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Pella-Tomlinson Nonlinear Statistical Model with Autocorrelated Errors

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SUMMARY

Linearized versions of Schaefer and Fox nonlinear models are extensively used for efficient fishery management. In this paper, generalization of their nonlinear versions, viz. Pella-Tomlinson model is studied in detail. As an illustration, this model with autocorrelated errors is applied to some catch-effort data by employing nonlinear estimation procedures. Finally, the maximum sustainable yield and the corresponding optimum effort to reap the same are also worked out.

Key words : Pella-Tomlinson model, Autocorrelated errors, Nonlinear estimation procedure, Maximum sustainable yield.

1. Introduction

Although 'nonlinear statistical models' play a very important role in almost all agricultural or biological disciplines, fisheries science can easily be singularly identified for their extensive applications due to the existence of wide population fluctuations and complex nonlinear inter-relationships among variables of interest. As fish is a renewable resource, it is of paramount importance to have an idea of sustainable yield in any fishery. This has generally been done on the basis of 'maximum sustainable yield' (MSY). Barber [1] cited its utility as a formal management objective, its simplicity and ability to be understood by the fishing industry, administrators, and managers.

Fish stocks do not automatically and invariably renew themselves simply because fish is called a renewable resource. Over-exploitation has led to commercial extinction of a number of fish stocks such as Japanese sardine, Peruvian anchovy, North sea herring. This tendency is also evident in our country where the spotted seerfish stock, which has been the commercial mainstay for fishermen in Palk Bay collapsed due to uncontrolled fishing. The

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same trend is visible in respect of prawn where catch per unit effort is declining. Devaraj and Martosubroto [2] recently brought out Proceedings of a Working Party in which various fish species are identified that have been either over- or under-exploited in our waters. In all such studies, Schaefer and Fox models are extensively used. The purpose of the present paper is to study the Pella-Tomlinson model, which is a generalization of these models. The situation where errors are autocorrelated is also considered along with an illustration for the same.

2. *Pella-Tomlinson Model*

Let N(t) be the fish biomass at time t, and let r, K denote respectively the intrinsic growth rate and carrying capacity. If E represents fishing effort, Pella-Tomlinson model is given by the differential equation:

$$\frac{dN}{dt} = \frac{r}{m} N \left[1 - \left(\frac{N}{K} \right)^{m} \right] - EN \tag{1}$$

where m is a parameter. Equation (1) can be solved exactly and so N (t) can be expressed as a nonlinear function of the parameters. If E = 0, this reduces to the well-known Richards model, which is quite often used in agriculture for describing the dynamics of a population (see e.g. Seber and Wild [7]). Further, (1) reduces to the well-known Schaefer, Fox, and hyperbolic models when m = 1, 0, −1 respectively. It appears that, for the application of Pella-Tomlinson model, a pre-requisite is the knowledge of population biomass N(t) for various values of t, which certainly is an extremely cumbersome task. However, this is not so and mere knowledge of catch and effort time-series data, which is readily available for most fisheries, is sufficient to estimate MSY, as shown below.

Assume that the fish population is in equilibrium, i.e. dN/dt = 0. Then, from (1), the equilibrium biomass N^* satisfies the relation

$$\frac{r}{m} N^* \left[1 - \left(\frac{N^*}{K} \right)^{m} \right] - EN^* = 0$$

which implies that

$$N^* = K(1 - mE/r)^{1/m}, \quad E < r/m \tag{2}$$

Therefore, the equilibrium catch is

$$Y = EN^* = KE(1 - mE/r)^{1/m}, \quad E < r/m \tag{3}$$

Thus the catch-effort relationship in the equilibrium situation is nonlinear. However, for Schaefer model (i.e. m = 1) this reduces to

$$Y = KE(1 - E/r), \quad E < r \tag{4}$$

which is a parabola. It may be noted that relationship between catch per unit effort (Y/E) and effort (E) is linear. Similarly, for Fox model (i.e. m = 0), the catch-effort relationship, using (3), is

$$Y = KEe^{-E/r} \tag{5}$$

Here again the relationship between catch per unit effort and effort can be converted to linearity on using the 'logarithmic transformation.' Getting rid of the nonlinearity for Schaefer or Fox model has, on one hand, increased tremendously the frequency of application of these models while, on the other, this has also led to a lot of confusion, as the methodology followed is statistically incorrect. In fact, all these models require 'nonlinear estimation procedures' for fitting catch-effort relationship [5].

Optimum Effort and Maximum Sustainable Yield:

The optimum effort E^* which maximizes the catch can be obtained by first determining dY/dE from (3) and then solving the equation dY/dE = 0 for which $d^2Y/dE^2 < 0$. Now from (3)

$$\frac{dY}{dE} = K\left(1 - \frac{mE}{r}\right)^{\frac{1}{m} - 1}\left[1 - \frac{(m+1)E}{r}\right] \tag{6}$$

Thus, the optimum effort is

$$E^* = r/(m+1), \quad m \neq -1 \tag{7}$$

Substituting this value of effort in (3), the maximum sustainable yield is

$$MSY = Kr/(m+1)^{1+1/m}, \quad m \neq -1 \tag{8}$$

3. Estimation of Parameters

In order to estimate the parameters of catch-effort curve given by (3), the usual procedure is to add an error term ε on the R.H.S. The main assumption is that ε's are independent. In view of the nonlinearity, it is imperative to apply 'Nonlinear estimation procedures' for fitting purposes. (see e.g. Seber and Wild [7]). Subsequently, it is desirable to carry out 'Residual analysis' in order to examine whether the assumption of independence of error term is or is not violated. This can be done by applying the well-known 'Run test.'

Autocorrelated Errors:

In the former situation, Durbin-Watson test on the residuals e_t (Lewis-Beck [3]) given by

$$d = \sum_{t=2}^{n} (e_t - e_{t-1})^2 / \sum_{t=1}^{n} e_t^2 \tag{9}$$

may be used to detect the presence of autocorrelation of order one. Positive, little, or negative autocorrelation is indicated according as d is near 0, 2, or 4. Then the error term can be expressed as

$$e_t = \phi e_{t-1} + \xi_t \quad , \ |\phi| < 1 \tag{10}$$

where ϕ is the AR parameter and ξ's are i.i.d. The estimation of parameters for (3) with above error structure can be carried out by using the Gauss method (see e.g. Seber and Wild [7]).

4. An Illustration

We consider the catch-effort data for Pacific Bigeye tuna from 1952 to 1987, as reported by Miyabe [4]. Levenberg-Marquardt iterative procedure available in NLIN option of SAS package [6] is used for estimating the parameters. In the first instance, Schaefer model given by (4) is fitted. However, residual analysis revealed that the assumption of independence of error terms is not satisfied. Thereafter, Schaefer model with AR (1) errors is tried using PROC model available in SAS package [6]. The estimate of optimum effort (2123.85) is found to be extremely high in comparison with the data for which the efforts ranged from 117.1 to 547.6. At the same time, the corresponding estimate of maximum sustainable yield is obtained as 282.09, which is too much on the lower side vis-a-vis the data values. Thus, even though Schaefer model could be fitted, the estimates did not have meaningful biological interpretation. The results for Fox model given by (3) are similar in nature. Therefore, neither Schaefer model nor Fox model satisfactorily describe the present data.

Subsequently, Pella - Tomlinson model given by (3) with additive i.i.d. error term is fitted and the results are given in second column of Table 1. Once again it is noticed that the assumption of independence of error terms is violated. The Durbin-Watson statistic value (0.73) shows the presence of autocorrelation of order one. Accordingly, (3) with AR (1) errors is fitted and the results are reported in last column of Table 1. A perusal indicates that the assumptions of independence and normality of errors is now satisfied as the calculated value of Shapiro-Wilk statistic (0.97) does not lie in the critical

Table 1. Fitting of Pella-Tomlinson Model

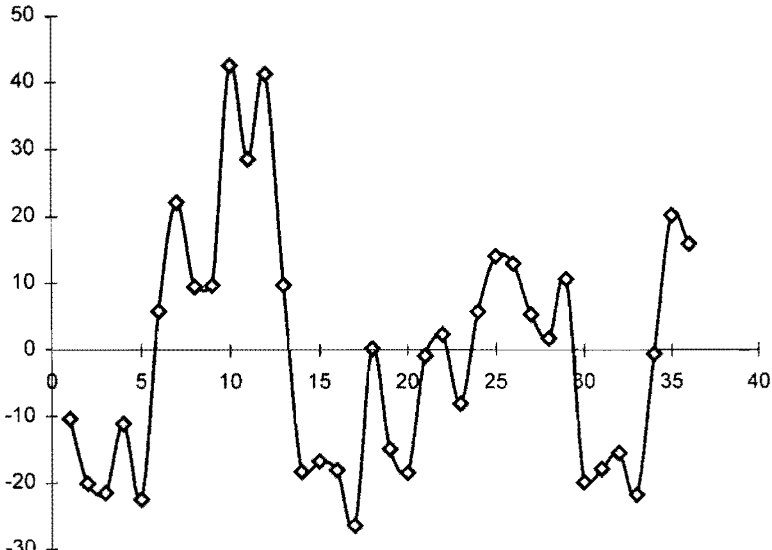
	Parameter estimate / Statistic	Error term i.i.d.	AR (1)
(i)	Parameters:		
	r	4705.73 (306.14)	6411.94 (950.10)
	k	0.35 (0.03)	0.34 (0.06)
	m	8.32 (0.60)	11.11 (1.84)
	ϕ	–	0.66 (0.13)
(ii)	Residual analysis:		
	Durbin-Watson d-statistic	0.73	–
	Run-test Z-statistic	–2.86	–1.64
	Shapiro-Wilk W-statistic	0.93	0.97
(iii)	Goodness of fit:		
	Mean Square Error	353.17	215.49

Note : The quantities within brackets ( ) indicate the corresponding asymptotic standard errors.

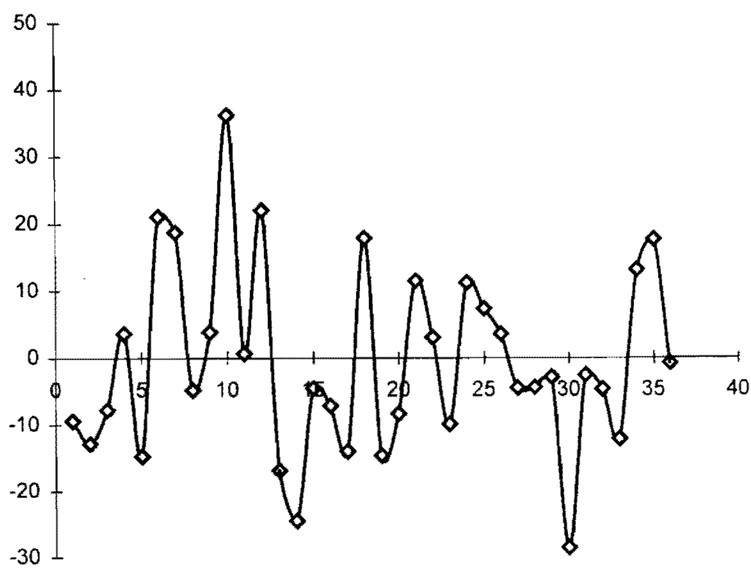
region at 5% level. In order to get visual insight, graphs of residuals for Pella-Tomlinson model are exhibited in Figs. 1(a) to 1(d). As the residuals in Fig. 1(b) do not constantly increase or decrease as time progresses, the assumption of homoscedasticity of error variances does not seem to be violated. Another point worth noting is that inclusion of AR(1) error structure has reduced the mean square error (MSE) from 353.17 to 215.49. This is also reflected visually by comparing Figs. 1(a) and 1(b). Further, AR(1) errors imply that, apart from random fluctuations, the catch for a particular year depends on the catch for the immediately preceding year, which seems quite plausible. A three-dimensional graph of fitted Pella-Tomlinson model along with data points is given in Fig. 1(d). Using (7) and (8), the maximum sustainable yield is estimated as 144198 metric tonnes and the optimum effort as 529.3 million hooks.

5. *Concluding Remarks*

Linearized versions of Schaefer and Fox models have been extensively used in fisheries. However, there are not many instances in literature dealing with applications of linearized version of Pella-Tomlinson model. The situation concerning the applications of the original nonlinear model by employing nonlinear estimation procedures is still worse. Perhaps the main reason is that global convergence to biologically meaningful values occurs very rarely. In this paper, we have considered the more realistic situation when the errors are autocorrelated. It is hoped that, in due course of time, researchers would start applying this model in other real life situations. However, there is a need to

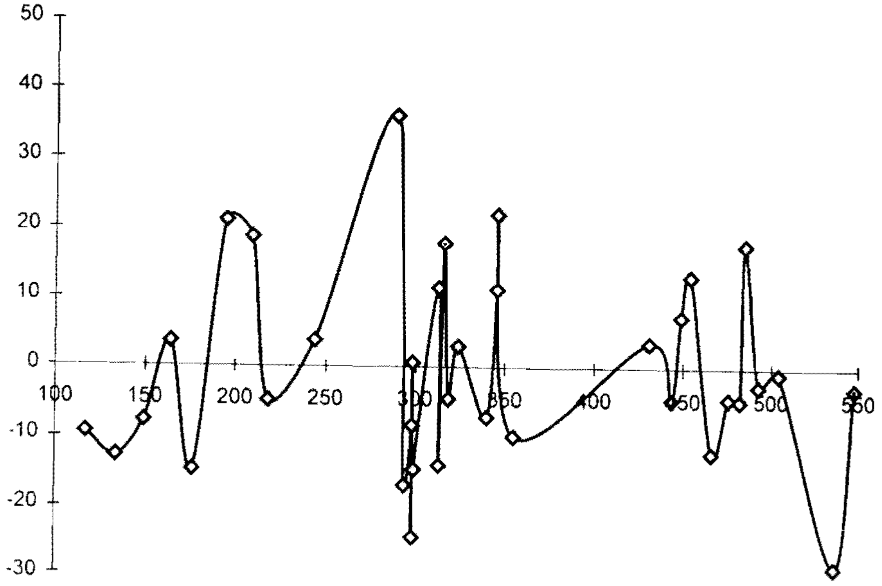


(a) Residuals for i.i.d. errors against time

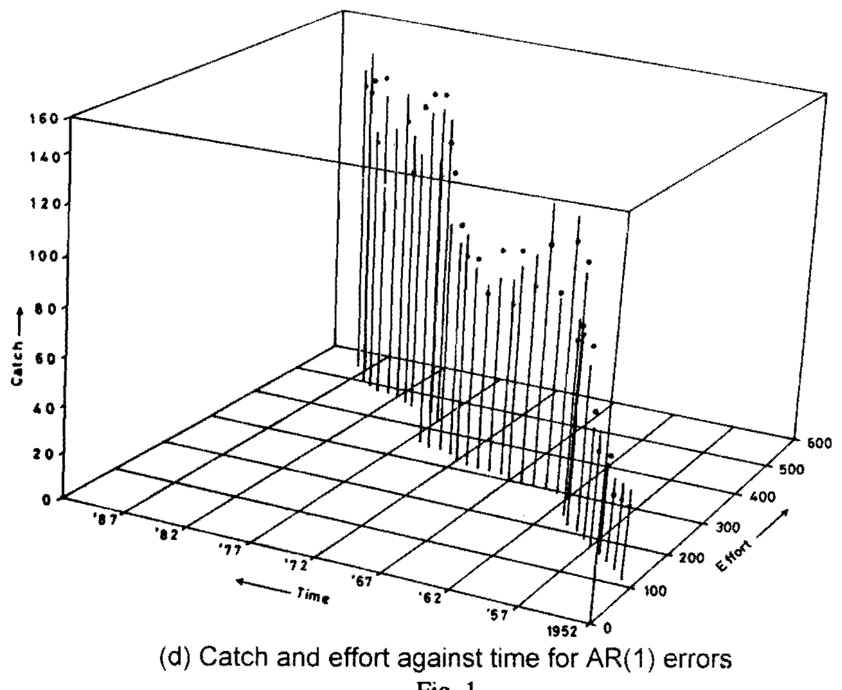


(b) Residuals for AR(1) errors against time

Fig. 1 : Graphs for Pella-Tomlinson model



(c) Residuals for AR(1) errors against effort



(d) Catch and effort against time for AR(1) errors

Fig. 1

study the nonequilibrium versions of Pella-Tomlinson model with i.i.d. or autocorrelated errors, which certainly is an extremely challenging task.

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