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Efficiencies of Certain Test Statistics Against Two-State Markov Chain

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SUMMARY

This paper discusses the asymptotic relative efficiency of a family of test statistics T_r, which include some of the well-known statistics, against the two-state Markov Chain alternative. Where a uniquely optimal statistic does not appear to exist, a relatively optimal statistic has been evaluated by considering the asymptotic relative efficiency of the statistics with respect to the preceding ones and choosing that statistic after which the relative gain in the efficiency is not significant, that is, not more than 5 percent. The statistics T_7 and $T_{n/5}$ emerge as the preferable ones but T_7 has a slight edge over $T_{n/5}$ in view of its higher efficiency as also greater convenience in its computation.

Keywords: Markov chain, Asymptotic relative efficiency, T-statistics, Optimum statistics, Sign test.

1. Introduction

Research work on complex Markov chains was started by the author way back in 1951, and he had evolved a test statistic based on the differences between adjacent and alternate observations for testing the randomness of a sequence of observations, (Singh [7]). When he endeavoured to enlarge the work by considering a number of test statistics based on the differences between the observations separated by more than one observation, he had to work out a large number of possible configurations for the differences and their frequencies. At that time, Dr. P.V. Sukhatme suggested about his paper [11] which proved to be of great help.

Now for testing the randomness of a sequence of n observations, Iyer and Singh [3] had considered T-statistics which, unlike many other statistics, have the advantage of being based on the differences between observations separated by a specified number of observations or less. Thus if r observations are taken such that any two of them are not separated by more than $(r - 2)$ observations, the statistic T_r is obtained as indicated in Section 2. Some of the well-known statistics have been shown to be the particular cases of the T-statistics.

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The normality, consistency, power and efficiency of the T- statistics for certain cases of large samples had been considered by Iyer and Singh [3] and Singh [10]. Besides, Singh [8] had made a comparative study of the empirical powers of some of the statistics including T-statistics against normal translation and dispersion alternatives in case of small and large samples.

The object of this paper is to discuss the asymptotic relative efficiencies of T-statistics against two-state markov chain alternative which is of great practical importance. Where a uniquely optimal statistics does not appear to exist, a relatively optimal statistic has been evaluated by considering the asymptotic relative efficiency of the statistic with respect to the preceding ones and choosing that statistic after which the relative gain in the efficiency is not significant, that is, not more than 5 percent. In fact these investigations have shown the behaviour of the test statistics for varying values of r from 2 to n (and not merely for r = 2 and n as is often done), thereby exploring the possibility of having more efficient statistics for intermediate values of r (between 2 and n) for this alternative.

2. *Definitions of Statistics*

Let $y_1, y_2 \ldots y_n$ be a sequence of n observations which can assume any of two values (or characters) A and B, y(ij) denote the transition or join between the i-th and j-th observations and

$$y(ij) = 1 \text{ if } y_i \text{ is A and } y_j \text{ is B so that the transition AB is obtained}$$

$$= 0 \text{ otherwise} \qquad (1)$$

The statistic T_r is then defined as

$$T_r = \sum_{\substack{1 \le i < j \le n \\ (j-i) \le (r-1)}} \sum y(ij) \qquad (2)$$

When r = 2, we get T_2 which is the same as the joins test.

If r = n/2 and only the differences between the observations separated by exactly $\left(\frac{n}{2} - 2\right)$ observations, are considered, one can extend Cox - Stuart unwighted sign test S_2 (1) as

$$S_2 = \sum_{i=1}^{n/2} y\,(i, n/2 + i) \qquad (3)$$

As mentioned by Iyer and Singh [3], the general expressions for the mean and variance of T_r hold good only so long as $r \leq (n/2+1)$. When $r > (n/2+1)$ we put $r = n - R$ and get the statistic T_{n-R} as

$$T_{n-R} = \sum_{\substack{1 \leq i < j \leq n \\ (j-i) \leq (n-R-1)}} \sum y(ij) \tag{4}$$

When r = n so that R = 0, we get

$$T_n = \sum_{1 \leq i < j \leq n} \sum y(ij) \tag{5}$$

In case all the observations of the sequence are different, Kendall's statistic τ is given in terms of T_n by the relation

$$\tau = \frac{4T_n}{n(n-1)} - 1 \tag{6}$$

If $R = \frac{1}{3}$ and only the differences between the observations separated by exactly $(n - \frac{n}{3} - 2)$ or $\left(\frac{2n}{3} - 2\right)$ observations are considered, Cox–Stuart's unweighted sign test $S_3(1)$ is extended as

$$S_3 = \sum_{i=1}^{n/3} y\left(i, \tfrac{2}{3}n + i\right) \tag{7}$$

3. Expectation of the Statistics for Two-State Markov Chain

Let a two-state (A, B) Markov chain be defined by the conditional probability matrix

$$\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \tag{8}$$

where p_1 and p_2 are the conditional probabilities for the i-th observation to be in state A when the (i-1)-th observation is in A or B respectively. That is,

$$\begin{aligned} P_i(A/i-1, A) &= p_1 \qquad & Q_i(B/i-1, A) &= q_1 \\ P_i(A/i-1, B) &= p_2 \qquad & Q_i(B/i-1, B) &= q_2 \end{aligned} \tag{9}$$

Now the probability for getting an AB transition $A_i B_j$ between the i-th and j-th observations (i < j) can be obtained by considering the conditional probability P_j (B/i, A) for the j-th observation to be B when the i-th is A, and the probability P_i (A) that the i-th observation is A. The asymptotic values of these probabilities have been found (2) to be

$$P_j(B/i, A) = q_1 (1 - \delta^{j-i}) / 1 - \delta \tag{10}$$

and

$$P_i(A) = \frac{p_2}{1 - \delta} \tag{11}$$

where

$$\delta = p_1 - p_2$$

Multiplying (10) and (11) we get the probability for an AB transition between the i-th and j-th observations (i < j) as

$$\frac{p_2 q_1}{(1 - \delta)^2} (1 - \delta^{j-i}) \tag{12}$$

Consider δ as the parameter of the Markov chain so that when $\delta = 0$ we get the random binomial sequence with complementary probabilities p_1 and q_1 in the null case.

To evaluate the expectation of T_r in the general case, rewrite (2) as

$$T_r = Y_1 + Y_2 + \ldots + Y_{r-1}$$

where

$$\begin{aligned} Y_1 &= y(12) + y(23) + y(34) + \ldots + y(n-1, n) \\ Y_2 &= y(13) + y(24) + \ldots + y(n-2, n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Y_{h-1} &= y(1, h) + y(2, h+1) + \ldots + y(n-h+1, n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Y_{r-1} &= y(1, r) + y(2, r+1) + \ldots\ldots + y(n-r+1, n) \end{aligned} \tag{13}$$

where y(ij) assumes value 1 if the transition is AB and zero otherwise.

Hence the expectation for any Y_{h-1} is

$$E(Y_{h-1}) = EY(1, h) + EY(2, h+1) + \ldots + EY(n-h+1, n)$$

$$= \sum_{1}^{n-h+1} \frac{p_2 q_1}{(1-\delta)^2} (i - \delta^{h-1}) \quad \text{from (12)}$$

$$= (n-h+1) \frac{p_2 q_1}{(1-\delta)^2} (1 - \delta^{h-1}) \tag{14}$$

Hence $$E(T_2) = E(Y_1) = (n-1)\,\frac{p_2 q_1}{1-\delta} \tag{15}$$

$$E(T_3) = E(Y_1 + Y_2)$$

$$= (n-1)\,\frac{p_2 q_1}{1-\delta} + (n-2)\,\frac{p_2 q_1}{1-\delta}(1+\delta) \tag{16}$$

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$$E(T_r) = \left\{\frac{1}{2}(r-1)(2n-r)\right\}\frac{p_2 q_1}{1-\delta}$$

$$+\left\{\frac{1}{2}(r-1)(2n-r)-(n-1)\right\}\frac{p_2 q_1 \delta}{1-\delta}$$

$$+\left\{\frac{1}{2}(r-1)(2n-r)-(n-1)-(n-2)\right\}\frac{p_2 q_1 \delta^2}{1-\delta}$$

$$+ \ldots\ldots + (n-r+1)\frac{p_2 q_1 \delta^{r-2}}{1-\delta}$$

$$\cong \left(\frac{1}{2}(r-1)(2n-r)\right)\frac{p_2 q_1}{1-\delta} + \left\{\frac{1}{2}(r-1)(2n-r)-(n-1)\right\}\frac{p_2 q_1 \delta}{1-\delta} \tag{17}$$

assuming that δ^2 and other higher powers of δ are negligible.

The expression (17) however holds good only so long as $r \leq (\frac{n}{2}+1)$. For $r \geq (\frac{n}{2}+1)$ we have the expectation for T_{n-R} from (4) and (12) as

$$E(T_{n-R}) = \frac{1}{2}(n-R-1)(n+R)\,\frac{p_2 q_1}{1-\delta}$$

$$+\left\{\frac{1}{2}(n-R-1)(n+R)-(n-1)\right\}\frac{p_2 q_1 \delta}{1-\delta}$$

$$+\left\{\frac{1}{2}(n-R-1)(n+R)-(n-1)-(n-2)\right\}\frac{p_2 q_1 \delta^2}{1-\delta}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(R+1)\,p_2 q_1\,\frac{\delta^{n-R-2}}{1-\delta}\;. \tag{18}$$

where R = 0 and n is large, the expression (18) reduces to

$$E(T_n) \cong \frac{n(n-1)p_2 q_1}{2(1-\delta)^2} - \frac{(n-1)p_2 q_1}{(1-\delta)^3} \tag{19}$$

which agrees with that given by Iyer and Ray [2].

If δ^2 and other higher powers of δ are omitted, we get from (18)

$$E(T_{n-R}) = \frac{1}{2}(n-R-1)(n+R)\frac{p_2 q_1}{1-\delta}$$

$$+\left\{\frac{1}{2}(n-R-1)(n+R)-(n-1)\right\}\frac{p_2 q_1 \delta}{1-\delta} \tag{20}$$

It may be added here that Singh [9] has shown that the distribution of any T_r would asymptotically tend to the normal form in the case of this Markov chain alternative.

4. *Asymptotic Relative Efficiency*

For evaluating the asymptotic relative efficiencies of various statistics, utilize the following result by Noether [5] who had generalized the work of Pitman [6].

Let E and σ denote the mean and standard deviation for any statistic t. Also, let

$$\theta = \theta_0 + \frac{k}{n^{\delta}} \qquad \delta > 0 \tag{21}$$

be the sequence of alternatives considered and for $\theta = \theta_0$

$$\frac{\partial E}{\partial \theta} = \frac{\partial^2 E}{\partial \theta^2} = \dots\dots\dots = \frac{\partial^{m-1} E}{\partial \theta^{m-1}} = 0, \ \frac{\partial^m E}{\partial \theta^m} > 0 \tag{22}$$

Furthermore, let

$$\left[\left(\frac{\partial^m}{\partial \theta^m} E(\theta)\right)_{\theta=\theta_0} \sigma(\theta_0)\right] \cong Cn^{m\delta} \tag{23}$$

Then the asymptotic relative efficiency (A.R.E.) of a statistic t_1 with respect to another statistic t_2 is given by

$$A(t_1, t_2) = \underset{n_1 \to \infty}{Lt} \frac{n_2}{n_1} = (C_1/C_2)^{\frac{1}{m\delta}}$$

$$= \underset{n \to \infty}{\mathrm{Lt}} \left[\frac{\left\{ \dfrac{\partial^m E_1(\theta)}{\partial \theta^m} \right\}_{\theta = \theta_0}}{\sigma_1(\theta_0)} \cdot \frac{\sigma_2(\theta_0)}{\left\{ \dfrac{\partial^m E_2(\theta)}{\partial \theta^m} \right\}_{\theta = \theta_0}} \right]^{\frac{1}{m\delta}} \tag{24}$$

When $m = 1$ and $\delta = \frac{1}{2}$, we get the Pitman's result from (24).

For all the T-statistics we find that $m = 1$. Hence for considering their A.R.E., evaluate the expression E'/σ or more conveniently, $\frac{E'^2}{V} = \phi$ say, where V stands for the variance and ϕ for the efficacy for $p_2 = q_1 = \frac{1}{2}$.

Putting $p_2 = q_1 = \frac{1}{2}$ in (17) and differentiating with respect to δ we get

$$\frac{\partial E}{\partial \delta}(T_r) = \left\{ \frac{1}{2}(r-1)(2n-r) \frac{1}{4(1-\delta)^2} \right\} + \frac{\frac{1}{2}(r-1)(2n-r)-(n-1)}{4(1-\delta)}$$

$$+ \left\{ \frac{1}{2}(r-1)(2n-r)-(n-1) \right\} \frac{\delta}{4(1-\delta)^2}$$

$$+ \text{ other terms involving } \delta^2 \text{ and other higher powers of } \delta$$

or $$E'(T_r) = \left[\frac{\partial E(T_r)}{\partial \delta} \right]_{\delta = 0} = \frac{1}{8}(r-1)(2n-r) + \frac{1}{8}(r-1)(2n-r) - \frac{(n-1)}{4}$$

$$= \frac{1}{4}(r-1)(2n-r) - \frac{(n-1)}{4} \tag{25}$$

$$\cong \frac{n(2r-3)}{4}$$

Also, in the null case p_1 (or p_2) $= q_1 = \frac{1}{2}$ and we have from Iyer and Singh [3]

$$V(T_r) \cong \frac{n(r-1)}{16} \tag{26}$$

Hence $$\phi(T_r) = \frac{E'^2(T_r)}{V(T_r)} = \frac{n^2(2r-3)^2}{16} \cdot \frac{16}{n(r-1)} \cong \frac{n(2r-3)^2}{r-1} \tag{27}$$

In particular, when $r = 2$, we get for T_2 or joins test

$$\phi(T_2) \cong n \tag{28}$$

Where $r = 3$, we have for T_3

$$\phi(T_3) \cong \frac{9}{2} n \tag{29}$$

which is more than that for the joins test.

Similarly for T_{r-1}

$$\phi(T_{r-1}) \cong \frac{n(2r-5)^2}{r-2} \tag{30}$$

Since $m = 1$ and $\delta = \frac{1}{2}$ for both T_r and T_{r-1}, the A.R.E. of T_r with respect to T_{r-1} is found from (27) and (30) as

$$A(T_r, T_{r-1}) = \phi(T_r)/\phi(T_{r-1}) \cong \frac{(r-2)(2r-3)^2}{(r-1)(2r-5)^2} \tag{31}$$

It can be easily seen that the ratio (31) would technically be always greater than 1 for all $r \geq 3$. But as Table 1 shows, this ratio starts with a maximum at $r = 3$, falls rapidly and then gradually tapers off to the limiting value of 1. This implies that any T_r is technically more efficient than the preceding ones or the efficiency increases as r increases but the actual gain in efficiency is not significant after a certain stage, say, after $r = 7$ where the relative gain is less than 5 percent. That is to say, the statistic T_7 can be taken to be a reasonably optimum statistic in this case.

Table 1: Values of $A(T_r, T_{r-1})$ for varying r

r	3	4	5	6	7	8	9	10
$A(T_r, T_{r-1})$	4.50	1.85	1.47	1.32	1.24	1.20	1.16	1.14

It had been assumed earlier that r takes the values like 2, 3, 4... etc in absolute terms. If, however, r is expressed as a sub-multiple of n, say $r = n/f$ where $f \geq 2$, we get

$$E'(T_{n/f}) \cong \frac{n^2(2f-1)}{4f^2} \tag{32}$$

and

$$V(T_{n/f}) \cong \frac{n^3}{24f^3} \tag{33}$$

Hence $$\phi(T_{n/f}) \cong \frac{3n(2f-1)^2}{2f} \tag{34}$$

If f = 2, we get $$\phi(T_{n/2}) = \frac{27n}{4} \tag{35}$$

If f = 3, we get $$\phi(T_{n/3}) = \frac{25n}{2} \tag{36}$$

which is much more than the expression (35) for $T_{n/2}$.

In fact, the expression (34) increases as f increases but no extremum seems to exist. Hence, evaluate a relatively optimum statistic by considering the asymptotic relative efficiency of $T_{n/(f+1)}$ with respect to $T_{n/f}$. For this purpose we have from (34)

$$\phi(T_{n/(f+1)}) \cong \frac{3n[2(f+1)^2-1]^2}{2(f+1)} \tag{37}$$

Since for both the statistics $T_{n/(f+1)}$ and $T_{n/f}$, m = 1 and $\delta = \frac{1}{2}$, we get

$$A(T_{n/(f+1)}, T_{n/f}) = \frac{f+1}{f} \tag{38}$$

In can be seen from Table 2 that, as in the previous case, the expression (38) is always greater than 1 but starting from a maximum of 1.5 for f = 2, it tapers off to the limiting value of 1. If, however, we consider the relative gain in the efficiency of $T_{n/(f+1)}$to be more than 5 percent (and less than 5 per cent afterwards) as compared to $T_{n/f}$, then we get f = 4 from (38). That is to say, $T_{n/5}$ can be considered to be a reasonably optimal statistic in this case.

Table 2: Values of A ($T_{n/(f+1)}$, $T_{n/f}$) for varying f

f	2	3	4	5	6	7
A ($T_{n/(f+1)}$, $T_{n/f}$)	1.50	1.33	1.25	1.20	1.17	1.14

It may be recalled that the statistic T_7 was found to be relatively optimum from Table 1 when assumed the integral values 2, 3, 4...etc. in absolute terms. Hence to compare the efficiencies of T_7 and $T_{n/5}$ we note from (26) and (34) (by putting f = 4 in the light of (38) and the subsequent para) that

$$\phi(T_7) = \frac{121n}{6} \tag{39}$$

and $$\phi(T_{n/5}) = \frac{147\,n}{8} \tag{40}$$

The expression (39) is slightly greater than (40). Hence it follows that T_7 has a slight edge over $T_{n/5}$ in view of its higher efficiency as also greater convenience in its computation.

If, however, Cox-Stuart's statistic S_2 is considered, we get from (3) and (12)

$$E(S_2) = \frac{n}{2} \cdot \frac{p_2 q_1}{(1-\delta)^2} \cdot (1 - \delta^{n/2})$$

$$\cong \frac{n}{2} \cdot \frac{p_2 q_1}{(1-\delta)^2} \tag{41}$$

After differentiating (41) with respect to δ and putting $p_2 = q_1 = \frac{1}{2}$ and $\delta = 0$, we have

$$E'(S_2) = \frac{n}{4} \tag{42}$$

Also $$V(S_2) = \frac{n}{8} \tag{43}$$

Hence $$\phi(S_2) \cong \frac{n}{2} \tag{44}$$

which is much less than the efficacies for the earlier T- statistics.

As stated earlier, the above results would hold good only so long as $r \leq \left(\frac{n}{2} + 1\right)$. For $r > \left(\frac{n}{2} + 1\right)$ we have from (18)

$$E'(T_{n-R}) = \left[\frac{\partial E}{\partial \delta}(T_{n-R})\right]_{\delta = 0}$$

$$\cong \frac{1}{4}\{(n - R - 1)(n + R) - (n - 1)\}$$

$$\cong \frac{n^2}{4} \tag{45}$$

Also in the null case p_1 (or p_2) $= q_1 = \frac{1}{2}$ and

$$V(T_{n-R}) \cong \frac{1}{6}\left\{\frac{1}{4}(n-R-1)(n-R-2)(n+2R)-\frac{n}{8}(n-1)(n-2)\right\}$$

$$\cong \frac{n^3}{48} \tag{46}$$

Hence for T_{n-R} we find from (45) and (46) that

$$\phi(T_{n-R}) = \frac{n^4}{16}\cdot\frac{48}{n^3} \cong 3n \tag{47}$$

It would be seen that the efficacy (47) is free from R which implies that, for all $r > \left(\frac{n}{2}+1\right)$, the efficacy assumes the same value of 3n. That is to say, for $r > \left(\frac{n}{2}+1\right)$, all the statistics including T_n are equally efficient. It would be further seen that any T_r excepting T_2, would be more efficient than any T_{n-R}.

If Cox-Stuart's statistics S_3 is considered, we get from (7) and (12)

$$E(S_3) = \frac{n}{3}\frac{p_2 q_1}{(1-\delta)^2}\left(1-\delta^{\frac{2n}{3}}\right) \tag{48}$$

$$\cong \frac{n}{3}\frac{p_2 q_1}{(1-\delta)^2} \tag{49}$$

Hence, as before,

$$E'(S_3) = n/6 \tag{50}$$

$$V(S_3) = n/12 \tag{51}$$

and

$$\phi(S_3) \cong n/3 \tag{52}$$

It therefore follows that the efficacy of S_3 is much less than that for any other T-statistics.

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