of freedom and $\theta_{13} = 1$ i.e., $\sigma_1^2 = \sigma_3^2$ and $\theta_{12} = 1$ i.e., $\sigma_1^2 = \sigma_2^2$ the bias of V expressed as a fraction of σ_1^2 lies between $(1 - \alpha_1)(1 - \alpha_2) - 1$ and $(1 - \alpha_1) + (1 - \alpha_3) + (1 - \alpha_1)(1 - \alpha_2)$.

Result 4.2 : For a given set of degrees of freedom and $\theta_{13} = 1$ i.e., $\sigma_1^2 = \sigma_3^2$ and $\theta_{12} = 1$ i.e., $\sigma_1^2 = \sigma_2^2$ the MSE of V expressed as a fraction of σ_1^4 has an upper bound

$$\left[2 + 2\left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right) + \left(1 + \frac{2}{n_3}\right)(1 - \alpha_1) + \left(1 + \frac{2}{n_2}\right)(1 - \alpha_3) - \left(1 - \frac{2}{n_{123}}\right)(1 - \alpha_1)(1 - \alpha_2)\right]$$

5. *Relative Efficiency*

Since V and V_A are rival estimators of σ^2 even though V, is in general, a biased estimator and V_A is an unbiased estimator of σ^2, it seems more appropriate to talk of the relative efficiency of V to V_A. We define the relative efficiency as follows :

$$\text{R.E.} = \frac{2\left(\frac{1}{n_1} + \frac{1}{n_2\,\theta_{12}^2} + \frac{1}{n_3\,\theta_{13}^2}\right)}{\frac{\text{MSE(V)}}{\sigma_1^4}}$$

For the data set considered in the next section, numerical values of these relative efficiencies have been assembled in Tables 5-8.

6. *Recommendations*

To study the behaviour of bias and relative efficiency of V, we have considered two sets of values of degrees of freedom viz., $n_1 = 6,\ n_2 = 3,\ n_3 = 6$ and $n_1 = 24,\ n_2 = 12,\ n_3 = 4$. The results for $\alpha = .10$ and $.25$ for different values of θ_{13} and θ_{12}, ranging from. .1(.1) 1.0 where $\theta_{13} = \frac{\sigma_1^2}{\sigma_3^2}$ and $\theta_{12} = \frac{\sigma_1^2}{\sigma_2^2}$ are summarised in tables given in the appendix.

Tables 1-4 show the bias in V expressed as a fraction of σ_1^2 for the two sets of degrees of freedom. For the first data set considered here at $\alpha = 10\%$ the bias is negative whenever $\theta_{12} \leq 8$ and $\theta_{12} \leq 6$ respectively over the whole range of θ_{13} *i.e.*, $\theta_{13} = .1(.1)1.0$. At $\alpha = 25\%$ it remains negative for $\theta_{12} \leq .2$ and $\theta_{13} \leq .3$.

For the second data set at $\alpha = 10\%$ the ranges of nuisance parameters change as $\theta_{12} \leq .6$, $\theta_{13} \geq .8$. Finally, at $\alpha = 25\%$ bias is negative for $\theta_{13} = .1$ and $\theta_{12} = .3$ and .4. Further for the remaining ranges of θ_{12}, θ_{13} the estimator is positively biased for both the data sets.

Referring Tables 5-8, relative efficiency of V increases as α increases. For $\alpha = 25\%$ V is always more efficient than V_A for all values of θ_{12} and $\theta_{13,}$ the gain in efficiency will be more if n_1 is taken large.

The values within parentheses in Tables 5-8 are those obtained by Singh and Gupta [5] for their estimator of error variance. A comparison of the values of relative efficiencies assembled in the tables of relative efficiencies indicates that our estimator performs better than that proposed by Singh and Gupta [5] and the unbiased estimator V_A for different values of level of significance and for the whole range of θ_{12}, considered here. It is recommended to use V for different values of α's considered here, for the first data set $\alpha = 25\%$ should be taken for whole range of the nuisance parameters .1(.1)1.0 and for the second data set in particular when $\theta_{13} \geq .7$, α should be taken 10%, however for $\theta_{13} \leq .7$ it is recommended to take $\alpha = 25\%$.

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APPENDIX

Table 1. Bias of the variance estimator V as a fraction of σ_1^2, $n_1 = 6$, $n_2 = 3$, $n_3 = 6$, $\alpha = .10$

θ_{12} \ θ_{13}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	–.324	–.476	–.521	–.491	–.426	–.349	–.271	–.198	–.131	–.071
0.2	–.391	–.508	–.547	–.525	–.472	–.408	–.343	–.280	–.222	–.169
0.3	–.366	–.461	–.495	–.479	–.436	–.382	–.325	–.270	–.218	–.171
0.4	–.298	–.378	–.411	–.400	–.364	–.316	–.266	–.216	–.170	–.126
0.5	–.215	–.285	–.316	–.309	–.279	–.237	–.192	–.146	–.103	–.063
0.6	–.130	–.193	–.223	–.220	–.194	–.156	–.115	–.073	–.033	.004
0.7	–.049	–.106	–.137	–.135	–.113	–.080	–.041	–.002	.035	.071
0.8	.025	–.028	–.058	–.058	–.039	–.008	.027	.063	.099	.133
0.9	.093	.043	.014	.009	.027	.055	.089	.123	.158	.190
1.0	.045	.077	.066	.068	.086	.113	.144	.178	.211	.242

Table 2. Bias of the variance estimator V as a fraction of σ_1^2, $n_1 = 6$, $n_2 = 3$, $n_3 = 6$, $\alpha = .25$

θ_{12} \ θ_{13}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	–.007	–.015	–.009	.012	.044	.082	.122	.163	.203	.242
0.2	–.006	–.002	–.008	.019	.048	.082	.119	.157	.194	.230
0.3	.034	.024	.026	.041	.066	.098	.132	.167	.201	.234
0.4	.070	.060	.059	.072	.095	.124	.155	.188	.220	.251
0.5	.111	.100	.098	.109	.129	.156	.185	.215	.245	.274
0.6	.153	.141	.139	.148	.166	.190	.218	.246	.275	.302
0.7	.194	.183	.179	.187	.203	.226	.251	.278	.305	.332
0.8	.234	.223	.219	.225	.240	.261	.285	.311	.336	.362
0.9	.273	.261	.256	.262	.275	.295	.317	.342	.367	.391
1.0	.308	.298	.292	.296	.309	.327	.349	.372	.396	.420

Table 3. Bias of the variance estimator V as a fraction of σ_1^2, $n_1 = 24$, $n_2 = 12$, $n_3 = 4$
$\alpha = .10$

θ_{12} \ θ_{13}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	–.035	–.082	–.107	–.105	–.082	.002	.013	.030	.103	.154
0.2	–.053	–.102	–.126	–.124	–.100	.001	.011	.029	.087	.137
0.3	–.102	–.147	–.170	–.165	–.140	–.002	.007	.025	.049	.100
0.4	–.146	–.190	–.210	–.204	–.178	–.003	.006	.024	.013	.063
0.5	–.163	–.205	–.224	–.218	–.192	.002	.013	.030	–.003	.046
0.6	–.149	–.189	–.208	–.202	–.178	.019	.029	.046	.004	.053
0.7	–.110	–.148	–.167	–.163	–.141	.046	.056	.072	.033	.080
0.8	–.055	–.092	–.111	–.109	–.089	.081	.090	.105	.075	.121
0.9	.005	–.029	–.049	–.048	–.031	.121	.130	.144	.124	.167
1.0	.0068	.035	.015	.014	.029	.164	.172	.186	.174	.216

Table 4. Bias of the variance estimator V as a fraction of σ_1^2, n_1= 24, n_2= 12, n_3= 4

$\alpha = .10$

θ_{12} \ θ_{13}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.002	.013	.030	.055	.084	.117	.151	.187	.222	.257
0.2	.001	.011	.029	.053	.082	.115	.150	.185	.221	.256
0.3	−.002	.007	.025	.050	.079	.111	.146	.181	.216	.251
0.4	−.003	.006	.024	.048	.077	.109	.143	.178	.213	.247
0.5	.002	.001	.030	.053	.082	.113	.146	.181	.215	.249
0.6	.019	.002	.046	.069	.096	.127	.159	.192	.226	.258
0.7	.046	.005	.072	.094	.120	.149	.181	.213	.245	.277
0.8	.081	.090	.105	.126	.152	.180	.210	.241	.272	.303
0.9	.121	.130	.144	.164	.188	.215	.244	.274	.304	.334
1.0	.164	.172	.186	.205	.228	.253	.281	.310	.339	.367

Table 5. Relative Efficiency of the variance estimator V to V_A, σ_1^2, n_1= 6, n_2= 3, n_3= 6

$\alpha = .10$

θ_{12} \ θ_{13}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.958	.954	.963	.969	.972	.972	.972	.971	.969	.968
0.2	.951	.924	.937	.953	.962	.966	.967	.965	.962	.959
0.3	.969	.939	.957	.987	1.009	1.022	1.027	1.028	1.025	1.021
0.4	.985	.970	1.000	1.050	1.095	1.126	1.144	1.151	1.152	1.149
0.5	.997	.996	1.043	1.120	1.196	1.257	1.298	1.322	1.332	1.332
0.6	1.003	1.015	1.077	1.179	1.292	1.393	1.470	1.520	1.547	1.558
0.7	1.007	1.027	1.098	1.220	1.368	1.512	1.633	1.720	1.774	1.801
0.8	1.009	1.032	1.109	1.243	1.416	1.599	1.764	1.893	1.979	2.025
0.9	.972	1.021	1.106	1.264	1.446	1.652	1.851	2.015	2.127	2.194
1.0	1.081	1.072	1.131	1.263	1.452	1.672	1.891	2.078	2.213	2.290

Table 6. Relative Efficiency of the variance estimator V to V_A, n_1= 6, n_2= 3, n_3= 6

$\alpha = .25$

θ_{12} \ θ_{13}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.999	.999	1.000	1.001	1.002	1.002	1.002	1.002	1.001	1.000
0.2	1.002	1.005	1.010	1.015	1.018	1.019	1.019	1.017	1.015	1.013
0.3	1.006	1.017	1.031	1.044	1.053	1.058	1.059	1.057	1.054	1.050
0.4	1.010	1.029	1.055	1.082	1.102	1.115	1.120	1.121	1.117	1.111
0.5	1.011	1.038	1.077	1.119	1.155	1.181	1.195	1.200	1.199	1.192
0.6	1.012	1.043	1.092	1.150	1.204	1.246	1.274	1.287	1.289	1.283
0.7	1.013	1.045	1.100	1.171	1.242	1.302	1.345	1.369	1.377	1.374
0.8	1.013	1.046	1.104	1.182	1.267	1.343	1.401	1.437	1.453	1.453
0.9	1.012	1.044	1.103	1.185	1.279	1.367	1.437	1.484	1.508	1.511
1.0	1.011	1.042	1.099	1.182	1.280	1.375	1.455	1.510	1.538	1.544

Table 7 Relative Efficiency of the variance estimator V to V_A, $n_1 = 24$, $n_2 = 12$, $n_3 = 4$

$\alpha = .10$

θ_{12} \ θ_{13}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.991	.982	.985	.994	1.003	1.010	1.015	1.017	1.017	1.017
0.2	.987	.961	.955	.970	.993	1.015	1.031	1.041	1.044	1.043
0.3	.984	.949	.932	.942	.970	1.003	1.032	1.052	1.061	1.062
0.4	.985	.949	.929	.938	.971	1.015	1.058	1.092	1.113	1.119
0.5	.987	.956	.940	.956	.999	1.060	1.126	1.185	1.227	1.249
0.6	.990	.965	.957	.982	1.040	1.123	1.220	1.315	1.394	1.444
0.7	.992	.973	.972	1.006	1.077	1.182	1.311	1.448	1.573	1.664
0.8	.994	.980	.984	1.023	1.104	1.224	1.377	1.548	1.715	1.846
0.9	.995	.984	.990	1.032	1.116	1.242	1.406	1.594	1.781	1.932
1.0	.996	.986	.993	1.034	1.116	1.240	1.401	1.586	1.769	1.911

Table 8. Bias of the variance estimator V to V_A, $n_1 = 24$, $n_2 = 12$, $n_3 = 4$, $\alpha = .25$

θ_{12} \ θ_{13}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	1.001 (1.000)	1.004	1.008	1.012	1.014 (1.001)	1.015	1.016	1.015	1.014	1.013 (1.001)
0.2	1.001 (1.004)	1.007	1.018	1.031	1.042 (1.009)	1.049	1.053	1.053	1.051	1.047 (1.009)
0.3	1.001 (1.008)	1.007	1.023	1.044	1.066 (1.033)	1.083	1.095	1.100	1.098	1.091 (1.033)
0.4	1.001 (1.011)	1.009	1.028	1.055	1.087 (1.063)	1.117	1.140	1.153	1.156	1.149 (1.073)
0.5	1.002 (1.014)	1.011	1.034	1.068	1.110 (1.104)	1.153	1.190	1.215	1.226	1.222 (1.130)
0.6	1.002 (1.015)	1.014	1.040	1.081	1.132 (1.146)	1.188	1.239	1.279	1.302	1.305 (1.200)
0.7	1.003 (1.015)	1.016	1.045	1.090	1.149 (1.183)	1.215	1.279	1.333	1.368	1.379 (1.275)
0.8	1.003 (1.015)	1.018	1.048	1.095	1.158 (1.208)	1.229	1.302	1.365	1.408	1.424 (1.345)
0.9	1.003 (1.014)	1.018	1.048	1.096	1.158 (1.219)	1.231	1.304	1.369	1.413	1.429 (1.397)
1.0	1.003 (1.033)	1.017	1.046	1.091	1.151 (1.216)	1.219	1.289	1.348	1.386	1.396 (1.428)

The bracket values refer to the values computed by Singh and Gupta [5].