Jour. Ind. Soc. Ag. Statistics
51(1), 1998 : 51–60

A Generalized Ridge-Cum-Stein-Rule Estimator in Linear Regression when Disturbances are Nonnormal

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(Received : November, 1996)

SUMMARY

A generalized ridge-cum-stein-rule estimator containing ridge estimator, stein-type estimators and some more as special cases, is proposed for the estimation of the vector of regression coefficients in linear regression model with nonnormal disturbances. The risk of the proposed estimator under general quadratic loss function is derived following small σ-asymptotic approach. A comparative study is carried out and some better estimators in the sense of having smaller risk are found.

Key words : Operational ridge estimator, Stein-rule estimator, Small σ-asymptotic, Risk, Efficient estimators.

1. Introduction

Let us consider the linear regression model

$$y = X\beta + u \tag{1.1}$$

where y is a $T \times 1$ vector of observations on the variable to be explained, X is a $T \times p$ full column rank martix of T observations on p explanatory variables, β is a $p \times 1$ vector of regression coefficients and u is a $T \times 1$ vector of disturbances whose elements $u_t\,(t = 1, 2, \ldots, T)$ are independently and identically distributed having first four moments finite, that is,

$$E(u_t) = 0,\ E(u_t^2) = \sigma^2,\ E(u_t^3) = \sigma^3\gamma_1,\ E(u_t^4) = \sigma^4(\gamma_2 + 3)$$

(where γ_1 and γ_2 are Pearson's measures of skewness and kurtosis of the distribution of disturbances respectively).

This paper concerns with some general small σ-asymptotic results (as $\sigma \to 0$) on the lines of Kadane [5]. For details regarding advantages of small-σ asymptotic approach over the large sample theory, one is referred to Kadane [5].

The ordinary least squares (OLS) estimator of β in (1.1) is given by

$$b = (X'X)^{-1} X'y \tag{1.2}$$

For D and B being known p x p positive definite matrices and $z = \frac{(y - Xb)'(y - Xb)}{b'Bb}$, let the operational generalized ridge-cum-stein rule estimator be defined as

$$\hat{\beta}_G = [I + kzD]^{-1} g(z)b \tag{1.3}$$

where k (> 0) is a characterizing scalar, z has atleast first m (≥ 4) moments finite and g(z) being a real valued function of the random variable z and satisfying the validity conditions of Taylor's (Maclaurin's) series expansion with its first two derivatives with respect to z being bounded in probability, is a bounded function of z such that

$$g(z = 0) = 1 \text{ and } g(z) = 0_p(1) \text{ as } \sigma \to 0$$

It may be easily verified that the operational ridge-type estimator $\hat{\beta}_R = [I + kz(X'X)^{-1}]^{-1} b$, the stein-type estimator $\hat{\beta}_S = [1 + kz]^{-1} b$ and the generalized estimator $\hat{\beta} = [I + kzD]^{-1} b$ in Vinod and Ullah [8] are the special cases of the proposed operational generalized ridge-cum-stein rule estimator $\hat{\beta}_G$ for $D = (X'X)^{-1}$, I, D respectively and $g(z) = 1$ in all the above cases.

Some new estimators which belong to the proposed class of estimators are

$$\hat{\beta}_{G1} = [I + kzD]^{-1} (1 + k_1 z)b \tag{1.4}$$

$$\hat{\beta}_{G2} = [I + kzD]^{-1} (1 + k_1 z)^{k_2} b \tag{1.5}$$

$$\hat{\beta}_{G3} = [I + kzD]^{-1} \{2 - (1 - k_1 z)^{k_2}\} b \tag{1.6}$$

where k, k_1, k_2 are the characterizing scalars to be chosen suitably so that $(1 + k_1 z)$; $(1 + k_1 z)^{k_2}$; $\{2 - (1 - k_1 z)^{k_2}\}$ and their first two derivatives with respect to z remain bounded.

2. *Properties of the Estimator* $\hat{\beta}_G$

For convenience, we introduce the following notations:

$$M = I - X(X'X)^{-1} X' ; \quad \overline{M} = I - M$$

$$N = I - (X'X)^{-1} X'(I * M)X ; \quad \overline{N} = I - N$$

$$q = n(n+2) + \gamma_2 \operatorname{tr}(M*M)\,;\; n = T - p \tag{2.1}$$

$$\phi_0 = \frac{\operatorname{tr} N (X'X)^{-1} QD}{\operatorname{tr}(X'X)^{-1} QD}; \quad \phi_1 = \frac{\operatorname{tr} N (X'X)^{-1} Q}{\operatorname{tr}(X'X)^{-1} Q} \tag{2.2}$$

$$\theta = \frac{\gamma_2}{n + \gamma_2} \tag{2.3}$$

where '*' denotes the Hadamard product of matrices.

The risk function of the estimator $\hat{\beta}_G$ under the general quadratic loss function $(\hat{\beta}_G - \beta)' Q (\hat{\beta}_G - \beta)$ is given by

$$\rho(\hat{\beta}_G) = E(\hat{\beta}_G - \beta)' Q (\hat{\beta}_G - \beta) \tag{2.4}$$

where Q is a positive definite symmetric matrix.

Theorem: The risk function of the estimator $\hat{\beta}_G$ to order $0(\sigma^4)$ when the disturbances are small in Kadane's sense [5] is given by

$$\begin{aligned}\rho(\hat{\beta}_G) = {} & \sigma^2 \operatorname{tr}(X'X)^{-1} Q - 2\sigma^3 k\gamma_1 \frac{\beta' D' Q (X'X)^{-1} X' (I*M) e}{\beta' B \beta} \\ & + \sigma^4 k\, r_o + 2\sigma^3 g'(0)\, \gamma_1 \frac{\beta' Q (X'X)^{-1} X' (I*M) e}{\beta' B \beta} \\ & + \sigma^4 g'(0) \left\{ r_1 - 2k \frac{\beta' D' Q \beta}{(\beta' B \beta)^2} q \right\}\end{aligned} \tag{2.5}$$

where $g'(0)$ is the first derivative of $g(z)$ with respect to z at the point $z = 0$, 'e' is a $T \times 1$ vector with all its elements equal to 1 and

$$\begin{aligned} r_0 = q \frac{\beta' D' QD\beta}{(\beta' B \beta)^2} \Bigg[k - \frac{2(n+\gamma_2)\beta' B\beta}{q\beta' D' QD\beta} \Bigg[& \{(1-\theta\phi_0) \operatorname{tr}(X'X)^{-1} QD\} \\ & - \frac{2\beta' B (X'X)^{-1} QD\beta}{\beta' B\beta} \left\{ 1 - \frac{\theta\beta' BN (X'X)^{-1} QD\beta}{\beta' B (X'X)^{-1} QD\beta} \right\} \Bigg]\Bigg] \end{aligned} \tag{2.6}$$

$$\begin{aligned} r_1 = q \frac{\beta' Q\beta}{(\beta' B\beta)^2} \Bigg[g'(0) + \frac{2(n+\gamma_2)\beta' B\beta}{q\beta' Q\beta} \Bigg[& \{(1-\theta\phi_1) \operatorname{tr}(X'X)^{-1} Q\} \\ & - \frac{2\beta' B (X'X)^{-1} Q\beta}{\beta' B\beta} \left\{ 1 - \frac{\theta\, \beta' BN (X'X)^{-1} Q\, \beta}{\beta' B (X'X)^{-1} Q\, \beta} \right\} \Bigg]\Bigg] \end{aligned} \tag{2.7}$$

Now we know that the risk associated with OLS estimator b under general quadratic loss function $(b-\beta)' Q (b-\beta)$ is

$$\rho(b) = E(b-\beta)' Q (b-\beta) = \sigma^2 \operatorname{tr}(X'X)^{-1} Q \tag{2.8}$$

Further, for the estimator

$$\hat{\beta} = [I + k z D]^{-1} b \tag{2.9}$$

in Vinod and Ullah [8] the risk for $\hat{\beta}$ is

$$\begin{aligned} \rho(\hat{\beta}) &= E(\hat{\beta}-\beta)' Q (\hat{\beta}-\beta) \\ &= \sigma^2 \operatorname{tr}(X'X)^{-1} Q - 2\sigma^3 k \gamma_1 \frac{\beta' D' (X'X)^{-1} X' (I*M) e}{\beta' B \beta} \\ &\quad + \sigma^4 k r_0 \end{aligned} \tag{2.10}$$

Thus, from (2.5), (2.8) and (2.10), we have

$$\begin{aligned} \rho(\hat{\beta}_G) &= \rho(\hat{\beta}) + \sigma^3 g'(0) \gamma_1 \frac{\beta' Q (X'X)^{-1} X' (I*M) e}{\beta' B \beta} \\ &\quad + \sigma^4 g'(0) r_1 - 2\sigma^4 g'(0) k \frac{\beta' D' Q \beta}{(\beta' B \beta)^2} q \end{aligned} \tag{2.11}$$

From the algebraic inequality Vinod and Ullah [8] we have

$$0 \le \eta_p \le \phi_i \le \eta_i \le 1; \; i = 0, 1 \tag{2.12}$$

where ϕ_0, ϕ_1 are given in (2.2) and η_1, η_p are the smallest and largest eigen values of the matrix N respectively. Also from Rao [6], we have

$$\min_\beta \frac{\beta' B \beta}{\beta' Q \beta} > \delta_p^* > 0 \tag{2.13}$$

and

$$\mu_1^* > \frac{\beta' B (X'X)^{-1} Q \beta}{\beta' B \beta} > 0 \tag{2.14}$$

where δ_p^* is the minimum eigen value of the matrix BQ^{-1} and μ_1^* is the maximum eigen value of the matrix $(X'X)^{-1} Q$.

From Vinod and Ullah [8] regarding $\theta = \dfrac{\gamma_2}{n + \gamma_2}$ in (2.3) we have

$$0 < \theta < 1, \quad \text{when } \gamma_2 > 0 \tag{2.15}$$

$$\theta < 0, \qquad \text{when} \quad \gamma_2 < 0;\ n \geq 2 \tag{2.16}$$

$$\theta < 0, \qquad \text{when} \quad \gamma_2 < 0;\ (1+\gamma_2) > 0;\ n = 1 \tag{2.17}$$

For the disturbances being symmetrically distributed (that is $\gamma_1 = 0$), from (2.11), we have

$$\rho(\hat{\beta}_G) - \rho(\hat{\beta}) = \sigma^4\, g'(0)\, r_1 - 2\sigma^4\, g'(0)\, k\, \frac{\beta' D' Q\beta}{(\beta' B\beta)^2}\, q \tag{2.18}$$

Also, from Vinod and Ullah [8], we know that $\hat{\beta}$ is superior to b in the sense of having smaller risk, if

$$0 < k \leq \frac{2(n+\gamma_2)}{q}\, \delta_p\ [(1-\theta\phi_o)\, \mathrm{tr}\, (X'X)^{-1} QD - 2\mu_1 (1-\theta\, \eta_j)]$$

or if

$$0 < k \leq \frac{2(n+\gamma_2)}{q}\, \delta_p\, \mu_1\ [(1-\theta\phi_o)\, d_o - 2(1-\theta\, \eta_j)] \tag{2.19}$$

where $j = p$ and 1 correspond to symmetric leptokurtic distributions ($\gamma_1 = 0,\ \gamma_2 > 0$) and symmetric platykurtic distributions ($\gamma_1 = 0,\ \gamma_2 < 0, n \geq 2, n = 1$ with $\gamma_2 + 1 > 0$) respectively, δ_p is the minimum eigen value of the matrix B $(D'QD)^{-1}$, μ_1 is the maximum eigen value of the matrix $(X'X)^{-1}$ QD and

$$d_o = \frac{1}{\mu_1}\, \mathrm{tr}\, (X'X)^{-1} QD \geq \frac{2 - 2\theta\, \eta_j}{(1-\theta\phi_o)} = 2 + \frac{2\theta\, (\phi_o - \eta_j)}{1-\theta\phi_o} > 0$$

In (2.18) since $\beta' D' Q\, \beta$ being positive definite is greater than zero, $\hat{\beta}_G$ is superior to $\hat{\beta}$ in the sense of having smaller risk if

$$k > 0,\ g'(0) > 0 \ \text{ and } \ \sigma^4\, g'(0)\, r_1 < 0$$

$$\text{or if } \ r_1 < 0 \ \text{ for } \ g'(0) > 0 \ \text{ and } \ k > 0 \tag{2.20}$$

or from (2.12) to (2.17), the condition (2.20) is satisfied if

$$k > 0;\ 0 < g'(0) \leq \frac{2(n+\gamma_2)}{q}\, \delta_p^*\ [(1-\theta\phi_1)\, \mathrm{tr}\, (X'X)^{-1} QD - 2\mu_1^* (1-\theta\, \eta_j)]$$

$$k > 0;\ 0 < g'(0) \leq \frac{2(n+\gamma_2)}{q}\, \delta_p^*\, \mu_1^*\ [(1-\theta\phi_1)\, d_1 - 2(1-\theta\, \eta_j)] \tag{2.21}$$

where $j = p$ and 1 correspond to symmetric leptokurtic distributions ($\gamma_1 = 0, \gamma_2 > 0$) and symmetric platykurtic distributions ($\gamma_1 = 0, \gamma_2 < 0, n \geq 2; n = 1$ with $\gamma_2 + 1 > 0$) respectively, and

$$d_1 = \frac{1}{\mu_1^*} \operatorname{tr}(X'X)^{-1} Q \geq \frac{2 - 2\theta\eta_j}{(1-\theta\phi_1)} = 2 + \frac{2\theta(\phi_1 - \eta_j)}{(1-\theta\phi_1)} > 0$$

Further, in the efficiency condition (2.21) of $\hat{\beta}_G$ over $\hat{\beta}$, if k is chosen according to the efficiency condition (2.19) of $\hat{\beta}$ over b, then the estimator $\hat{\beta}_G$ will be more efficient than both the estimators $\hat{\beta}$ and b.

Some Remarks

(a) For normal distribution ($\gamma_2 = 0$), the dominance conditions for $\hat{\beta}_G$ over $\hat{\beta}$ in (2.21) reduces to

$$k > 0;\ 0 < g'(0) \leq \frac{2\delta_p^* \mu_1^*}{(n+2)} (d_1 - 2)$$

$$d_1 = \frac{1}{\mu_1^*} \operatorname{tr}(X'X)^{-1} Q > 2 \qquad (2.22)$$

which ensures that the estimators of the class $\hat{\beta}_G$ satisfying the dominance condition (2.22) have smaller risk than that of the estimator $\hat{\beta}$.

(b) For skewed distributions ($\gamma_1 \neq 0$), we have

$$\rho(\hat{\beta}_G) - \rho(\hat{\beta}) = 2\sigma^3 g'(0)\, \gamma_1 \frac{\beta' Q (X'X)^{-1} X' (I*M) e}{\beta' B \beta} + \sigma^4 g'(0) r_1 - 2\sigma^4 g'(0) k \frac{\beta' D' Q \beta}{(\beta' B \beta)^2} q$$

$$= 2\sigma^3 g'(0)\, n\, \beta' Q \operatorname{Cov}(b, \bar{s}) + \sigma^4 g'(0) r_1 - 2\sigma^4 g'(0) k \frac{\beta' D' Q \beta}{(\beta' B \beta)^2} \qquad (2.23)$$

where $\bar{s} = \frac{u'Mu}{n} = \frac{y'My}{n}$ is the disturbance estimator and $\operatorname{Cov}(b, \bar{s}) = (X'X)^{-1} X' E(u'Mu.u) = \gamma_1 (X'X)^{-1} X (I*M) e$ is the covariance between b and $\bar{s}$. For all the elements of Cov $(b, \bar{s})$ and β being of opposite signs and $g'(0) > 0$ making the coefficient

$2g'(0)\, n\, \beta' Q\, \mathrm{Cov}(b, \bar{s})$ of σ^3 negative, and for $\gamma_2 > 0$, $\gamma_2 < 0$, $\gamma_2 = 0$ and ranges of $g'(0)$ given by the conditions (2.21) and (2.22) making the coefficient $g'(0)\, r_1$ of σ^4 negative, we observe from (2.23) that $\hat{\beta}_G$ dominates over $\hat{\beta}$ in the sense of having smaller risk, that is $\rho(\hat{\beta}_G) - \rho(\hat{\beta}) < 0$ for the skewed distribution of disturbances.

(c) In particular, for the estimator $\hat{\beta}_{G1} = [I + kzD]^{-1}(1 + k_1 z)\, b$ belonging to the class $\hat{\beta}_G$ of estimators, we have $g'(0) = k_1$ so that for normal distribution, by choosing $k > 0$ and $k_1 > 0$ such that

$$k > 0\, ; 0 < k_1 \le \frac{2\delta_p^* \mu_1^*}{(n+2)} (d_1 - 2)$$

$$d_1 = \frac{1}{\mu_1^*} \operatorname{tr}(X'X)^{-1} Q > 2 \tag{2.24}$$

satisfying (2.22), we see that the estimator $\hat{\beta}_{G1}$ is better than $\hat{\beta}$ {see Vinod and Ullah [8]} in the sense of having smaller risk. Similar remarks follow for the estimators $\hat{\beta}_{G2}$ and $\hat{\beta}_{G3}$ also for normal or nonnormal disturbances.

(d) From Vinod and Ullah [8], for normal distribution the dominance condition of $\hat{\beta}$ over b is

$$0 < k < \frac{2\delta_p \mu_1}{(n+2)} (d_o - 2) \tag{2.25}$$

Hence, incorporating (2.25) in the efficiency condition (2.22) of $\hat{\beta}_G$ over $\hat{\beta}$, the estimator $\hat{\beta}_G$ is more efficient than both the estimator $\hat{\beta}$ and the OLS estimator b if

$$0 < k \le \frac{2\delta_p \mu_1}{(n+2)} (d_o - 2)\, ; \; 0 < g'(0) \le \frac{2\delta_p^* \mu_1^*}{(n+2)} (d_1 - 2) \tag{2.26}$$

3. Derivation of Results

For $u = \sigma v$, the model (1.1) can be rewritten as

$$y = X\beta + \sigma v \tag{3.1}$$

where σ is small approaching to zero and $v_1, v_2, \ldots, v_T$ are independently and identically distributed with

$$E(v_t) = 0,\; E(v_t^2) = 1,\; E(v_t^3) = \gamma_1,\; E(v_t^4) = (\gamma_2 + 3) \tag{3.2}$$

Further, we have

$$(y - Xb)'(y - Xb) = \sigma^2 v' Mv \tag{3.3}$$

$$b - \beta = \sigma (X'X)^{-1} X' v \tag{3.4}$$

and

$$\frac{1}{b' B b} = \frac{1}{\beta' B \beta + 2\sigma\beta' B (X'X)^{-1} X' v + \sigma^2 v' X (X'X)^{-1} B (X'X)^{-1} X' v}$$

$$= \frac{1}{\beta' B \beta} - \frac{2\sigma\beta' B (X'X)^{-1} X' v}{(\beta' B \beta)^2} \qquad \{\text{upto order } 0(\sigma)\} \tag{3.5}$$

Now, we have

$$\hat{\beta}_G = [I + kzD]^{-1} g(z) b$$

$$= [I - kzD] g(z) b + \ldots \qquad \{\text{Following Vinod and Ullah [8]}\}$$

Expanding g(z) about the point z = 0 in second order Taylor's series and noting g(z = 0) = 1, we have

$$\hat{\beta}_G = \{ g(0) + z g'(0) + \frac{z^2}{2!} g''(z^*) \} b - k z \{g(0) + z g'(0) + \frac{z^2}{2!} g''(z^*) \} Db + \ldots$$

$$= b + z \{g'(0) - kD\} b - z^2 g'(0) kD b + \frac{z^2}{2!} g''(z^*) b - \frac{z^3}{2!} g''(z^*) k Db + \ldots \tag{3.6}$$

where $g''(z^*)$ is the second derivative of g(z) with respect to z at the point $z = z^*$; $z^* = zw$, $0 < w < 1$. Using (3.3), (3.4) and (3.5) in (3.6) and retaining terms upto order $0(\sigma^3)$, we have

$$\hat{\beta}_G - \beta = \sigma \xi_1 + \sigma^2 k \xi_2 - \sigma^2 g'(0) \xi_2^* + \sigma^3 k \xi_3 - \sigma^3 g'(0) \xi_3^* \tag{3.7}$$

where

$$\xi_1 = (X'X)^{-1} X' v \tag{3.8}$$

$$\xi_2 = \frac{v' Mv}{\beta' B\beta} D\beta \; ; \; \xi_2^* = \frac{v' Mv}{\beta' B\beta} \beta \tag{3.9}$$

$$\xi_3 = -\frac{v' Mv}{\beta' B\beta} \left[D (X'X)^{-1} X'v - \frac{2\beta' B (X'X)^{-1} X' v}{\beta' B\beta} D\beta \right] \tag{3.10}$$

$$\xi_3^* = -\frac{v' M v}{\beta' B \beta}\left[(X'X)^{-1} X' v - \frac{2\beta' B (X'X)^{-1} X' v}{\beta' B \beta}\beta\right] \tag{3.11}$$

Hence from (3.7) the bias of $\hat{\beta}_G$ to order $0(\sigma^2)$ is

$$\text{Bias}(\hat{\beta}_G) = E(\hat{\beta}_G) - \beta = -\frac{\sigma^2 kn}{\beta' B\beta} D\beta + \frac{\sigma^2 g'(0)\, n}{\beta' B \beta}\beta \tag{3.12}$$

$$\{\text{since } E(v) = 0 \text{ and } E(v' Mv) = \operatorname{tr} M = n = T - p\}$$

The risk associated with $\hat{\beta}_G$ upto order σ^4, is

$$\rho(\hat{\beta}_G) = E(\hat{\beta}_G - \beta)' Q (\hat{\beta}_G - \beta)$$

$$= \sigma^2 E(\xi'_1\; Q\,\xi_1) + 2\sigma^3 k\, E(\xi'_1 Q\, \xi_2) - 2\sigma^3 g'(0)\, E(\xi'_1 Q\, \xi_2^*)$$

$$+ 2\sigma^4 k\, E(\xi'_1 Q\, \xi_3) - 2\sigma^4 g'(0)\, E(\xi'_1 Q\, \xi_3^*) + \sigma^4 k^2 E(\xi'_2 Q\, \xi_2)$$

$$+ \sigma^4 \{g'(0)\}^2 E(\xi_2^{*\prime}\; Q\,\xi_2^*) - 2\sigma^4 g'(0)\, k\, E(\xi'_2 Q\, \xi_2^*) \tag{3.13}$$

It can be verified that

$$E(\xi'_1 Q\, \xi_1) = \operatorname{tr}(X'X)^{-1} Q \tag{3.14}$$

$$E(\xi'_1 Q\, \xi_2) = -\frac{1}{\beta' B\beta}\beta' D' Q (X'X)^{-1} X' E(v' Mv.\, v)$$

$$= -\gamma_1 \frac{\beta' D' Q (X'X)^{-1} X' (I*M)\, e}{\beta' B \beta} \tag{3.15}$$

$$E(\xi'_1 Q\, \xi_2^*) = -\gamma_1 \frac{\beta' Q (X'X)^{-1} X' (1*M)\, e}{\beta' B \beta} \tag{3.16}$$

$$E(\xi'_2 Q\, \xi_2) = q\frac{\beta' D' QD\beta}{(\beta' B \beta)^2};\; E(\xi'_2 Q\, \xi_2^*) = q\frac{\beta' D' Q\beta}{(\beta B\beta)^2} \tag{3.17}$$

$$E(\xi_2^{*\prime} Q\, \xi_2^*) = q\frac{\beta' Q \beta}{(\beta' B \beta)^2} \tag{3.18}$$

$$E(\xi'_1 Q\, \xi_3) = -\frac{1}{\beta' B \beta}\,[n \operatorname{tr}(X'X)^{-1} QD + \gamma_2 \operatorname{tr}(X'X)^{-1} QD\,\overline{N}$$

$$-\frac{2}{\beta' B \beta}\,\{n\,\beta' B (X'X)^{-1} Q D \beta$$

$$+ \gamma_2\, \beta' B\, \overline{N} (X'X)^{-1} Q D \beta\,\}] \tag{3.19}$$

$$E(\xi'_1 Q \xi^*_3) = -\frac{1}{\beta' B \beta}[n \operatorname{tr}(X'X)^{-1} Q + \gamma_2 \operatorname{tr}(X'X)^{-1} Q \bar{N}$$
$$- \frac{2}{\beta' B \beta}\{n \beta' B (X'X)^{-1} Q \beta + \gamma_2 \beta' B \bar{N} (X'X)^{-1} Q \beta\}] \tag{3.20}$$

After substituting from (3.14) to (3.20) in (3.13) and noting that $\bar{N} = I - N$, we get the required expression for the risk $\rho(\hat{\beta}_G)$ to order σ^4 as given in (2.5).

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