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On Totally Balanced Change-over Designs

Cini Varghese and V.K. Sharma
Indian Agricultural Statistics Research Institute, New Delhi- 110012
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SUMMARY

In this paper a series of change-over designs balanced for the first order residual effects of treatments has been proposed. These designs use v experimental units for $2v - 1$ periods for v treatments and estimate any contrast between direct effects and also between residual effects with the same variance. Efficiency of these designs has been compared with some of the existing designs and their robustness against the last v ($1 \leq v \leq 2v - 2$) missing observations on an experimental unit has been studied.

Key words : Change-over designs, Direct effect, Residual effect, Cumulative effect, Totally balanced, Efficiency, Robustness.

1. Introduction

A change-over design (COD), permitting the estimation of first order residual effects, is called *balanced*, if the variance of any estimated elementary contrast among the direct effects is constant, say α, and the variance of any estimated elementary contrast among the residual effects is also constant, say, β. The constants α and β may not be equal. However, if $\alpha = \beta$, the design is called *totally balanced* (Dey and Balachandran, [2]). Dey and Balachandran [2] constructed a class of balanced COD from a series of BIB designs by using a pre-period. Sharma [5] gave a class of change-over designs balanced for first residuals which require v experimental units for 2v periods, if v treatments are to be tested. These designs estimate direct and residual effects with nearly the same precision and can be advantageously used in the situations wherein the experimental units are relatively scarce or expensive.

Here, we propose a series of totally-balanced change-over designs considering the first order residual effects of treatments and compare their efficiencies with some of the existing designs. The robustness of these designs against last v ($1 \leq v \leq 2v - 2$) missing observations on an experimental unit has also been examined.

2. Construction

For constructing a totally balanced design for v treatments in v units and $(2v-1)$ periods, we proceed as follows :

Denote the v treatments under test by 0, 1, 2, ... v - 1. We form a sequence of 2v elements by interlacing the elements of the sequence $\{0, 1, 2, 3, \ldots, v-1\}$ with the elements of its reverse $\{v-1, v-2, \ldots, 2, 1, 0\}$, viz.

$$\{0, v-1, 1, v-2, 2, v-3, \ldots, v-2, 1, v-1, 0\}$$

Delete one of the two middle terms having the same value either $\frac{v}{2}$ or $\frac{(v-1)}{2}$ according as v is even or odd. The sequence thus formed is called the initial sequence. On developing this sequence mod (v), we get an array of v rows and $2v-1$ columns. If the rows of this array represent the experimental units and columns the periods, we get a design with $(2v-1)$ periods and v experimental units. However, this design is not variance-balanced. In order to make this design variance-balanced, a pre-period is added with treatments exactly the same as those in the last period. Thus the required change-over design [COD $(v, 2v-1, v)$] is a pre-period design with $2v-1$ periods and v experimental units for v treatments.

An interesting feature of this design is that it is symmetrical about the middle v^{th} period and is seen to be, in Section 4, totally balanced.

3. Illustration

Let v = 4. Then the number of periods, $p = 2v-1 = 7$ and the number of experimental units, $s = v = 4$. The initial sequence is $\{0, 3,1,2,1,3,0\}$. On developing this sequence mod (4) and adding a pre-period with treatments (0, 1, 2, 3), we get the following totally balanced COD (4, 7, 4) :

		Preiods							
	›	0	1	2	3	4	5	6	7
Units	1	0	0	3	1	2	1	3	0
	2	1	1	0	2	3	2	0	1
	3	2	2	1	3	0	3	1	2
	4	3	3	2	0	1	0	2	3

4. Analysis

Let Y_{hijk} represents the observation from the k^{th} experimental unit in the h^{th} period when treatment i is applied to it and is proceeded by the j^{th} treatment

in $(h-1)^{th}$ period. The linear fixed effects model for the COD $(v, 2v-1, v)$ can be written as :

$$Y_{hijk} = \mu + \pi_h + \psi_k + \tau_i + \rho_j + \varepsilon_{hijk}$$

$$(h = 1, 2 \ldots, 2v-1;\ i, j, k = 1, 2, \ldots v) \qquad (4.1)$$

where μ, π_h, ψ_k, τ_i, and ρ_j represent the general mean, effect of the h^{th} period, effect of the k^{th} unit, direct effect of treatment i and residual effect of the treatment j, respectively. ε_{hijk} are random errors assumed to be identically and independently distributed with $N(0, \sigma^2)$.

Let G = total of all observations

P_h = total of observations recorded in the h^{th} period

T_i = total of observations receiving treatment i

R_i = total of observations immediately following treatment i

S_k = total of observations recorded on the k^{th} experimental unit

Minimization of the residual sum of squares

$$\sum_{h,k} (Y_{hijk} - \mu - \pi_h - \tau_i - \rho_j - \psi_k)^2 \qquad (4.2)$$

with respect to the parameters yields a set of normal equations which under the restrictions

$$\Sigma \pi_h = \Sigma \tau_i = \Sigma \rho_j = \Sigma \psi_k = 0 \qquad (4.3)$$

can be solved to give :

$$\hat{\mu} = \frac{G}{v(2v-1)} \qquad (4.4)$$

$$\hat{\pi}_h = \frac{1}{v} P_h - \frac{G}{v(2v-1)} \qquad (4.5)$$

$$\hat{\tau}_i = \frac{(2v-2)T'_i + R'_i}{2v(2v-3)} \qquad (4.6)$$

$$\hat{\rho}_i = \frac{T'_i + (2v-2)R'_i}{2v(2v-3)} \qquad (4.7)$$

with $T'_i = T_i + \frac{S_i}{2v-1} - \frac{2G}{2v-1}$ and $R'_i = R_i + \frac{S_{(i)}}{2v-1} - \frac{2G}{2v-1}$

Here, $S_{(i)}$ denotes the total of observations recorded on the experimental unit that receives treatment i in the middle v^{th} period.

The variances of the estimated elementary contrasts between direct, residual and cumulative effects (cumulative effect = direct effect + residual effect) of treatments are :

$$V(\hat{\tau}_i - \hat{\tau}_m) = V(\hat{\rho}_i - \hat{\rho}_m) = \frac{4(v-1)}{2v(2v-3)}\sigma^2 \quad (4.8)$$

and

$$Cov(\hat{\tau}_i - \hat{\tau}_m, \hat{\rho}_i - \hat{\rho}_m) = \frac{1}{2v(2v-3)}\sigma^2 \quad (4.9)$$

$$V(\hat{\tau}_i + \hat{\rho}_i - \hat{\tau}_m - \hat{\rho}_m) = \frac{4v-3}{v(2v-3)}\sigma^2 \quad (4.10)$$

It can be noticed that the estimated contrasts among direct and residual effects of treatments have the same variance. Partitioning of degrees of freedom and total sum of squares for the balanced COD (v, 2v – 1, v) is given in Table-1. In this design, direct effects are non-orthogonal to the residual effects,

Table 1. Partitioning of total sum of squares

Source	d.f.	Sum of Squares
Periods	2 (v – 1)	$\sum_h \frac{P_h^2}{v} - \frac{G^2}{v(2v-1)}$
Units	(v – 1)	$\sum_k \frac{S_k^2}{2v-1} - \frac{G^2}{v(2v-1)}$
Direct effects ignor. residual effects	(v – 1)	$\frac{(2v-1)}{4v(v-1)} \sum_i T'^2_i$
Residual effects elim. direct effects	(v – 1)	$\frac{v(2v-3)}{(v-1)} \sum_i \hat{\rho}_i^2$
or		
Residual effects ignor. direct effects	(v – 1)	$\frac{(2v-1)}{4v(v-1)} \sum_i R'^2_i$
Direct effects elim. residual effects	(v – 1)	$\frac{v(2v-3)}{(v-1)} \sum_i \hat{\tau}_i^2$
Error	2(v – 1)(v – 2)	By subtraction
Total	v(2v – 1) – 1	$\sum_h \sum_k Y^2_{hijk} - \frac{G^2}{v(2v-1)}$

therefore, the total treatment sum of squares $\Sigma \hat{\tau}_i T'_i + \Sigma \hat{\rho}_i R'_i$ with $2(v-1)$ d.f. has been partitioned into two-ways as shown in Table 1.

5. An Illustration

The analysis of the totally balanced COD is illustrated through synthetic data on milk production of dairy cows, for the COD with parameters $v = 5$, $p = 9$ and $n = 5$. Each period is of 5 weeks duration and the data given are the average productions of fat-corrected milk in kilograms.

Table 2. Fat-corrected milk (kg) of dairy cows

		Units				
		1	2	3	4	5
Pre-period	0	(0) 9.76	(1) 16.42	(2) 10.63	(3) 16.48	(4) 7.66
	1	(0) 12.45	(1) 15.49	(2) 7.92	(3) 14.20	(4) 10. 47
	2	(4) 11.85	(0) 9.80	(1) 14.33	(2) 6.71	(3) 16.42
	3	(1) 16.25	(2) 9.68	(3) 17.33	(4) 8.54	(0) 8.03
	4	(3) 20.77	(4) 9.98	(0) 7.62	(1) 12.60	(2) 12.26
Periods	5	(2) 12.42	(3) 19.83	(4) 10.25	(0) 11.30	(1) 18.49
	6	(3) 19.03	(4) 9.87	(0) 8.32	(1) 16.66	(2) 11.56
	7	(1) 15.14	(2) 10.55	(3) 15.92	(4) 9.11	(0) 8.59
	8	(4) 13.11	(0) 9.95	(1) 13.90	(2) 9.48	(3) 20.18
	9	(0) 9.18	(1) 17. 94	(2) 10.40	(3) 14.39	(4) 8.94

Figures in parentheses represent the treatment numbers.

Here

$G = 567.21$

$T_0 = 85.24, T_1 = 140.80, T_2 = 90.98, T_3 = 158.07, T_4 = 92.12$

$R_0 = 127.19, R_1 = 118.87, R_2 = 112.22, R_3 = 96.83, R_4 = 113.10$

$S_1 = 130.20, S_2 = 113.09, S_3 = 105.99, S_4 = 102.99, S_5 = 114.94$

$P_1 = 60.53, P_2 = 59.11, P_3 = 59.83, P_4 = 63.23, P_5 = 72.29$

$P_6 = 65.44, P_7 = 59.31, P_8 = 66.62, P_9 = 60.85$

$S_{(0)} = 102.99, S_{(1)} = 114.94, S_{(2)} = 130.20, S_{(3)} = 113.09, S_{(4)} = 105.99$

Also $T'_0 = -29.3633, T'_1 = 27.5244, T'_2 = -20.6000, T'_3 = 44.5889$

$T'_4 = -22.1500$

and $R'_0 = 12.5867, R'_1 = 5.5944, R'_2 = -0.3600, R'_3 = -16.6511$
$R'_4 = -1.1700$

Solutions for various effects from (4) to (7) are :

$\hat{\mu} = 12.6047$

$\hat{\pi}_1 = -0.4987, \hat{\pi}_2 = -0.7827, \hat{\pi}_3 = -0.6387, \hat{\pi}_4 = 0.0413, \hat{\pi}_5 = 1.8533$
$\hat{\pi}_6 = 0.4833, \hat{\pi}_7 = -0.7427, \hat{\pi}_8 = 0.7193, \hat{\pi}_9 = -0.4347$

$\hat{\tau}_0 = -3.1760, \hat{\tau}_1 = 3.2256, \hat{\tau}_2 = -2.3594, \hat{\tau}_3 = 4.8580, \hat{\tau}_4 = -2.5481$

$\hat{\rho}_0 = 1.0190, \hat{\rho}_1 = 1.0326, \hat{\rho}_2 = -0.3354, \hat{\rho}_3 = -1.2660, \hat{\rho}_4 = -0.4501$

The analysis of variance is given in Table 3.

Table 3. Analysis of variance for the COD (5, 9, 5)

Source	d.f.	S.S.	M.S.	F
Periods	8	30.9861	3.8733	4.6419
Units	4	49.7750	12.4438	14.9133
Direct effects ignor. residual effects	4	508.8321	127.2080	
Residual effects elim. direct effects	4	35.1965	8.7991	10.5454**
or				
Residual effects ignor. direct effects	4	52.7040	13.1760	
Direct effects elim. residual effects	4	491.3246	122.8312	147.2078**
Error	24	20.0258	0.8344	
Total	44	644.8155	12.12	

** indicate significant at 1% level.

Estimates of the variances of the estimated elementary contrasts between direct effects and also between residual effects are :

$$\hat{V}(\hat{\tau}_i - \hat{\tau}_m) = \hat{V}(\hat{\rho}_i - \hat{\rho}_m) = 0.1907$$

Also, $$\hat{Cov}(\hat{\tau}_i - \hat{\tau}_m, \hat{\rho}_i - \hat{\rho}_m) = 0.0119$$

and $$\hat{V}(\hat{\tau}_i + \hat{\rho}_i - \hat{\tau}_m - \hat{\rho}_m) = 0.4053$$

6. Comparison with Some Existing Designs

The efficiency of the proposed designs has been compared with some of the existing designs having more or less the same number of observations or periods like (a) William's designs [6], (b) designs by Sharma [5], and (c) the class of designs studied by Quenouille [4], Berenblut [1] and Patterson [3].

As the number of observations vary from design to design, efficiency of the proposed design relative to the existing design is given by

$$E = \frac{n_1 V_1}{n_2 V_2} \tag{4.11}$$

where n_1 and V_1 are the total number of observations and variance of estimated elementary treatment contrasts for the existing design and n_2 and V_2 have the same meaning for the proposed design.

$n_1 = v^2$ or $2v^2$ according as v is even or odd, for design (a)

$= 2v^2$ for design (b)

$= 2v^3$ for design (c) and

$n_2 = v(2v-1)$ for the present design.

Table 4. Efficiency of the proposed design compared to existing designs

Efficiencies for	t	3	4	5	6	7	8	9
Direct effects	(a)	1.1244	1.0476	1.0262	1.0172	1.0120	1.0093	1.0072
	(b)	0.9310	0.9702	0.9825	0.9894	0.9929	0.9950	0.9961
	(c)	0.8996	0.9524	0.9718	0.9820	0.9873	0.9907	0.9928
Residual effects	(a)	2.0240	1.5238	1.3499	1.2627	1.2120	1.1740	1.1489
	(b)	1.1559	1.1286	1.1040	1.0876	1.0748	1.0656	1.0579
	(c)	1.1169	1.1083	1.0923	1.0794	1.0691	1.0613	1.0546
Cumulative effects	(a)	2.0000	1.5385	1.3724	1.2855	1.2318	1.1955	1.1687
	(b)	1.1000	1.0989	1.0865	1.0752	1.0660	1.0589	1.0529
	(c)	0.8967	0.9511	0.9716	0.9813	0.9868	0.9904	0.9925

Table 4 shows that the proposed change-over design is more efficient than the William's design for estimation of all the treatment effects. If the interest of the experimenter centers on the estimation of residual effects of treatments then the suggested design is better than the designs (b) and (c).

7. Robustness Against Missing Observations

Robustness of the proposed COD $(v, 2v-1, v)$ against last v $(1 \leq v \leq 2v-2)$ missing observations on an experimental unit has been examined using the efficiency criterion. If $\mathbf{C_d}$ is the information matrix of the original design **d** and $\mathbf{C_{d^*}}$ is that of the residual design **d*** after the missing

observations, then the efficiency of the residual design relative to the original is given by

$$E = \frac{\text{Harmonic mean of nonzero eigen values of } C_{d*}}{\text{Harmonic means of nonzero eigen values of } C_d}$$

These efficiencies have been worked out for practical range of $v (\leq 9)$ and are given in Table 5.

Table 5. Efficiencies of residual designs relative to the original design

S.No.	Parameters			No. of missing observation v	Efficiencies for		
	v	p	s		direct effects	residual effects	direct and residual together
1	3	5	3	1	1.000	0.8889	0.8889
				2	0.8889	0.6400	0.6809
				3	0.7273	0.3137	0.3678
				4	0.4444	0.2500	0.2500
2	4	7	4	1	1.0000	0.9364	0.9364
				2	0.9364	0.8825	0.8825
				3	0.8825	0.8017	0.8825
				4	0.8290	0.6772	0.8150
				5	0.7393	0.5798	0.6042
				6	0.6307	0.5730	0.5730
3	5	9	5	1	0.9636	0.9636	0.9635
				2	0.9305	0.9305	0.9305
				3	0.9011	0.9011	0.9011
				4	0.8733	0.8610	0.8671
				5	0.8321	0.8059	0.8188
				6	0.7797	0.7530	0.7661
				7	0.7368	0.7066	0.7214
				8	0.7054	0.7054	0.7054
4	6	11	6	1	0.9769	0.9769	0.9769
				2	0.9552	0.9552	0.9552
				3	0.9351	0.9351	0.9351
				4	0.9166	0.9166	0.9166
				5	0.8991	0.8926	0.8959
				6	0.8750	0.8619	0.8684
				7	0.8455	0.8270	0.8385
				8	0.8182	0.8026	0.8103
				9	0.7940	0.7744	0.7840
				10	0.7740	0.7740	0.7740

5	7	13	7	1	0.9842	0.9842	0.9842
				2	0.9690	0.9690	0.9690
				3	0.9546	0.9546	0.9546
				4	0.9410	0.9410	0.9410
				5	0.9283	0.9283	0.9283
				6	0.9162	0.9123	0.9142
				7	0.9002	0.8927	0.8964
				8	0.8813	0.8733	0.8773
				9	0.8632	0.8543	0.8587
				10	0.8460	0.8356	0.8408
				11	0.8301	0.8164	0.8232
				12	0.8162	0.8162	0.8162
6	8	15	8	1	0.9885	0.9885	0.9885
				2	0.9774	0.9774	0.9774
				3	0.9667	0.9667	0.9667
				4	0.9564	0.9564	0.9564
				5	0.9465	0.9465	0.9465
				6	0.9372	0.9372	0.9372
				7	0.9283	0.9283	0.9270
				8	0.9168	0.9122	0.9145
				9	0.9037	0.8987	0.9012
				10	0.8909	0.8854	0.8881
				11	0.8785	0.8723	0.8754
				12	0.8665	0.8591	0.8628
				13	0.8552	0.8451	0.8501
				14	0.8450	0.8450	0.8450
7	9	17	9	1	0.9913	0.9913	0.9913
				2	0.9828	0.9828	0.9828
				3	0.9746	0.9746	0.9746
				4	0.9665	0.9665	0.9665
				5	0.9588	0.9588	0.9588
				6	0.9514	0.9514	0.9513
				7	0.9442	0.9442	0.9442
				8	0.9373	0.9356	0.9364
				9	0.9287	0.9256	0.9272
				10	0.9190	0.9157	0.9174
				11	0.9095	0.9059	0.9077
				12	0.9002	0.8962	0.8982
				13	0.8911	0.8865	0.8888
				14	0.8822	0.8766	0.8794
				15	0.8738	0.8660	0.8699
				16	0.8659	0.8659	0.8659

Table 5 shows that for a small number of treatments, the design is robust for small number of missing observations at the end of the sequence. However, if the number of treatments exceeds 6, the design remains robust against missing of last $2v-2$ observations on a unit.

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