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A Series of Balanced Ternary Change Over Designs

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SUMMARY

In this paper an attempt has been made to construct a series of balanced ternary change over designs along with the method of analysis. In the method of construction, it is considered that a treatment symbol should occur in a given sequence at most twice but the same treatment will not appear in two consecutive periods. Various combinatorial requirements along with their parametric relationships for a ternary change over design to be balanced are written and discussed. According to Saha [2] a set of difference sets corresponding to initial sequences are defined satisfying certain conditions for getting balanced change over design. Two designs for v = 7 and v = 11 from the Series of Balanced Ternary Change Over Designs are developed.

Key words: Ternary change over designs, Analysis, Combinatorial requirements, Construction of balanced ternary change over designs.

1. Introduction

A change over design (with the presence of only first order residual effects) is said to be balanced if every estimable elementary contrast among direct effects is estimated with the same variance and every estimated elementary contrast among residual effects is estimated with the same variance. Quite a few balanced change over designs are available in the literature ([1], [2]). However, all these designs are binary in the sense that no treatment symbol occurs in a given sequence more than once. If this condition is relaxed, it will perhaps be possible to obtain some more balanced change over designs. The purpose of this paper is to investigate this aspect. A series of balanced change over designs are actually constructed.

2. Analysis of 'Ternary' Change Over Designs

We consider designs in which a treatment symbol appears in a given sequence at most twice, but, the same treatment does not appear in two consecutive periods. In conformity with the nomenclature used in block designs, we may call such designs as 'ternary' change over designs.

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We assume that in the design under consideration, there are v treatments, b sequences (i.e. experimental units) and each sequence has length k (i.e. k periods). As (i) each treatment occurs in every period a constant number of times usually denote by t, say, and (ii) any two consecutive treatments in any sequence are distinct.

It is therefore, obvious that treatment is replicated exactly r = tk times in the design. It is assumed that t = (b/v) is an integer.

For the analysis, we consider the usual homoscedastic fixed effects model

$$Y_{ijlm} = \mu + s_i + p_j + t_l + r_m + e_{ijlm} \tag{2.1}$$

where Y_{ijlm} is the observation corresponding to a treatment l immediately preceded by treatment m given to sequence i in period j (i = 1, 2, ..., b; j = 1, 2, ..., k; l, m = 1, 2, ..., v, l ≠ m), μ is a general mean effect, s_i the effect of the i^{th} sequence (unit), p_j the effect of the j^{th} period, t_l the direct effect of the l^{th} treatment, r_m the residual effect of the m^{th} treatment and e_{ijlm} is the random error component, assumed to have a zero mean and constant variance σ^2. We shall consider the case in which there are no residual effects present in the 1st period, so that when j = 1, the suffix m and r_m are dropped. For convenience the symbols s_i, p_j, t_l etc. will denote the least squares estimators of these effects as well. Let T_l and R_l denote the totals of observations which contain the direct effect of l and residual effect of treatment l, respectively, (l = 1, 2, ..., v).

Taking the restrictions (i), (ii) above and using $\Sigma_l\, t_l = 0 = \Sigma_m\, r_m = \Sigma_j\, p_j = \Sigma_i\, s_i$ the least square normal equations for estimating direct and residual effects for a design are as follows:

$$G = bk\,\mu \tag{2.2}$$

$$T_l = r\mu + \sum_i s_i^{(l)} + rt_l + \sum_{m=1}^{v} \gamma_l^m\, r_m \tag{2.3}$$

$$R_l = (r - b/v)\mu + (r - b/v)\, r_l + \sum_{\substack{m=1 \\ m \neq l}}^{v} \gamma_m^l\, t_m + (b/v) \sum_{j \neq 1} p_j + \sum_i s_i^{[l]} \tag{2.4}$$

where γ_l^m is the number of times, treatment l has been immediately preceded by m (in the sequences) in the whole design, $\sum_i s_i^{(l)}$ is the sum of those sequence effects which contain treatment l in any period, $\sum_i s_i^{[l]}$ is the sum of those sequence effects which contain treatment l in any but the final period. Note that if treatment l appears in a given sequence twice, that particular sequence effect will appear twice in the sum $\sum_i s_i^{(l)}$. Similar convention will be followed for the sum $\sum_i s_i^{[l]}$. G is the grand total of all observations.

Let $\Sigma_i S_i^{(l)}$ denote the sum of the totals of those sequences which contain treatment l in any period, and $\sum_i S_i^{[l]}$, the sum for those sequences which contain treatment l in any but the final period with the convention that if treatment l appears twice in a sequence, that particular sequence total will appear twice in the sum $\sum_i S_i^{(l)}$; similar convention is followed for $\sum_i S_i^{[l]}$ also. We further restrict out attention to the class of designs for which $\sum_{i=1}^{b} s_{li}^{2}$ is a constant for all l = 1, 2, ..., v, where s_{li} is the number of times the l^{th} treatment appears in the i^{th} sequence. Then, we have the following normal equations:

$$\sum_i S_i^{(l)} = rk\ \mu + k \sum_i s_i^{(l)} + Rt_l + \sum_{m=1,\ m \neq l}^{v} \lambda_{l_m}\ t_m + (R - b/v)\, r_l + \sum_{m=1,\, m \neq l}^{v} (\lambda_{lm} - \beta_m^l)\, r_m \qquad (2.5)$$

$$\sum_i S_i^{[l]} = r(k-1)\,\mu + k \sum_i s_i^{[l]} + (R - b/v)\, t_l + \sum_{m=1,\ m \neq l}^{v} (\lambda_{l_m} - \beta_l^m)\, t_m + (R - b/v)\, r_l + \sum_{m=1,\ m \neq l}^{v} (\lambda_{lm} - \beta_l^m - \beta_m^l)\, r_m \qquad (2.6)$$

where $R = \sum_{i=1}^{b} s_{li}^2\ \lambda_{lm}$ is number of times the treatment pair (l, m) has occured together in sequences in the design and β_l^m is the number of times the treatment pair (l, m) has occured in sequences with l in the last period in the whole design. The quantities λ_{lm} and β_l^m may be called as functions of ternary frequencies of pair of treatments.

Since $\sum_i^b s_{li} = r$ for all l, $\sum_l s_{li} = k$ for all i, (l = 1, 2, ..., v; i = 1, 2, ...b) and $\sum_i s_{li}^2 = R$ for all l, it follows that

$$\lambda_{ll'} = \sum_{i=1}^{b} s_{li}\, s_{l'i}$$

is also a constant, λ, say $l \neq l'; l, l' = 1, 2, \ldots, v$. It is clear that a sum of a product of ternary frequencies of a pair of treatments when added over-all the sequences is equivalent to combinatorial requirement IV mentioned on next page

It also amounts to

$$\lambda = (rk - R)/(v-1) \tag{2.7}$$

Also $\hat{p}_1 = P'_1/b - G/bk$, where P'_1 is the total of all observations in the first period. Thus eliminating $\sum_i s_i^{(l)}$ and $\sum_i s_i^{[l]}$ by making use of (2.5), (2.6), (2.7) and $\sum_{j \neq 1} p_j = -p_1$ in (2.3) and (2.4), we have

$$P_l = (r - R/k)\, t_l - \lambda/k \sum_{m = 1, m \neq l} t_m - 1/k\,(R - b/v)\, r_l$$
$$- 1/k \sum_{m = 1, m \neq l} (\lambda - \beta_m^l - k\, \gamma_l^m)\, r_m \tag{2.8}$$

$$Q_l = (r - R/k - b/v + b/vk)\, r_l - 1/k \sum_{m = 1, m \neq l} (\lambda - \beta_m^l - \beta_l^m)\, r_m$$
$$- 1/k\,(R - b/v)\; t_l - 1/k \sum_{m = 1, m \neq l} (\lambda - \beta_l^m - k\, \gamma_m^l)\; t_m \tag{2.9}$$

where

$$P_l = T_l - (1/k) \sum_i s_i^{(l)} \tag{2.10}$$

and $Q_l = R_l - (1/k) \sum_i S_i^{[l]} + (p'_l/v - G/vk), l = 1, 2, \ldots, v$ (2.11)

From (2.8) and (2.9), it is now clear that if the design satisfies some further constancy restrictions imposed on the parameters γ_l^m and β_l^m the normal equations (2.8) and (2.9) are further simplified. Suppose the design satisfies the following conditions.

(a) $\gamma_l^m = \gamma$ for all l, m $(l \neq m)$, l, m = 1, 2,..., v (2.12)

(b) $\beta_l^m = \beta$ for all l, m $(l \neq m)$, l, m = 1, 2,..., v

under (2.12 (a), (b) the normal equations (2.8) and (2.9) reduce to

$$P_l = A\ t_l + B\ r_l \quad (2.13)$$

$$Q_l = C\ t_l + D\ r_l$$

where A, B, C and D are some constants depending on the design parameters only. Constant restrictions serve the two-fold purposes of attaining the simplicity of analysis and the balance. By virtue of the conditions (2.13), the resultant design becomes balanced as well.

We thus have the following combinatorial requirements for a ternary change-over-design to be balanced.

I. Each treatment occurs in a given period an equal number of times. This number is usually denote by t and clearly we then have b = vt.

II. It is defined that s_{li} is the number of times the l^{th} treatment occurs in the i^{th} sequence (l = 1, 2,...,v). Since any treatment occurs in a given sequence at most twice and the same treatment does not appear in two consecutive periods then the ternary nature of the design is brought out by imposing the conditions that $0 \leq s_{li} \leq 2$ for all l and i. Also it is indicated that $\sum_i s_{li} = r$ for all l and $\sum_l s_{li} = k$ for all i.

III. Every treatment is followed and preceded by every other treatment equally frequently, say γ number of times.

IV. Every ordered pair of distinct treatments should occur as an ordered pair of consecutive treatments in a sequence say λ number of times.

V. Every ordered pair of distinct treatments should occur as an ordered pair of consecutive treatments in the same number of curtailed sequences formed by omitting the final period.

VI. In those sequences in which a given treatment occurs in the final period, the other treatments occur equally frequently say β number of times.

VII. In those sequences in which a given treatment occurs in any but the final period, each other treatment occurs equally often in the final period say β number of times.

As explained earlier, conditions I and III are the most important combinatorial requirements for a change-over-design to be balanced with respect of direct and first residual effects. Clearly from requirement I, (b/v) is an integer. Combinatorial requirements II and IV have already been explained. Requirement III implies that $b(k-1) = \gamma v(v-1)$ i.e. $b(k-1)/v$ is a multiple of $(v-1)$. As $s_{li} = 0$, 1 and 2 and every treatment occurs equal number of times in a period. It clearly shows that $\sum_i s_{li}^2 = R$ by virtue of condition IV.

VIII. If condition V holds then conditons VI and VII are automatically satisfied. If final period is omitted, then every ordered pair of distinct treatments should occur, say λ' number of times in the curtailed sequences where $\lambda' = (r-t)(k-1)-(R^2-t)/(v-1)$. Every treatment followed and preceded by every other treatment equally frequently, $(\gamma-1)$ number of times. One can say that condition III gives balance with respect to previous treatments. The remaining conditions give balance in current treatments and preceeding treatments with respect to periods. Thus if a design satisfies combinatorial requirements from I to VII, it is ternary balanced. The same type of conditions for binary balanced were also explained by Patterson [1]. These combinatorial conditions are necessary conditions for a change over design to be variance balanced for the direct and first order residual treatment effects.

In the next section, we shall actually construct a series of balanced ternary change-over-designs.

3. *A Series of Balanced Ternary Change-Over-Designs*

Suppose $v = (4t + 3)$ is a prime or a prime power. We construct a series of balanced ternary change-over designs with parameters

$$v = 4t+3,\ b = (4t+2)\ (4t+3),\ r = 4(4t+2),\ k = 4 \qquad (3.1)$$

Let x be a primitive element of GF (v) and consider the initial sequences

$$(0, x^{2v}, 0, x^{2v+2}),\ (0, -x^{2v},\ 0, -x^{2v+2}),\ v = 0, 1, 2, \ldots, 2t \qquad (3.2)$$

Then to prove that these (4t + 2) initial sequences will give rise to a balanced ternary change-over design with parameters as given in (3.1), we reproduce certain results from Saha [2] on the method of difference for construction of balanced binary change-over designs, with suitable modifications.

Suppose 0, 1, 2,..., v-1 be the v elements of module representing the treatments. Suppose $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ is an initial sequence, containing k elements of module (not necessarily distinct). The difference set $(\psi_0, \psi_1, \ldots, \psi_{k-1})$ corresponding to the initial sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ is defined as

$$\alpha_0 = \psi_0,\ \alpha_1 = \psi_0 + \psi_1, \alpha_2 = \psi_0 + \psi_1 + \psi_2, \ldots, \alpha_{k-1} = \psi_0 + \psi_1 + \ldots + \psi_{k-1}$$

The additions are taken mod v. Without any loss of generality, we may take $\psi_0 = 0$, and therefore write the difference set $(0, \psi_1, \psi_2, \ldots, \psi_{k-1})$ as $(\psi_1, \psi_2, \ldots, \psi_{k-1})$.

Suppose $(\psi_{1i}, \psi_{2i}, \ldots, \psi_{\overline{k-1}\,i})$, (i = 1, 2,.., b/v), be a set of b/v difference sets corresponding to b/v initial sequences. From the following triangular arrays as

$$d_i = \begin{bmatrix} \psi_{1i} & \psi_{1i} + \psi_{2i} & \psi_{1i} + \psi_{2i} + \psi_{3i} & \ldots & \psi_{1i} + \psi_{2i} + \ldots + \psi_{\overline{k-1}\,i} \\ & \psi_{2i} & \psi_{2i} + \psi_{3i} & \ldots & \psi_{2i} + \ldots + \psi_{\overline{k-1}\,i} \\ & & \psi_{3i} & \ldots & \psi_{3i} + \ldots + \psi_{\overline{k-1}\,i} \\ & & & & \psi_{\overline{k-1}\,i} \end{bmatrix}$$

$$i = 1, 2, \ldots, b/v$$

let the sets $d_{\gamma i}, d_{\eta i}$ and $d_{\beta i}$ be defined as

$$d_{\gamma i} = (\psi_{1i}, \psi_{2i}, \psi_{3i}, \ldots, \psi_{\overline{k-1}\,i})$$

$$d_{\eta i} = (\psi_{1i}, \psi_{1i} + \psi_{2i}, \psi_{1i} + \psi_{2i} + \psi_{3i}, \ldots, \psi_{1i} + \psi_{2i} + \ldots \psi_{\overline{k-1}\,i})$$

$$d_{\beta i} = (\psi_{1i} + \psi_{2i} + \ldots + \psi_{\overline{k-1}\,i}, \psi_{2i} + \psi_{3i} + \ldots + \psi_{\overline{k-1}\,i}, \ldots, \psi_{\overline{k-1}\,i})$$

$$i = 1, 2, \ldots, b/v$$

Further, let the totality of b/v sets $[d_i]$ be called D and D_γ, D_β and D_η be the totality or the sets $[d_{\gamma i}]$, $[d_{\beta i}]$ and $[d_{\eta i}]$ respectively.

Then, as observed by Saha [2], if the b/v initial sequences satisfy the following conditions, we get a balanced change over design:

(i) D and –D (the additive inverses of the elements of D) contain every non-zero element of the module equally frequently.

(ii) D_γ contains every non-zero element of mod v equally frequently.

(iii) D_β contains every non-zero element of mod v equally frequently.

(iv) For every i = 1, 2, ... b/v, $d_{\eta i}$ does not contain any non-zero element of mod v repeated.

Against this background we now show that the initial sequences (3.2) generated a balanced change-over design.

For the series of designs under consideration, we have

$$D_\gamma = [\pm\ x^{2v}, \pm x^{2v+2}, \pm x^{2v}], v = 0, 1, 2, \ldots, 2t$$

It is seen easily that among the elements $\pm\ x^{2v}$ every non-zero element of GF (v) occurs precisely once and similarly, among the elements $\pm x^{2v+2}$, every non-zero element of GF(v) occurs precisely once. Thus, in D_γ, every non-zero element of GF(v) appears thrice, giving rise to the value of $\gamma_l^m = \gamma = 3$.

Again,

$$D_\beta = [\pm x^{2v+2}, \pm x^{2v+2}, \pm x^{2v}(x^2 - x^0)]$$

$$= [\pm x^{2v+2}, \pm x^{2v+2}, \pm x^{2v+q}], x^q = x^2 - x^0$$

Thus, in D_β, every non-zero element of GF(v) appears thrice and we have $\beta_l^m = \beta = 3$ for all $l \neq m$; $l, m = 0, 1, 2, ..., v-1$.

For the designs under consideration, we can easily see that

$$R = \sum_i s_{\eta i}^2 = 6\,(4t + 2)$$

Thus, from (2.7), we have

$$\lambda = (rk - R)/(v - 1)$$

$$= [4.4\,(4t + 2) - 6\,(4t + 2)]/(4t + 2)$$

$$= 10$$

Finally, it is seen that

$$D_\eta = [\pm x^{2v}, \pm x^{2v+2}, 0, 0],\ v = 0,\ 1, 2, \ldots, 2t$$

Thus for every value of $v = 0, 1, 2, \ldots, 2t$, $d_{\eta i}$ does not contain any non-zero element of GF(v) repeated.

Hence the requirements of balance are met and the proof is complete.

To illustrate the ideas, we give two examples:

(i) $v = 7$, the initial sequences in this case are (0, 1, 0, 2), (0, 2, 0, 4), (0, 4, 0, 1), (0, 6, 0, 5), (0, 5, 0, 3) and (0, 3, 0, 6).

The full design is given in Table 1 below.

(ii) $v = 11$, the initial sequences in this case are (0, 1, 0, 4), (0, 4, 0, 5), (0, 5, 0, 9), (0, 9, 0, 3), (0, 3, 0, 1), (0, 10, 0, 7), (0, 7, 0, 6), (0, 6, 0, 2), (0, 2, 0, 8) and (0, 8, 0, 10). Now the full design may be easily developed.

Table 1: Sequences

	1	0 1 2 3 4 5 6	0 1 2 3 4 5 6	0 1 2 3 4 5 6	0 1 2 3 4 5 6	0 1 2 3 4 5 6	0 1 2 3 4 5 6
Period	2	1 2 3 4 5 6 0	2 3 4 5 6 0 1	4 5 6 0 1 2 3	6 0 1 2 3 4 5	5 6 0 1 2 3 4	3 4 5 6 0 1 2
	3	0 1 2 3 4 5 6	0 1 2 3 4 5 6	0 1 2 3 4 5 6	0 1 2 3 4 5 6	0 1 2 3 4 5 6	0 1 2 3 4 5 6
	4	2 3 4 5 6 0 1	4 5 6 0 1 2 3	1 2 3 4 5 6 0	5 6 0 1 2 3 4	3 4 5 6 0 1 2	6 0 1 2 3 4 5

REFERENCES

[1] Patterson, H.D., 1952. The construction of balanced designs for experiments involving sequences of treatments. *Biometrika*, **39**, 32-48.

[2] Saha, G.M., 1970. Some contribution to design and analysis of experiments involving sequences of experiments involving sequences of treatments. Ph.D. thesis, IARI, New Delhi.