Pooling Procedures – A New Perspective[1]

Rakesh Srivastava
Saurashtra University, Rajkot - 360 005

SUMMARY

A conditional-guess testimator of the mean life time in the two parameter exponential distribution utilizing conditional information on it, with guessed guarantee has been suggested. Expressions for bias, mean square error and relative efficiency of the proposed testimator have been obtained. It is claimed that this testimator fairs better than neverpool estimator in certain range of life ratio. Recommendations regarding its use have been attempted.

Key Words : Exponential distribution, Censored samples, Preliminary test, Conditional-guess testimator, Bias, Mean square error, Relative efficiency, Significance level, Life ratio.

1. Introduction

In life testing research the simplest and the most widely exploited model is the exponential distribution with probability density function :

$$f(x_i\,;A_i,\theta_i) = \begin{cases} \dfrac{1}{\theta_i}\exp.[-(x_i - A_i)/\theta_i] & ;\ x_i \geq A_i\,,\ \theta_i > 0;\ (i = 1, 2) \\ 0 & ;\ \text{otherwise} \end{cases} \tag{1.1}$$

Here, θ_i is the average life time of the item and it also acts as a scale parameter, A_i is the guarantee period or location parameter or it represents the threshold or shift parameter or within which no failure can occur.

Davis [6] examined different types of data and the exponential distribution appears to fit most of the situations quite well. Epstein [9] remarks that the exponential distribution plays as important a role in life testing experiments as the part played by the normal distribution in agricultural experiments on effects of different treatments on the yield. For a situation where the failure rate appears to be more or less constant, the exponential distribution would be an adequate choice, in the context of life testing and reliability experiments. Exponential distribution also occurs in several other contexts, such as the waiting

---

1 The paper was presented on 1st July 1997 at a special session of the Society held at IASRI, New Delhi and adjudged best for "ISAS Young Scientist Award, 1996".

time problems. Maguire, Pearson and Wynn [15] studied mine accidents and showed that the time intervals between accidents follow exponential distribution.

Suppose that the life times of certain equipments produced follow (1.1). The problem investigated here, is concerned with a production setup of equipments, before and after some modification. It is beneficial to resort to censoring in life testing experiments. Specifically, let us consider that r_2 observations from a sample of n_2 equipments produced before modification is available. A similar set of r_1 observations from a sample of n_1 equipments produced after modification(s) is also available. Here, the producer is interested in estimating the average life of the equipments produced after modification(s) in the production procedure, hoping that the average life will increase after modification(s). Further, in addition to this information, the experimenter may have a guess value A_1^o of A_1 either from his past experience or from any other reliable source.

Our object is to provide an estimator of θ_1, however without loss of generality one may provide an estimator of θ_2, using the sample information available from two type II censored samples and utilizing additional information(s) about the population parameters. Here, we have two uncertainties about the population parameters. First, it is suspected that the average life will increase after some modification(s), i.e. $\theta_1 \geq \theta_2$; where θ_2 is the average life time without any modification(s) and further, it is suspected that after modification(s) A_1 may attain a value A_1^o (say) which is desired specification, i.e. $A_1 \geq A_1^o$; it implies that we have some guess on A_1 in the form of a point guess value A_1^o.

To resolve these uncertainties we apply preliminary test of significance (PTS). The PT procedure is more advantageous for life data problems where generally, testing happens to be destructive and expensive. The effect of using a PTS for subsequent estimation was first considered by Bancroft [1] who investigated the bias and mean square error of variance estimator after a PTS of equality of two variances. The PT procedure for estimation has been studied by many authors, such as, Bancroft [2], Kitagawa [13], Singh and Gupta [17] for analysis of variance models. Richards [16] attempts to analyse the consequences of using a PTS, to determine whether to use a one parameter or two parameter exponential distribution. Bancroft and Han [3] and Han, Rao, Ravichandran [12] has compiled a bibliography on procedures involving PTS.

As we wish to incorporate the available information(s) on the basis of the outcome of a PTS and not indiscriminately, the problem considered here has two separate tests for deciding whether to use or not to use the available information(s). This may be done by testing $H_o : \theta_1 = \theta_2$ against $H_1 : \theta_1 > \theta_2$ and $H'_o : A_1 = A_1^o$ against $H'_1 : A_1 > A_1^o$, both hypotheses may be tested at some preassigned level of significance. We have taken the same level of significance for both the hypotheses. Consequently, we may have following possibilities: accept H_o and H'_o; accept H_o and reject H'_o ; reject H_o and accept H'_o; reject H_o and H'_o. Finally, the proposed testimator may be constructed by utilizing one or both the available information(s) depending upon the outcome of PTS. As the proposed estimator may incorporate the conditional or guess or both the informations, based on a test of significance, the proposed estimator may be termed as CONDITIONAL-GUESS TESTIMATOR ($\hat{\theta}_{CG}$, (say)).

The statistics for testing the hypothesis H_o under various situations relating to some informations on A_i (i = 1, 2) have been given by Epstein and Tsao [7]. If $\theta_1 = \theta_2$, a MVU estimator of average life for each of the situations using both sets of observations has been given by Epstein and Sobel [8]. The MVU estimator of average life on the basis of a single set of observations has been given by Epstein [10]. Gupta and Singh [11] proposed a preliminary test estimator for life data. We have taken the test statistics for testing the hypothesis H'_o from Bain and Engelhardt [5]. This study is an attempt to combine the two lines of estimation procedure, viz., estimation under conditional specification and estimation utilizing a guess information.

The plan of the paper is as follows : we outline our procedure for estimation in section-two, in the third section we have obtained bias of the proposed testimator. Section-four deals with mean square error. In the fifth section we have obtained relative efficiency of the proposed testimator with neverpool estimator. Finally section-six comprises of numerical computations and recommendations regarding the application of the proposed testimator.

2. *Conditional-guess Testimator* $\hat{\theta}_{CG}$ *of* θ_1

Let $x_{11} \leq x_{12} \leq \ldots \leq x_{1r_1}$ be the failure times of the first r_1 items in a life test in which n_1 items were placed on test. Further, let $x_{21} \leq x_{22} \leq \ldots \leq x_{2r_2}$ be the failure times of the first r_2 items in another life test in which n_2 items were placed on test. Suppose that the underlying distribution of each x_{ij} $(i = 1, 2; j = 1, 2, \ldots, r_i)$ is two parameter exponential $f(x_i; A_i, \theta_i)$ $i = 1, 2$.

We are interested in estimating θ_1 when it is suspected that $\theta_1 \geq \theta_2$ and $A_1 \geq A_1^o$. To incorporate the available information(s), we resort to the PTS and test the hypotheses viz., $H_o: \theta_1 = \theta_2$ against $H_1: \theta_1 > \theta_2$ and $H'_o: A_1 = A_1^o$ against $H'_1: A_1 > A_1^o$ using the test statistics proposed by Epstein and Tsao [7] and Epstein and Sobel [8], Bain and Engelhardt [5].

Thus, depending upon the outcome of PTS, our estimator can be proposed as follows

$$\hat{\theta}_{CG} = \begin{cases} \dfrac{u_1^o + v_2}{r_1 + r_2 - 1} & \text{; If } H_o \text{ and } H'_o \text{ are accepted i.e. if } f_1 < \beta_1 \text{ and } U > \alpha \\ \dfrac{u_1 + v_2}{r_1 + r_2 - 2} & \text{; If } H_o \text{ is accepted and } H'_o \text{ is rejected i.e. if } f'_1 < \beta'_1 \text{ and } U \leq \alpha \\ \dfrac{u_1^o}{r_1} & \text{; If } H_o \text{ is rejected and } H'_o \text{ is accepted i.e. if } f_1 \geq \beta_1 \text{ and } U > \alpha \\ \dfrac{u_1}{r_1 - 1} & \text{; If } H_o \text{ and } H'_o \text{ are rejected i.e. if } f'_1 \geq \beta'_1 \text{ and } U \leq \alpha \end{cases}$$

where

$$f_1 = \frac{(r_2 - 1)}{r_1} \frac{u_1^o}{v_2}; \beta_1 = F(2r_1, 2r_2 - 2; \alpha_1)$$

$$f'_1 = \frac{(r_2 - 1)}{(r_1 - 1)} \frac{u_1}{v_2}; \beta'_1 = F(2r_1 - 2, 2r_2; \alpha_1)$$

$$u_1^o = \sum_{j=1}^{r_1} (x_{1j} - A_1^o) + (n_1 - r_1)(x_{1r_1} - A_1^o)$$

$$u_1 = \sum_{j=1}^{r_1} (x_{1j} - x_{11}) + (n_1 - r_1)(x_{1r_1} - x_{11})$$

$$v_2 = \sum_{j=1}^{r_2} (x_{2j} - x_{21}) + (n_2 - r_2)(x_{2r_2} - x_{21})$$

$$U = \left\{\frac{s - n\, x_{1:n}}{s - n\, A_1^o}\right\}^{r-1} = \left\{\frac{r\,\hat{\theta}}{r\,\hat{\theta} + n\, x_{1:n} - n\, A_1^o}\right\}^{r-1} \sim \text{UNIF}(0, 1)$$

Here, F (m, n; α) is the upper 100 α% point of the F - distribution with (m, n) degrees of freedom and $\chi^2(2r_1, \alpha)$ is the value of central chi-square variate satisfying the following relation.

$$\alpha = \int_0^z f(\chi^2_{2r_1})\, d(\chi^2_{2r_1}) \qquad ; z = \chi^2(2r_1 ; \alpha)$$

where $f(\chi^2_{2r_1})$ is the probability density function of the central chi-square variate with $2r_1$ degrees of freedom.

3. The Bias of $\hat{\theta}_{CG}$

The bias of $\hat{\theta}_{CG}$ is defined as

$$\text{BIAS}(\hat{\theta}_{CG}) = E(\hat{\theta}_{CG}) - \theta_1 \tag{3.1}$$

In order to evaluate BIAS ($\hat{\theta}_{CG}$), first we have to find E ($\hat{\theta}_{CG}$). Now, the expected value of $\hat{\theta}_{CG}$ is given by

$$E(\hat{\theta}_{CG}) = E_1 \Pr.(f_1 < \beta_1 \text{ and } U \geq \alpha) + E_2 \Pr.(f'_1 < \beta'_1 \text{ and } U < \alpha)$$
$$+ E_3 \Pr.(f_1 \geq \beta_1 \text{ and } U \geq \alpha) + E_4 \Pr.(f'_1 \geq \beta'_1 \text{ and } U < \alpha) \tag{3.2}$$

where,

$$E_1 = E\left(\frac{u_1^o + v_2}{r_1 + r_2 - 1} \mid f_1 < \beta_1 \text{ and } U \geq \alpha\right)$$

$$E_2 = E\left(\frac{u_1 + v_2}{r_1 + r_2 - 2} \mid f'_1 < \beta'_1 \text{ and } U < \alpha\right)$$

$$E_3 = E\left(\frac{u_1^o}{r_1} \mid f_1 \geq \beta_1 \text{ and } U \geq \alpha\right)$$

$$E_4 = E\left(\frac{u_1}{r_1 - 1} \mid f'_1 \geq \beta'_1 \text{ and } U < \alpha \right)$$

keeping in view that mutually independent statistics $2u_1^o/\theta_1$ follows chi-square distribution with $(2r_1)$ degrees of freedom and $2v_2/\theta_2$ follows chi-square distribution with $(2r_2 - 2)$ d.f. Therefore, the joint density function of u_1^o and v_2 is given by

$$f(u_1^o, v_2) = C_1 (u_1^o)^{r_1 - 1} (v_2)^{r_2 - 2} \exp.\left(-\left\{ \frac{u_1^o}{\theta_1} + \frac{v_2}{\theta_2} \right\} \right)$$

where $$C_1 = (\Gamma(r_1)\Gamma(r_2 - 1)\,\theta_1^{r_1}\theta_2^{r_2 - 1})^{-1} \tag{3.3}$$

We make the following transformations to evaluate the expected value of $\hat{\theta}_{CG}$.

Here, for the evaluation of E_1 we use (3.3.a) and for E_3 we use the transformation defined in (3.3.b).

$$x = u_1^o + v_2 \quad \text{and} \quad y = u_1^o/v_2 \tag{3.3.a}$$

$$u_1^o = u_1^o \quad \text{and} \quad y = u_1^o/v_2 \tag{3.3.b}$$

simplifying (3.3.a) we get $u_1^o = \frac{xy}{(1+y)}$ and $v_2 = \frac{x}{(1+y)}$

Similarly, we get $v_2 = u_1^o/y$ from (3.3.b).

Therefore, the joint density function of x and y is given by

$$g_1(x, y) = C_1 \frac{x^{(r_1 + r_2 - 2)}\, y^{(r_1 - 1)}}{(1+y)^{r_1 + r_2 - 1}} \exp.\left(-\frac{x}{1+y}\left\{ \frac{y}{\theta_1} + \frac{1}{\theta_2} \right\} \right) \tag{3.4.a}$$

and the joint density function of u_1^o and y is given by

$$f(u_1^o, y) = C_1 \frac{(u_1^o)^{r_1 + r_2 - 2}}{(y)^{r_2}} \exp.\left(-\frac{u_1^o}{y}\left\{ \frac{y}{\theta_1} + \frac{1}{\theta_2} \right\} \right) \tag{3.4.b}$$

using the transformations, limits of integration in (3.4.a) become

$$0 \leq x < \infty \quad ; 0 \leq y < z_1 \tag{3.5.a}$$

and the limits of integration in (3.4.b) are given by

$$0 \leq u_1^o < \infty \qquad ; z_1 \leq y < \infty \tag{3.5.b}$$

where $z_1 = r_1 \beta_1/(r_2 - 1)$

We know that the mutually independent statistics $2u_1/\theta_1$ follows chi- square distribution with $(2r_1 - 2)$ degrees of freedom and $2v_2/\theta_2$ follows chi-square distribution with $(2r_2 - 2)$ degrees of freedom. Therefore the joint density function of u_1 and v_2 is given by

$$f(u_1, v_2) = C_2 (u_1)^{r_1 - 2} (v_2)^{r_2 - 2} \exp. \left(- \left\{ \frac{u_1}{\theta_1} + \frac{v_2}{\theta_2} \right\} \right)$$

where,

$$C_2 = (\Gamma(r_1 - 1) \Gamma(r_2 - 1) \theta_1^{r_1 - 1} \theta_2^{r_2 - 1})^{-1} \tag{3.3}$$

Again, we make the following transformations to evaluate the expected value of $\hat{\theta}_{CG}$. Here, for the evaluation of E_2 we use (3.3.c) and for the evaluation of E_4 we use the transformations defined in (3.3.d).

$$x = u_1 + v_2 \qquad \text{and} \qquad y = u_1/v_2 \tag{3.3.c}$$

$$u_1 = u_1 \qquad \text{and} \qquad y = u_1/v_2 \tag{3.3.d}$$

simplifying (3.3.c) we get $u_1 = \frac{xy}{(1+y)}$ and $v_2 = \frac{x}{(1+y)}$

Similarly, we get $v_2 = u_1/y$ from (3.3.d).

Therefore, the joint density function of x and y is given by

$$g_2(x, y) = C_2 \frac{x^{(r_1 + r_2 - 3)} \, y^{(r_1 - 2)}}{(1+y)^{r_1 + r_2 - 2}} \exp. \left(- \frac{x}{1+y} \left\{ \frac{y}{\theta_1} + \frac{1}{\theta_2} \right\} \right) \tag{3.4.c}$$

and the joint density function of u_1 and y is given by

$$f(u_1, y) = C_2 \frac{(u_1)^{r_1 + r_2 - 3}}{(y)^{r_2}} \exp. \left(- \frac{u_1}{y} \left\{ \frac{y}{\theta_1} + \frac{1}{\theta_2} \right\} \right) \tag{3.4.d}$$

using the transformations, limits of integration in (3.4.c) become

$$0 \leq x < \infty \qquad ; 0 \leq y < z_2 \tag{3.5.c}$$

and the limits of integration in (3.4.d) are given by

$$0 \le u_1 < \infty \qquad ; z_2 \le y < \infty \tag{3.5.d}$$

where $z_2 = (r_1 - 1)\beta'_1/(r_2 - 1)$

also, we have

$$\Pr.(U < \alpha) = \alpha \text{ and } \qquad \Pr.(U \ge \alpha) = 1 - \alpha \tag{3.5.e}$$

with the above transformations and limits of integration (3.2) can be written as

$$\begin{aligned} E(\hat{\theta}_{CG}) = & \int_{y=0}^{z_1} \int_{x=0}^{\infty} \int_{u=\alpha}^{1} \frac{u_1^o + v_2}{r_1 + r_2 - 1} g_1(x, y) f(u)\, du\, dx\, dy \\ & + \int_{y=0}^{z_2} \int_{x=0}^{\infty} \int_{u=0}^{\alpha} \frac{u_1 + v_2}{r_1 + r_2 - 2} g_2(x, y) f(u)\, du\, dx\, dy \\ & + \int_{y=z_1}^{\infty} \int_{u_1^o=0}^{\infty} \int_{u=\alpha}^{1} \frac{u_1^o}{r_1} f(u_1^o, y) f(u)\, du\, du_1^o\, dy \\ & + \int_{y=z_2}^{\infty} \int_{u_1=0}^{\infty} \int_{u=0}^{\alpha} \frac{u_1}{r_1 - 1} f(u_1, y) f(u)\, du\, du_1\, dy \end{aligned} \tag{3.6}$$

where $U \sim \text{UNIF}(0, 1)$

Now, integrating expression (3.6) using the following results:

(i) $\displaystyle\int_0^{C_o} \frac{y^{m-1}}{(a+by)^{m+n}}\, dy = \frac{B_{\alpha_o}(m, n)}{a^n\, b^m} \qquad ; \alpha_o = \frac{b\, C_o}{a + b\, C_o}$

(ii) $\displaystyle\int_c^{\infty} \frac{y^{m-1}}{(a+by)^{m+n}}\, dy = \frac{1}{a^n\, b^m} \left(B(m, n) - B_{C_o}(m, n) \right); C_o = \frac{b\, C}{a + b\, C}$

(iii) $\displaystyle I_{x_o}(m, n) = \frac{B_{x_o}(m, n)}{B(m, n)}$

we get the expression for bias of $\hat{\theta}_{CG}$ as a fraction of θ_1 as given below

$$B_1 = \frac{\text{BIAS}(\hat{\theta}_{CG})}{\theta_1}$$

$$B_1 = \frac{\alpha(r_2-1)}{r_1+r_2-2}\left\{\phi\, I_{x'_1}(r_1-1, r_2) - I_{x'_1}(r_1, r_2-1)\right\}$$
$$+\frac{(1-\alpha)(r_2-1)}{r_1+r_2-1}\left\{\phi\, I_{x_1}(r_1, r_2) - I_{x_1}(r_1+1, r_2-1)\right\} \quad (3.7)$$

where

$$x_1 = \frac{\phi\, r_1\, \beta_1}{\phi\, r_1\, \beta_1 + (r_2-1)}; \quad x'_1 = \frac{\phi\,(r_1-1)\,\beta'_1}{\phi\,(r_1-1)\,\beta'_1 + (r_2-1)}; \phi = \theta_2/\theta_1$$

As a partial check on (3.7),

If we take $\beta'_1 = 0$ and $\alpha = 1$, then $x'_1 = 0$, it implies that always reject both the hypotheses H_o and H'_o. In this case our testimator reduces to

$$\hat{\theta}_{CG} = \frac{u_1}{r_1-1}$$

from (3.7) $B_1 = 0 = E(\hat{\theta}_{CG}) - \theta_1$

Thus, this case is analogous to the neverpool estimator which is an unbiased estimator of θ_1.

Again, if we take the limits $\beta'_1 \to \infty$ and $\alpha = 1$, it implies that never reject H_o and always reject H'_o. In this case our testimator reduces to

$$\hat{\theta}_{CG} = \frac{u_1+v_2}{r_1+r_2-2}$$

From (3.7) $B_1 = \dfrac{r_2-1}{r_1+r_2-2}(\phi-1) = E(\hat{\theta}_{CG}) - \theta_1$

Thus, this case is analogous to the alwayspool estimator proposed by Gupta and Singh [11] for situation-3 (when nothing is known about guarantees).

4. *Mean Square Error of* $\hat{\theta}_{CG}$

In order to evaluate the mean square error of $\hat{\theta}_{CG}$ we use the relation given below

$$MSE(\hat{\theta}_{CG}) = E(\hat{\theta}^2_{CG}) - 2\theta_1\, BIAS(\hat{\theta}_{CG}) - \theta_1^2 \quad (4.1)$$

Thus, we need the expression for $E(\hat{\theta}^2_{CG})$. The term BIAS $(\hat{\theta}_{CG})$ is already evaluated in section-3, and for the evaluation of $E(\hat{\theta}^2_{CG})$ we follow the same method which has been used for evaluating the $E(\hat{\theta}_{CG})$. After some simplification we get final expression for $E(\hat{\theta}^2_{CG})$ as follows

$$E(\hat{\theta}^2_{CG}) = \theta_1^2 \Bigg[\frac{r_1+1}{r_1} + \frac{\alpha}{r_1(r_1-1)} + \frac{(1-\alpha)\phi^2 r_2(r_2-1)}{(r_1+r_2-1)^2} I_{x_1}(r_1, r_2+1)$$

$$+ \frac{2(1-\alpha)\phi r_1(r_2-1)}{(r_1+r_2-1)^2} I_{x_1}(r_1+1, r_2) + \frac{\alpha\phi^2 r_2(r_2-1)}{(r_1+r_2-2)^2} I_{x'_1}(r_1-1, r_2+1)$$

$$+(1-\alpha)\left(\frac{r_1(r_1+1)}{(r_1+r_2-2)^2} - \frac{(r_1+1)}{r_1}\right) I_{x_1}(r_1+2, r_2-1) + \frac{2\alpha\phi(r_1-1)(r_2-1)}{(r_1+r_2-2)^2}$$

$$* \; I_{x'_1}(r_1, r_2) + \alpha\left\{\frac{r_1(r_1-1)}{(r_1+r_2-2)^2} - \frac{r_1}{r_1-1}\right\} I_{x'_1}(r_1+1, r_2-1) \Bigg] \qquad (4.2)$$

substituting the values of $E(\hat{\theta}^2_{CG})$ and BIAS $(\hat{\theta}_{CG})$ from (4.2) and (3.7), in (4.1) and then after simplification we obtain the MSE $(\hat{\theta}_{CG})$ expressed as a fraction of θ_1^2 as given below

$$M_1 = \frac{\text{MSE}(\hat{\theta}_{CG})}{\theta_1^2}$$

$$M_1 = \frac{1}{r_1} + \frac{\alpha}{r_1(r_1-1)} + (1-\alpha)\left\{\frac{r_1(r_1+1)}{(r_1+r_2-2)^2} - \frac{(r_1+1)}{r_1}\right\} I_{x_1}(r_1+2, r_2-1)$$

$$+ \frac{(1-\alpha)\phi(r_2-1)}{(r_1+r_2-1)^2}\left\{\phi r_2 I_{x_1}(r_1, r_2+1) + 2r_1 I_{x_1}(r_1+1, r_2)\right\}$$

$$+\alpha\left\{\frac{r_1(r_1-1)}{(r_1+r_2-2)^2} - \frac{r_1}{r_1-1}\right\} I_{x'_1}(r_1+1, r_2-1)$$

$$+ \frac{\alpha\phi(r_2-1)}{(r_1+r_2-2)^2}\left\{\phi r_2 I_{x'_1}(r_1-1, r_2+1) + 2(r_1-1) I_{x'_1}(r_1, r_2)\right\}$$

$$- \frac{2\alpha(r_2-1)}{(r_1+r_2-2)}\left\{\phi I_{x'_1}(r_1-1, r_2) - I_{x'_1}(r_1, r_2-1)\right\}$$

$$- \frac{2(1-\alpha)(r_2-1)}{(r_1+r_2-1)} \left\{ \phi\, I_{x_1}(r_1, r_2) - I_{x_1}(r_1+1, r_2-1) \right\} \tag{4.3}$$

As a partial check on (4.3),

Let $\beta'_1 = 0$ and $\alpha = 1$, that is the above expression reduces to $M_1 = 1/(r_1 - 1) = \mathrm{MSE}(\hat{\theta})/\theta_1$. Which is same as mean square error of the neverpool estimator.

If we take $\beta'_1 \to \infty$ and $\alpha = 1$, we get

$$M_1 = \frac{1}{(r_1+r_2-2)^2} \left(\{ (r_1-1) + (r_2-1)\phi^2 \} + (r_2-1)^2 (1-\phi)^2 \right)$$

which can be shown very easily to be equal to MSE (AP. EST.)/θ_1^2 where, MSE (AP. EST.) has been obtained by Gupta and Singh [11] for situation-3 as alwayspool estimator of θ_1.

5. *Efficiency of* $\hat{\theta}_{CG}$

As preliminary test estimators are in general biased and neverpool estimators are always unbiased and since the two are competing estimators of θ_1, it is more appropriate to talk of the relative efficiency of the proposed estimator to the neverpool estimator which may be defined as follows

$$\mathrm{RE} = \frac{\text{MSE (neverpool estimator)}}{\text{MSE (conditional–guess testimator)}}$$

$$\mathrm{RE} = \frac{1}{r_1} \left[\frac{1}{r_1} + \frac{\alpha}{r_1(r_1-1)} + (1-\alpha) \left\{ \frac{r_1(r_1+1)}{(r_1+r_2-2)^2} - \frac{(r_1+1)}{r_1} \right\} I_{x_1}(r_1+2, r_2-1) \right.$$

$$+ \frac{(1-\alpha)\phi(r_2-1)}{(r_1+r_2-1)^2} \left\{ \phi\, r_2\, I_{x_1}(r_1, r_2+1) + 2r_1\, I_{x_1}(r_1+1, r_2) \right\}$$

$$+ \alpha \left\{ \frac{r_1(r_1-1)}{(r_1+r_2-2)^2} - \frac{r_1}{r_1-1} \right\} I_{x'_1}(r_1+1, r_2-1)$$

$$+ \frac{\alpha\phi(r_2-1)}{(r_1+r_2-2)^2} \left\{ \phi\, r_2\, I_{x'_1}(r_1-1, r_2+1) + 2(r_1-1)\, I_{x'_1}(r_1, r_2) \right\}$$

$$- \frac{2\alpha(r_2-1)}{(r_1+r_2-2)}\left\{\phi\, I_{x'_1}(r_1-1, r_2) - I_{x'_1}(r_1, r_2-1)\right\}$$

$$\left. - \frac{2(1-\alpha)(r_2-1)}{(r_1+r_2-1)}\left\{\phi\, I_{x_1}(r_1, r_2) - I_{x_1}(r_1+1, r_2-1)\right\}\right]^{-1} \quad (5.2)$$

From the above expression we observe that RE is a function of $r_1, r_2, \phi, \beta_1, \beta'_1$ and α. In which r_1 and r_2 are determined and fixed in advance by the experimenter taking cost and time into consideration. The average life ratio ϕ is in general unknown. Hence, the only parameter(s) at our disposal is level(s) of significance for both the hypotheses.

We have taken the same level(s) of significance for both the hypotheses. However, possibly one can take different level(s) for both the hypotheses. The best choice for level of significance could be proposed using minimax regret criterion. It remains a future task.

To study the behaviour of mean square error of the proposed testimator $\hat{\theta}_{CG}$, we have considered four sets of values given in Table 5.1.

Table 5.1

r_1	r_2
16	8
16	12
4	8
4	12

We have taken α = 0.01, 0.05, 0.10 and 0.25; ϕ = 0.1 (0.1)1.0 for each r_1 and r_2.

6. *Discussion on Numerical Results*

As discussed earlier, relative efficiency is a function of r_1, r_2, ϕ, α_1 and α. We have calculated the relative efficiencies for the setup given in section-5. Tables of relative efficiencies (RE) have been given in the appendix. From these tables, we put forward the criteria for the application of the proposed testimator as follows

1. Table showing the effective ranges of ϕ where RE > 1.

Sample size	Life ratio	Level(s) of significance (α)			
		1%	5%	10%	25%
$r_1 = 4$ $r_2 = 8$	ϕ	0.5 to 1.0	0.5 to 1.0	0.5 to 1.0	0.5 to 1.0
$r_1 = 4$ $r_2 = 12$	ϕ	0.6 to 1.0	0.5 to 1.0	0.5 to 1.0	0.5 to 1.0
$r_1 = 16$ $r_2 = 8$	ϕ	0.1 to 1.0	0.1 to 1.0	0.1 to 1.0	0.7 to 1.0
$r_1 = 16$ $r_2 = 12$	ϕ	0.1 to 1.0	0.1 to 1.0	0.1 to 1.0	0.6 to 1.0

2. The proposed testimator performs uniformly better compared to neverpool estimator for the whole range of life ratio (ϕ) considered here, that is, (0.1 to 1.0), for larger censoring fraction after modification, irrespective of level of significance. However, when $0.5 \leq \phi \leq 1.0$, the proposed testimator fairs uniformly better irrespective of r_1, r_2 and α.

3. When $\phi < 0.5$, the magnitude of RE increases as α increases for the values of α considered here, however, RE decreased at $\alpha = 25\%$. Just reverse happens to the magnitude of RE when $\phi > 0.5$.

4. The relative efficiency attains its maximum at $\phi = 0.9$ when $r_2 > r_1$, also it occurs at $\phi = 1.0$ for $r_1 > r_2$.

5. On the basis of tables given in appendix, we observed that for $r_1 < r_2$ the gain in efficiency for the proposed estimator is maximum compared to $r_1 > r_2$, in terms of its magnitude. Although, r_1 and r_2 are fixed in advance by the experimenter but it is empirically shown that $r_1 \leq 3r_2$ implies maximum gain in efficiency. So, while, deciding for censoring, these should serve as the guide line for choosing r_1 and r_2.

REFERENCES

[1] Bancroft, T.A., 1944. On biases in estimation due to the use of preliminary test of significance. *Ann. Math. Statist.*, **15**, 17-32.

[2] Bancroft, T.A., 1964. Analysis and inference for incompletely specified models involving the use of preliminary test(s) of significance. *Biometrics*, **20**, 427-442.

[3] Bancroft, T.A. and Han, C.P., 1977. Inference based on conditional specification : A note and a bibliography, *Int. Statist. Rev.*, **45**, 117-127.

[4] Bartholomew, D.J., 1987. A problem in life testing, *J. Am. Statist. Assoc.*, **52**, 350-355.

[5] Bain, L.J. and Engelhardt, M., 1991. *Statistical analysis of reliability and life testing models*, Marcel Dekker Inc.

[6] Davis, D.J., 1952. The analysis of some failure data, *J. Am. Stat. Assoc.*, **47**, 113-160.

[7] Epstein, B. and Tsao, C.K., 1953. Some tests based on ordered observations from two exponential populations, *Ann. Math. Stat.*, **24**, 458-466.

[8] Epstein, B. and Sobel, M., 1954. Some theorems relevant to life testing from an exponential distribution, *Ann. Math. Stat.*, **25**, 373-381.

[9] Epstein, B., 1958. Exponential distribution and its role in life testing, *Ind. Qual. Control*, **15**, 4-6.

[10] Epstein, B., 1960. Estimation of parameters of two parameter exponential distribution from censored sample, *Technometrics*, **2**, 403-406.

[11] Gupta, V.P. and Singh, U., 1985. Preliminary test estimator for life data, Microelectron, *Reliab.*, **25**, 881-887.

[12] Han, C.P., Rao, C.V., Ravichandran, J., 1988. Inference based on conditional specification : A second bibliography, *Commun. Stat.—Theory Methods*, **17(6)**, 1945-1964.

[13] Kitagawa, T., 1963. Estimation after preliminary tests of significance, University of California publications in statistics, **3**, 147-186.

[14] Lawless, J.F., 1983. Statistical methods in reliability, *Technometrics*, **25**, 305-316.

[15] Maguire, B.A., Pearson, E.S. and Wynn, A.H.A., 1952. The time intervals between industrial accidents, *Biometrika*, **39**, 168-180.

[16] Richards, D.O., 1963. Incompletely specified models in life testing, Unpublished Ph.D. thesis, Iowa State University.

[17] Singh, U. and Gupta, V.P., 1976. Bias and mean square error of an estimation procedure after a preliminary test of significance for three-way layout in anova model - II, *Sankhya*, **B 38**, 164-178.

[18] Varde, S.D., 1970. Estimation of reliability of a two parameter exponential component series system, *Technometrics*, **12**, 867-875.

APPENDIX

Table-1: Relative efficiency of $\hat{\theta}_{CG}$ w.r.t. neverpool estimator $r_1 = 4, r_2 = 8$

Life ratio	Level(s) of significance (α)			
φ	0.01	0.05	0.10	0.25
0.1	0.955232	0.974069	0.964306	0.922978
0.2	0.885448	0.945970	0.953273	0.925335
0.3	0.883529	0.943009	0.957016	0.938157
0.4	0.930384	0.973602	0.984830	0.965575
0.5	1.00454	1.02878	1.03221	1.00584
0.6	1.08993	1.09440	1.08983	1.05796
0.7	1.17053	1.15848	1.14589	1.09945
0.8	1.22921	1.20543	1.18707	1.13407
0.9	1.25065	1.22166	1.20093	1.14689
1.0	1.22683	1.19916	1.17967	1.13058

Table-2: Relative efficiency of $\hat{\theta}_{CG}$ w.r.t. neverpool estimator $r_1 = 4, r_2 = 12$

Life ratio	Level(s) of significance (α)			
φ	0.01	0.05	0.10	0.25
0.1	0.966062	0.976602	0.965171	0.923053
0.2	0.872616	0.943120	0.951925	0.924982
0.3	0.836045	0.922189	0.946141	0.936281
0.4	0.866712	0.939222	0.965622	0.963380
0.5	0.946315	0.992965	1.01285	1.00753
0.6	1.05995	1.07391	1.08170	1.06495
0.7	1.19063	1.16771	1.16014	1.12702
0.8	1.31269	1.25360	1.23029	1.18034
0.9	1.39203	1.30584	1.27032	1.20856
1.0	1.39872	1.30242	1.26166	1.19746

Table-3: Relative efficiency of $\hat{\theta}_{CG}$ w.r.t. neverpool estimator $r_1 = 16, r_2 = 8$

Life ratio	Level(s) of significance (α)			
ϕ	0.01	0.05	0.10	0.25
0.1	0.999286	0.996675	0.993377	0.983607
0.2	0.999024	0.996675	0.993398	0.983621
0.3	1.00007	0.992832	0.993818	0.983817
0.4	1.0031	0.999134	0.995183	0.984559
0.5	1.00748	1.00215	0.997568	0.986048
0.6	1.01236	1.00586	1.00066	0.988203
0.7	1.01702	1.00968	1.00401	0.990757
0.8	1.02087	1.0131	1.00715	0.993358
0.9	1.02349	1.01568	1.00966	0.995639
1.0	1.02454	1.01706	1.01117	0.997254

Table-4: Relative efficiency of $\hat{\theta}_{CG}$ w.r.t. neverpool estimator $r_1 = 16, r_2 = 12$

Life ratio	Level(s) of significance (α)			
ϕ	0.01	0.05	0.10	0.25
0.1	0.999317	0.996677	0.993378	0.983607
0.2	0.998524	0.996603	0.993370	0.983614
0.3	0.998177	0.996985	0.993746	0.983823
0.4	1.00304	1.00013	0.996105	0.985081
0.5	1.01474	1.00778	1.00197	0.988521
0.6	1.03163	1.01965	1.01153	0.994687
0.7	1.05066	1.03405	1.02363	1.00320
0.8	1.06840	1.04852	1.03634	1.01292
0.9	1.08145	1.06035	1.04733	1.02217
1.0	1.08681	1.06688	1.05420	1.04107