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Further Versions of the Convolution Equation[1]

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SUMMARY

Many authors seem to be unaware of the importance or the existence of the earlier papers of Choquet and Deny [4] and Deny [6] on its convolution equation, even when these have links with their own papers. Lau and Rao [9] and Davies and Shanbhag [5] among others have established variants or extended versions of the results of Choquet and Deny [4] and Deny [5], and have given various applications of these. The recent monograph of Rao and Shanbhag [15] should provide the reader with the relevant details of the literature in this connection. In the present paper, we make some further observations on the literature on integral equations, pointing out, in particular, that certain results of Laczkovich [8] and Baker [3] are essentially simple corollaries to, or variants of corollaries to, the general theorem of Deny [6] or the theorem of Choquet and Deny [4]. In the process of doing this, we are led to some new versions of the existing results on integral equations.

Key words: Integrated Cauchy equation, Exchangeability, de Finetti's theorem, Choquet–Deny theorem, Deny's theorem, Lau-Rao-Shanbhag theorems.

1. Introduction

Choquet and Deny [4] showed that if S is a locally compact Abelian topological group, then a bounded continuous function $H : S \to \mathbf{R}$ satisfies

$$H(x) = \int_S H(x+y)\,\mu(dy),\ x \in S \tag{1.1}$$

with μ as a probability measure, only if $H(x+y) = H(x)$ for each $y \in \text{supp}[\mu]$ and each $x \in S$. With an additional condition that S is second countable, Deny [6] gave an extended version of this result. He showed that

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if $H : S \to \mathbf{R}_+$, with H as continuous without necessarily being bounded, satisfies (1.1) with μ as a Radon measure (that is not necessarily a probability measure) such that the smallest closed subgroup of S generated by supp $[\mu]$ equals S itself, then (assuming it to be trivially so when $H \equiv 0$) H has an integral representation as a weighted average of μ-harmonic exponential functions. (A function $e : S \to \mathbf{R}_+$ is called an exponential function if it is continuous and satisfies $e(x+y) = e(x)\,e(y)$ for all $x, y \in S$; an exponential function e is referred to as μ-harmonic if $\int_S e(x)\,\mu(dx) = 1$.)

Shanbhag [18] and Lau and Rao [9] have given variants of Deny's theorem in the cases when $S = N_0 (= \{0, 1, \ldots\})$ and $S = \mathbf{R}_+$ respectively, and, under some mild conditions extended versions of Deny's theorem, subsuming the Lau-Rao-Shanbhag theorems, to the case when S is a semigroup are given by Davies and Shanbhag [5], Lau and Zeng [10] and Shanbhag [19]. Rao and Shanbhag [15] have discussed and unified many results on functional equations including some of these.

Laczkovich [8] and, more recently, Baker [3] have given certain results, on functional equations. Indeed, one of these results, namely Theorem 2 of Laczkovich, is a direct corollary to Deny's theorem, while, among the remaining, Theorems 1 and 3 of Laczkovich and Theorem 1 of Baker are slight variations of corollaries to Deny's theorem or the Choquet-Deny theorem; the authors of the two papers cited here do not seem to be aware of this. In the present paper, we show explicitly how these latter results of Laczkovich and Baker are either linked with or follow as consequences of the two celebrated theorems from the prior literature, referred to. In the process of doing this, taking a clue from Theorem 1 of Laczkovich, we observe the validity of certain versions of the results of Deny, Lau and Rao, and Davies and Shanbhag, with apriori local integrability condition (with respect to Lebesgue or Haar measure) relative to the functions involved in them deleted. Included among corollaries of these versions is essentially Theorem 1 of Laczkovich, which does not insist apriori that the local integrability condition in question be met. We also make here some observations on characterizations of stable laws, that are linked with the results on functional equations.

2. *Some Known Specialized Results of Functional Equations*

The following are some specialized results that are to be used in, or of relevance to, our present discussion.

Theorem 1. Let H be a non-negative real locally integrable Borel measurable function of $\mathbf{R}_+$, other than a function which is identically equal to 0 almost everywhere [L], such that it satisfies

$$H(x) = \int_{\mathbf{R}_+} H(x+y)\,\mu(dy) \text{ for almost all [L] } x \in \mathbf{R}_+ \tag{2.1}$$

for some σ-finite measure μ on (the Borel σ-field of) $\mathbf{R}_+$ with $\mu(\{0\}) < 1$ (yielding trivially that $\mu(\{0\}^c) > 0$), where L corresponds to Lebesgue measure. Then, either μ is arithmetic with some span λ and

$$H(x + n\lambda) = H(x)\,b^n, n = 0, 1 \ldots \text{ for almost all [L] } x \in \mathbf{R}_+$$

with b such that

$$\sum_{n=0}^{\infty} b^n \mu(\{n\lambda\}) = 1$$

or μ is nonarithmetic and

$$H(x) \propto \exp\{\eta x\} \text{ for almost all [L] } x \in \mathbf{R}_+$$

with η such that

$$\int_{R_+} \exp\{\eta x\}\,\mu(dx) = 1$$

Corollary 1. Let $\{(v_n, w_n) : n = 0, 1, \ldots\}$ be a sequence of vectors with non-negative real components such that $v_n \neq 0$ for at least one n, $w_0 < 1$, and the largest common divisor of the set $\{n : w_n > 0\}$ is unity. Then

$$v_m = \sum_{n=0}^{\infty} v_{m+n} w_n, \quad m = 0, 1, \ldots$$

if and only if

$$v_n = v_0 b^n, \; n = 0, 1, 2, \ldots \text{ and } \quad \sum_{n=0}^{\infty} w_n b^n = 1$$

for some $b > 0$.

Theorem 2. Let H be a nonnegative real locally integrable Borel measurable function on **R**, other than a function which is identically equal to zero almost everywhere [L], such that it satisfies

$$H(x) = \int_{\mathbf{R}} H(x+y)\,\mu(dy) \quad \text{for almost all [L] } x \in \mathbf{R} \tag{2.2}$$

for some σ-finite measure μ on **R** satisfying $\mu(\{0\}) < 1$ or equivalently $\mu(\{0\}^c) > 0$ (with L as Lebesgue measure). Then, either μ is non-arithmetic and

$$H(x) = c_1 \exp\{\eta_1 x\} + c_2 \exp\{\eta_2 x\} \text{ for almost all [L] } x \in \mathbf{R}$$

or μ is arithmetic with some span λ and

$$H(x) = \xi_1(x) \exp\{\eta_1 x\} + \xi_2(x) \exp\{\eta_2 x\} \text{ for almost all [L] } x \in \mathbf{R}$$

with c_1 and c_2 as non-negative real numbers, ξ_1 and ξ_2 as periodic non-negative Borel measurable functions having period λ, and η_i, $i = 1, 2$ as real numbers such that $\int_{\mathbf{R}} \exp\{\eta_i x\}\,\mu(dx) = 1$. (We allow here the case of $\eta_1 = \eta_2$; in this case, we take $c_2 = 0$ and $\xi_2 \equiv 0$.)

Corollary 2. Let $\{(v_n, w_n) : n = 0, \pm 1, \ldots\}$ be a sequence of two-component vectors with non-negative real components such that $w_0 < 1$ and at least one $v_n \neq 0$. Then

$$v_m = \sum_{n=-\infty}^{\infty} w_n v_{n+m}, \quad m = 0, \pm 1, \ldots$$

if and only if

$$v_m = B(m)\, b^m + C(m)\, c^m, \quad m = 0, \pm 1, \ldots$$

and $\sum_{m=-\infty}^{\infty} w_m b^m = \sum_{m=-\infty}^{\infty} w_m c^m = 1$ for some $b, c > 0$ and non-negative periodic functions B, C with the largest common divisor of $\{m : w_m > 0\}$ as their common period.

Theorem 1 is due to Lau and Rao [9] and has several interesting proofs including that based on exchangeability, given by Alzaid, Rao and Shanbhag [1]. Theorem 2 is essentially a corollary to Deny's theorem and has been arrived at via an alternative argument based on a random walk approach in some recent articles; see Ramachandran and Lau [12] or Rao and Shanbhag [15] for the details of the references. Corollary 1 is a slightly extended version of Shanbhag's [18] lemma and Corollary 2 is a result obtained recently by Ramachandran [11] via real variables techniques different from those employed by Deny [6].

Davies and Shanbhag [5], Shanbhag [19], and Rao and Shanbhag [15] have given general results subsuming versions of Deny's theorem as well as the Lau-Rao theorem, via arguments based on exchangeability, amongst other things. As illustrated in Rao and Shanbhag [15], these results have applications in various topics in probability and statistics.

As a corollary to Theorem 1.2.5 in Rao and Shanbhag [15] or its specialized version appearing in Davies and Shanbhag [5], it follows that if S is a countable Abelian semigroup with zero element, $v : S \to \mathbf{R}_+$ and $w : S \to \mathbf{R}_+$ are functions satisfying

$$v(x) = \sum_{y \in S} v(x+y)\, w(y),\ x \in S \tag{2.3}$$

then, for each $x \in S$

$$v(x+2y)\, v(x+2z) \geq (v(x+y+z))^2,\ y, z \in S^*(w) \tag{2.4}$$

where $S^*(w)$ is the smallest subsemigroup of S, with zero element, containing $\{ x \in S : w(x) > 0 \}$. The exchangeability argument used by the earlier authors to arrive at more general forms of the inequality is of elementary nature and it simplifies considerably in the present special case; one could also arrive at the result appearing here via celebrated de Finetti's theorem for exchangeable random elements, though the argument in this case would be less elementary. (Note that (2.4) follows trivially for each x with $v(x) = 0$.)

Indeed de Finetti's theorem tells us something more about (2.3). To illustrate this, we may proceed as follows. From (2.3), we get

$$v(x) = \sum_{y \in S^*(w)} v(x+y)\, w^*(y),\ x \in S \tag{2.5}$$

where

$$w^*(y) = \sum_{k=1}^{\infty} 2^{-k} \sum_{\{(y_1, \ldots, y_k) : \sum_1^k y_i = y\}} \Pi_{i=1}^k\, w(y_i),\ y \in S^*(w)$$

(We allow here some or all of the $w^*(y)$'s to be equal to ∞). If $v(0) = 0$, we get from (2.5), the restriction of v to $S^*(w)$ to be identically equal to zero. Consider now the case of $v(0) \neq 0$ and define a sequence $\{X_n : n = 1, 2, \ldots\}$ of exchangeable random elements with values in $S^*(w)$, such that

$$P\{X_1 = x_1, \ldots, X_n = x_n\} = \frac{v(x_1 + \ldots + x_n)}{v(0)} \Pi_{i=1}^{n} w^*(x_i)$$

$$x_1, \ldots, x_n \in S^*(w); n = 1, 2, \ldots \tag{2.6}$$

We have then, in view of de Finetti's theorem (and the fact that $w^*(y)$'s are all nonzero), that for all $x, y \in S^*(w)$

$$0 = \frac{1}{v(0)} \{v((x+y)+(x+y)) - 2v((x+y)+x+y) + v(x+y+x+y)\}$$

$$= E\{((P\{X_1 = x+y|\tau\}/w^*(x+y)) - (P\{X_1 = x|\tau\}/w^*(x))$$

$$(P\{X_1 = y|\tau\}/w^*(y))^2\}$$

where τ is the tail or invariant σ-field of $\{X_n\}$. Hence, it follows that

$$(P\{X_1 = x|\tau\}/w^*(x))(P\{X_1 = y|\tau\}/w^*(y))$$

$$= P\{X_1 = x+y|\tau\}/w^*(x+y), x, y \in S^*(w), \text{ a.s}$$

which implies, in view of (2.6), that there exists a collection $\{e(x, .) : x \in S^*(w)\}$ of τ-measurable non-negative real-valued measurable functions such that

$$e(x, .)\, e(y, .) = e(x+y, .), x, y \in S^*(w) \tag{2.7}$$

and

$$v(x) = v(0)\, E(e(x, .)), x \in S^*(w) \tag{2.8}$$

In view of (2.3), (2.7) and (2.8), we then get on appealing of Fubini's theorem that

$$0 = E\{1 - \sum_{x \in S^*(w)} e(x, .)\, w(x))^2\}$$

implying that

$$\sum_{x \in S^*(w)} e(x,.)\, w(x) = 1 \qquad \text{a.s} \tag{2.9}$$

Modifying the definition of $e(x, .)$, $x \in S^*(w)$, slightly on a null set, we can then produce a version of it so that (2.7), (2.8) and (2.9) "a.s" deleted are met. Consequently, if we equip S or $S^*(w)$ with discrete topology, we get that the restriction of v to $S^*(w)$ has an integral representation in terms of μ^*- harmonic exponential functions, where μ^* is the measure determined on $S^*(w)$ by $\{ w(x), x \in S^*(w) \}$ and exponential or μ^*-harmonic exponential functions are defined in the obvious way. (Note that we allow here an exponential or μ^*-harmonic exponential function to have 0 as one of its values.)

The result that we have obtained above was established via Choquet's theorem by Rao and Shanbhag [15] via a somewhat related approach by Ressel [17]. As a consequence of the result if follows that if $S^*(w) = S$, then we have either $v \equiv 0$ or $v(0) \neq 0$ and there is an integral representation for v in terms of μ^*- harmonic exponential functions; Corollary 2 in the case when there exist at least one negative n for which $w_n > 0$ and at least one positive n for which $w_n > 0$ follows trivially as a corollary to this result and hence to de Finetti's theorem, and in other cases it follows as a corollary to Corollary 1, which in turn, is also a corollary to de Finetti's theorem as observed by Rao and Shanbhag [16]. Following essentially the technique of Remark 4.4.3, (iv) on pages 100 and 101 in Rao and Shanbhag [15] (see also a relevant comment in Remark 5 in the next section), one can produce an integral representation for v in terms of μ-harmonic functions, where μ is the measure determined on S by $\{ w(x), x \in S \}$ when $S^*(w) \neq S$, v is not identically equal to zero and the following structural condition is met.

Condition: Let $w(0) < 1$ and, for every $x \in S + (\text{supp}\,[\mu] \setminus \{0\})$, there exist $k \geq 1$ and $y_1, \ldots, y_k \in S^*(w)$ such that $x + \sum_{i=1}^{r-1} y_i \in S + y_r$, $r = 1, 2, \ldots, k$ and $x + \sum_{i=1}^{k} y_i \in S^*(w)$.

(To understand the implications of the condition better and to have a clearer idea of the role played by de Finetti's theorem or its weaker versions, in the problem of solving general integral equations relative to semigroups of the type discussed here, the reader is referred to Rao and Shanbhag [15].)

In this case, the extension $e(x, .)$, $x \in S$, of $e(x, .)$ $x \in S^*(w)$ to S can also be seen to be such that for each $x \notin S^*(w)$

$$e(x, .) = (1 - w(0))^{-1} \sum_{d \in D} w(d) (e(x + d + y_d^*, .)/e(y_d^*, .))$$

where $D = \{x \in S : x \neq 0, w(x) > 0\}$, y_d^* is $\sum_{i=1}^{k} y_i$ of the condition above in the case when the point x of the condition is our $x + d$, and the ratio under the summation is to be understood as zero for each (d, ω) for which $e(y_d^*, \omega) = 0$.

Obviously, Deny's theorem in the case when the group is countable follows as a corollary to the last result that we have referred to above; note that in the present case we can choose $e(x, .)$ to be nonvanishing for each $x \in S^*(w)$ and take $e(x, .) = e(x_1, .)/e(x_2, .)$ whenever $x = x_1 - x_2$ with $x_1, x_2 \in S^*(w)$. This specialized version of Deny's theorem, in turn, gives Laczkovich's [8] Theorem 2 as an obvious corollary; without being aware of Deny's work, Laczkovich proved this theorem via the Krein-Milman theorem.

Before moving to the next section, let us make the following specific remarks:

Remark 1: As mentioned earlier, (2.4) also follows because of de Finetti's theorem. This is so because $v(0) = 0$ implies the inequality trivially when $x = 0$ and if $v(0) \neq 0$, in the notation considered before, the theorem gives

$$P\{X_1 = y, X_2 = y\} P\{X_1 = z, X_2 = z\} \geq (P\{X_1 = y, X_2 = z\})^2, \; y, z \in S^*(w)$$

implying (2.4) for $x = 0$; on applying the partial result to $v_x(.) = v(x + .)$ for each x, we then arrive at the general inequality.

Remark 2: In the case of arithmetic μ, the conclusion of Theorem 1 holds without the requirement of H being a locally integrable Borel measurable function and with "almost all [L]" in the representation for H replaced by "all", provided we replace "almost all [L]" in (2.1) by "all" and delete "almost everywhere [L]" appearing immediately after "identically equal to 0"; analogous remark also applies to Theorem 2.

3. *Comments on the Papers by Laczkovich and Baker*

We now make specific comments on the two papers referred to.

3a Laczkovich [8]:

This paper contains 3 theorems namely Theorems 1, 2 and 3. Theorem 2 of the paper is, as mentioned by us in our previous section, an obvious corollary to Deny's theorem, while Theorem 3 in the cited reference is a corollary to Theorem 1 given therein. We shall therefore restrict ourselves now to discussing Theorem 1 of the paper in question and getting it via (2.4) essentially as a simple corollary to Theorem 2 of Section 2, which, in turn, is a corollary to Deny's theorem.

Suppose $f : \mathbf{R} \to \mathbf{R}_+$ is a Borel measurable function satisfying

$$f(x) = \sum_{i=1}^{k} A_i f(x + a_i), \ x \in \mathbf{R} \tag{3.1}$$

where $A_1, \ldots, A_k$ are known positive real numbers and $a_i, i = 1, 2, \ldots, k$ are known distinct non-zero real numbers. We can indeed express (3.1) as

$$f(x) = \int_{\mathbf{R}} f(x + y)\,\mu(dy), \ x \in \mathbf{R} \tag{3.2}$$

where μ is the measure on $\mathbf{R}$ such that it is concentrated on $\{a_1, \ldots, a_k\}$ and satisfies $\mu(\{a_i\}) = A_i, \ i = 1, 2, \ldots, k$.

If μ in (3.2) is arithmetic with span λ, then, on using Theorem 2 (and recalling the relevant observation in Remark 2), we get easily that (3.2) is equivalent to the condition that f is either identically equal to zero or of the form

$$f(x) = \xi_1(x) \exp\{\eta_1 x\} + \xi_2(x) \exp\{\eta_2 x\} + , \ x \in \mathbf{R} \tag{3.3}$$

where $\xi_i, \eta_i, i = 1, 2$ are as in Theorem 2.

Consider now the case when μ is non-arithmetic, and assume that f is not a function that is equal to zero almost everywhere [L]. Let S be the subgroup of **R**, generated by supp [μ]. For each $x' \in \mathbf{R}$, define the function $v_{x'}$ on S such that

$$v_{x'}(.) = f(x' + .)$$

(2.4) with obvious notational alteration is valid for each $v_{x'}$. From the inequality, we can then easily conclude for each $x' \in \mathbf{R}$, $y \in S$ and $m \in \{1, 2, \ldots\}$ that

$$v_{x'}((m-1)y)\, v_{x'}((m+1)y) \geq (v_{x'}(my))^2 \tag{3.4}$$

and hence

$$f(x' + (m-1)y)\; f(x' + (m+1)y) \geq (f(x' + my))^2 \tag{3.5}$$

(This follows on observing that (2.4) in the case when S is a group implies

$$v((m-1)(y-z))\, v((m+1)(y-z)) \geq (v(m(y-z)))^2,\ y, z \in S^*(w)$$
$$m = 1, 2, \ldots$$

on taking $x = (m-1)(y-z) - 2z$). From (3.5), we can conclude inductively that for each x', $f(x' + my) = 0$, for each m and y whenever $f(x') = 0$ and $f(x' + my)/f(x') \geq (f(x' + y)/f(x'))^m$ for each m and y whenever $f(x') \neq 0$. (In view of the known properties of moment sequences, one could also get this last result directly via the specialized version of Deny's theorem in the case of a countable group, or its corollary appearing as Theorem 2 in Laczkovich [8]; our argument given here is obviously of more elementary nature.) As for any real δ,

$$f_\delta^*(x) = \exp\{\delta x\} f(x),\ x \in \mathbf{R}$$

satisfies (3.1) with A_i replaced by $A_i \exp\{-\delta a_i\}$ for $i = 1, 2, \ldots, k$, whenever f satisfies (3.1), it is clear that there is no loss of generality in assuming $A_1 > 1$ in the problem of identifying the solution to (3.1). Assume then this to be so; this gives immediately

$$f(x + na_1) \leq f(x),\ x \in \mathbf{R},\ n = 1, 2, \ldots \tag{3.6}$$

Given any positive integers m and n, we have a sequence $\{y_r : r = 1, 2, \ldots\}$ of points of S, such that it converges to $\frac{na_1}{m}$. Because of Laczkovich's elementary Lemma 2 and what we have seen, it then follows that for almost all [L]$x \in \mathbf{R}$, and m and n as stated

$$f\left(x+\frac{na_1}{m}\right)\begin{cases} = 0 & \text{if } f(x) = 0 \\ \le f(x)\left(\dfrac{f(x+na_1)}{f(x)}\right)^{m^{-1}} \le f(x), & \text{if } f(x) \ne 0 \end{cases}$$

implying that $f\left(x+\frac{n}{m}a_1\right) \le f(x)$. Applying the cited Lemma 2 once more, we can hence conclude that if $y \in \mathbf{R}_+$

$$f(x+ya_1) \le f(x) \text{ for almost all [L] } x \in \mathbf{R} \tag{3.7}$$

This gives in view of Fubini's theorem that

$$f(x+ya_1) \le f(x) \text{ for almost all [L] } y \in \mathbf{R} \text{ for almost all [L] } x \in \mathbf{R}$$

implying that f is locally integrable. This, in turn, implies that the conclusion of Theorem 2 (appearing in Section 2) now holds with f in place of H.

Remark 3: Essentially the same argument as above, but without any reference to Laczkovich's [8] Theorem 2, implies that if the measure μ in Theorem 2 is concentrated on a countable set, then the conclusion of the theorem holds even when H is not assumed apriori to be locally integrable; this latter result obviously subsumes the result arrived at above concerning Theorem 1 of Laczkovich. The analogue of the result corresponding to Theorem 1 (i.e. the Lau-Rao theorem) also holds. (These modified results hold even without the apriori restriction that μ be σ-finite.)

Remark 4: If we are prepared to use the specialized version of Deny's theorem in the case of a countable group, in place of the inequality (2.4), then the local integrability of f above, or of H of the result relative to Theorem 2, mentioned in Remark 3, follows via a shorter argument: Lemma 2 of Laczkovich and the Jensen inequality imply, in this case, immediately that if $y \in [0, 1]$, then

$$f(x+y) \le (f(x))^{1-y}(f(x+1))^y \text{ for almost all [L] } x \in \mathbf{R} \tag{3.7'}$$

giving, in view of Fubini's theorem, that

$$f(x+y) \le (f(x))^{1-y}(f(x+1))^y \text{ for almost all [L] } y \in [0,1] \text{ for almost all } x \in \mathbf{R}$$

It is worth pointing out at this stage that a version of Deny's theorem referred to here gives easily as its corollary the specialized version of Theorem 2 in the case where μ is concentrated on a countable set, and hence, as a by-product of what we have just seen, also now its modified version mentioned in Remark 3. The analogue of this statement with the result for the countable semigroup meeting the structural condition stated in Section 2, in place of the version of Deny's theorem, holds for Theorem 1; indeed, the result relative to the countable semigroup, on applying a slightly modified version of the argument implied here, yields the following version of a result of Davies and Shanbhag [5] (or of Corollary 3.4.5 in Rao and Shanbhag [15]):

Theorem 3: Let $n \geq 1$, $S = \Pi_{i=1}^{n} S_i$ with $S_i = \mathbf{Z}$ or N_0 or $-N_0$ or $\mathbf{R}$ or $\mathbf{R}_+$ or $-\mathbf{R}$, and let G be the smallest subgroup of $\mathbf{R}^n$ containing S. Further, let λ be the restriction to S of a Haar measure on G, $h : S \to \mathbf{R}_+$ be a Borel measurable function and μ be a measure on S with $\mu(\{0\}) < 1$ such that it is concentrated on a countable set. Assume that there is a dense subset or equivalently a countable dense subsemigroup $\hat{S}$ of S such that for every $x \in \hat{S} + (B \setminus \{0\})$, there exists $k \geq 1, y_1, \ldots, y_k \in S^*$ for which $x + \sum_{i=1}^{r-1} y_i \in \hat{S} + y_r$, $r = 1, 2, \ldots, k$ and $x + \sum_{i=1}^{k} y_i \in S^*$, where S^* is the subsemigroup, with zero element, of S generated by B and $B = \{y \in S : \mu(\{y\}) > 0\}$. Then

$$h(x) = \int_S h(x+y)\, \mu(dy) \text{ for almost all } [\lambda]\ x \in S$$

implies that

$$h(x) = \int_{[-\infty, \infty]^n} \exp\{\langle x, y \rangle\}\, v(dy) \text{ for almost all } [\lambda]\ x \in S$$

where v is a measure on $[-\infty, \infty]^n$ such that it is concentrated on the set of points y at which $\langle ., y \rangle$ is a (well defined) function from S to $[-\infty, \infty)$ such that $\int_S \exp\{\langle x, y \rangle\}\, \mu(dx) = 1$. (We define here $0(\pm\infty) = 0$ and $e^{-\infty} = 0$).

Remark 5: One can also arrive at the inequality (3.7') via a minor variant of the argument based on (2.4), given above to establish (3.7). The argument in question, with certain modifications, can further be applied to obtain the modified version of (3.7') corresponding to the integral equation of Theorem 3. In the case when S of Theorem 3 is a group of such that $\hat{S} = S^*$, the modifications needed are simple, while in the case when the S is not so, one can provide the necessary modifications involving the following observations: If (2.3) holds and the structural condition of Section 2 is met, then appealing to (2.4) with $z = 0$ and (2.5), it can be seen (in the notation met earlier in Section 2) that given $x \in S$ and $d \in D$ and integers $m, m' > 0$, we have, for each $x' \in S$

$$v(m'x + x' + d) = \sum_{n=0}^{\infty} \sum_{z_1 \in S^{**}(w)} \cdots \sum_{z_n \in S^{**}(w)} v\left(m'x + x' + d + m(d + y_d^*) + \sum_1^n z_i\right) \left(\prod_{i=1}^{n} w^*(z_i)\right) w^*(m(d + y_d^*))$$

where $S^{**}(w) = S^*(w) \setminus \{m(d + y_d^*)\}$ (and y_d^* obviously is $\sum_1^k y_i$ of the structural condition mentioned when the point x of the conditions is our $x + d$). (Incidentally, if we take $m, m' = 1$, we get from the assertion, in view of (2.3), the first equation on page 101 in Rao and Shanbhag [15]; this follows even when we do not involve Ressel's [17] result in our analysis. Also it is worth pointing out here that the above equation concerning v holds even with "y_d^*" in place of "$d + y_d^*$".) This follows on noting among other things that the term under the summation sign with respect to n, with $w^*(m(d + y_d^*))$ deleted, in the equation above tends to zero as $n \to \infty$, and hence using (2.4), with $z = 0$, inductively that the term with "$v(m'x + x' + d + \sum_i^n z_i)$" in place of "$v(m'x + x' + d + m(d + y_d^*) + \sum_1^n z_i)$", tends to zero as $n \to 0$. The assertion with "$\geq$" in place of "$=$" holds if we replace "$m, m' > 0$" by "$m > 0$ and $m' = 0$". In view of (2.4), what we have observed here, with $m = 2$, then implies that

$$v(m'x + x' + d)\, v((m' + 2)x + x' + d) \geq (v((m' + 1)x + x' + d))^2$$
$$x, x' \in S, d \in D, m' = 0, 1, 2, \ldots$$

which, in turn, implies, because of (2.3), that

$$v(m'x + x')\, v((m' + 2)x + x') \geq (v((m' + 1)x + x'))^2$$
$$x, x' \in S, \quad m' = 0, 1, 2, \ldots$$

Remark 6: Under the stated assumptions, the equation

$$\sum_{i=1}^{k} A_i \exp\{\lambda a_i\} = 1$$

in λ with λ as real has either no roots or one root or two roots only. From what appears in Laczkovich [8], one gets the impression that the author of the paper was unaware of the fact while writing the paper; note that he uses the notation "N" for the number of roots without clarifying what its possible values are.

3b Baker [3]

Let $f: (0,\infty) \to \mathbf{R}_+$ such that sufficiently close to zero it is positive and

$$f(x) = \Pi_{j=1}^{N} [f(\beta_j x)]^{\gamma_j}, \ x \in (0, \infty) \tag{3.8}$$

where $0 < \beta_j < 1$ and $\gamma_j > 0$ for $j = 1, 2, \ldots, N$ are known constants. From (3.8), it is immediate that f is positive on the whole of $(0, \infty)$. Clearly, we have here a unique real number α such that

$$\sum_{j=1}^{N} \beta_j^{\alpha} \gamma_j = 1 \tag{3.9}$$

If we now assume for some real c, the function $\frac{\log f(x)}{x^{\alpha}} + c, x \in (0, \infty)$ to be such that sufficiently close to zero and hence, in view of (3.8), everywhere nonpositive or nonnegative continuous, then we have from either of Theorems 1 and 2, for some Φ

$$f(x) = \exp \{\Phi (\log x) x^{\alpha}\}, x \in (0, \infty) \tag{3.10}$$

where Φ is a periodic function with period λ if the subgroup of $\mathbf{R}$ generated by $\{\log \beta_1, \ldots, \log \beta_N\}$ is the lattice $\{n\lambda : n \in \mathbf{Z}\}$, for some $\lambda > 0$, and Φ is identically equal to a constant if the subgroup in question is not of the form stated. If we take "bounded" in place of "continuous", then, in view of the specialized version of the Choquet-Deny theorem corresponding to a countable group, (3.10) still holds with Φ as periodic function with periods $\log \beta_i$, $i = 1, 2, \ldots, N$.

Consider now the case with α as a positive integer and $f^*: \mathbf{R} \to \mathbf{R}_+$ such that

$$f^*(x) = \Pi_{j=1}^{N} (f^*(\beta_j x))^{\gamma_j}, \ x \in \mathbf{R} \tag{3.11}$$

and let f^* have α–th order derivative at zero. Since $\sum_{j=1}^{N} \gamma_j > 1$ and $f^*(x) \to f^*(0)$ as $x \to 0$, it follows that if $f^*(0) = 0$, we have $f^*(x) \leq \epsilon$ for all $0 < \epsilon < 1$ and $x \in \mathbf{R}$, implying that $f^* \equiv 0$. In view of the stated condition, it also follows that if $f^*(0) > 0$, we have $f^*(x) > 0$ for all $x \in \mathbf{R}$ and $f^*(0) = 1$. (To see that $f^*(0) = 1$, note that $f^*(0) = (f^*(0))^{\sum_{i}^{N} \gamma_j}$). In the case when $f^*(0) = 1$ (with obviously $f^*(x) > 0$ for all $x \in \mathbf{R}$), we can define

$$F(x) = \log f^*(x), x \in \mathbf{R}$$

In view of the condition concerning the α-th order derivative at 0 of f^*, we see that f^* is differentiable $\alpha - 1$ times in a neighbourhood of the origin; this, in turn, implies because of (3.11) that f^* and hence F is differentiable $\alpha - 1$ times everywhere on $\mathbf{R}$. Denoting for each $r = 0, 1, \ldots, \alpha - 1$, the r–th order derivative of F by $F^{(r)}$, we see that

$$F^{(r)}(x) = \sum_{j=1}^{N} \gamma_j \beta_j^r F^{(r)}(x\beta_j), \; x \in \mathbf{R}, r = 0, 1, \ldots, \alpha - 1 \tag{3.12}$$

implying because of (3.9) that $F^{(r)}(0) = 0$, $r = 0, 1, \ldots, \alpha - 1$. As $f^*(x) = e^{F(x)}, x \in \mathbf{R}$ and f^* has the α–th order derivative at zero, it then immediately follows that $F^{(\alpha-1)}(x)/x$ has a limit as $x \to 0$. As for $x \neq 0$

$$F^{(\alpha-1)}(x)/x = \lim_{x \to \infty} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} (\Pi_{m=1}^{n} \gamma_{jm} \beta_{jm}^{\alpha}) F^{(\alpha-1)}(x\beta_{j1} \cdots \beta_{jn}) / (x\beta_{j1} \cdots \beta_{jn})$$

in view of (3.9), we can immediately conclude that for some real c

$$F^{(\alpha-1)}(x) = cx, x \in \mathbf{R} \setminus \{0\} \tag{3.13}$$

Since $F^{(r)}(0) = 0$, $r = 0, 1, \ldots, \alpha - 1$, (3.13) implies that $F(x) = \frac{cx^\alpha}{\alpha!}$, $x \in \mathbf{R}$ or equivalently that $f^*(x) = \exp\{cx^\alpha/\alpha!\}, x \in \mathbf{R}$. (The results that we have arrived at are valid even when $\mathbf{R}$ is replaced by $\mathbf{R}_+$.)

The latter set of results that we have observed here give Baker's Theorem 1. Obviously we have reproduced now some of the steps of Baker in a simplified form. But, what we have revealed shows that Theorem 1 of Baker is linked with the result concerning f given in this section. To see the link better, note that the function $\exp\{F^{(\alpha-1)}(x)\}, x \in (0,\infty)$ or $\exp\{F^{(\alpha-1)}(-x)\}, x \in (0,\infty)$ meets the requirements of f relative to $\alpha = 1$ (with obvious notational alterations) and in this case $F^{(\alpha-1)}(x)/x, x \in (0,\infty)$ (or $F^{(\alpha-1)}(-x)/x, x \in (0,\infty)$) is bounded close to zero; indeed any periodic function $\Phi(z), z \in \mathbf{R}$, has a limit as $z \to -\infty$ or as $z \to \infty$ only if it is identically equal to a constant.

Although Baker's result is appealing as a result on functional equations, it cannot be considered to be of much importance in characterization problems involving characteristic functions of stable distributions that are nonnormal; on the other hand the restrictions to $(0,\infty)$, of characteristic functions of symmetric stable laws (or except in the case of $\alpha = 1$, of moment generating functions of extreme stable laws) satisfy functional equations of the form (3.8) with $\alpha \in (0, 2]$ (and $\alpha \in (0, 2)$, in the case of nonnormal laws).

For various applications of the Choquet-Deny and Deny type functional equations in characterization problems of stable laws, the reader is referred to Ramachandran and Lau [12] and Rao and Shanbhag [15]. In particular, Rao and Shanbhag [15] have given through Theorems 6.4.1 and 6.4.6 in their Chapter 6 certain characterizations of discrete and continuous multivariate stable laws, involving the following results from Hardy [7; page 37] in Number Theory: if m is an integer greater than or equal to 2 and p is a positive integer with none of $p, p^2, \ldots, p^{m-1}$ as a perfect m^{th} power (i.e. as the m^{th} power of an integer), then $a_0 + a_1\xi + \ldots + a_{m-1}\xi^{m-1} = 0$ with $a_0, a_1, \ldots, a_{m-1}$ as rational numbers and $\xi = p^{1/m}$ if and only if $a_0 = a_1 = \ldots = a_{m-1} = 0$. (Incidentally, the statement of the result as appearing in Hardy [7] is slightly inaccurate and what is given here is a corrected version of it.) These latter characterization have obvious generalizations in view of the general nature of the afore mentioned results from Number Theory. For example, it is now an easy exercise to see via the result that the characterization results of stable distributions referred to here hold if in place of the power 2 we take any integer power greater than or equal to 2.

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