

## **On Generalized Probability Distribution and Generalized Mixture Distribution**

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### **SUMMARY**

A generalized probability distribution is developed which served as a general formula for some well known probability distributions such as weibull, exponential, rayleigh, gamma, log-normal, pareto, maxwell, generalized laplace etc. Besides aforesaid probability distributions, a family of new distributions can also be derived from it. The statistical properties viz. moments and moment generating function have been worked out for generalized probability distribution (G.P.D.) A mixture (or compounded) distribution by mixing gamma distribution with G.P.D., called generalized probability distribution of mixture (G.P.D.M.) has also been obtained. Cauchy's, beta, t, reversed generalized logistic distributions are shown to be members of G.P.D.M.

*Key words:* Gamma distribution, Generalized probability distribution, Generalized logistic distribution, Mixture distribution, Weibull distribution.

### **1. Introduction**

Statistical distributions such as weibull, exponential, log normal, gamma etc. have proved to be of considerable use not only to explain a large number of data sets for ecological and biological populations [Pinder *et al.* [11]], but also in the development of theory of reliability. In reliability, these distributions are mainly used as lifetime distribution of a component or a device. In recent years, many generalization of gamma and weibull distributions are suggested either by introducing more parameter/s or by assuming a parameter of the distribution as random variable and then compounding it. The main contributors are: Lindley and Singpurwalla [8], Srivastava [13], Lee and Gross [6], [7], Bondesson [5], Pham and Alnhana [10], Al-Mutairi and Agarwal [4], [3] and Agarwal and Kalla [2].

In this paper, a generalized probability distribution (G.P.D.) is developed which serves as a general model of the life length of a component or device. Almost all well known commonly used lifetime distributions such as exponential, gamma, weibull, log normal, logistic, pareto, rayleigh, maxwell

etc. and a family of new distributions (depending upon physical considerations of the device) can be derived from it. The closed form of moments and moment generating function are worked out for G.P.D. A mixture (or compounded) distribution by mixing gamma distribution with G.P.D., called generalized probability distribution of mixture (G.P.D.M.) has also been given. It serves as a general formula for mixing any probability distribution, which is a member of G.P.D., with gamma distribution. Cauchy's, t, reversed generalized logistic distributions and a family of new distributions can be derived from G.P.D.M.

### 2. Generalized Probability Distribution

In order to define G.P.D., consider the following form of confluent hypergeometric function of variable Z as:

$$f(Z; \alpha, \lambda, p, A, C) = \alpha^{\lambda+1}(Z)^\lambda e^{-pZ} M(A, C, -\alpha Z) \tag{2.1}$$

where  $\alpha, \lambda, p$  and  $A$  are constants;  $C \neq 0, -1, -2, \dots$  and  $0 \leq Z \leq \infty$

$M(A, C, Z)$  is confluent hypergeometric function [Abramowitz and Stegun [1]].

*Theorem 1.* A random variable Z follows a G.P.D., if its probability density function (p.d.f.) is given by

$$f(Z; \alpha, \lambda, p, A, C) = K^{-1} \alpha^{\lambda+1} (z)^\lambda e^{-pz} M(A, C, -\alpha z) \tag{2.2}$$

where, the constant

$$K = \left(\frac{\alpha}{p}\right)^{\lambda+1} \Gamma(\lambda+1) F\left(A, \lambda+1; C; -\frac{\alpha}{p}\right); \alpha+p > 0; \lambda > -1 \tag{2.3}$$

$F(a, b, d; -x)$  is Gauss hypergeometric function (Mathai [9]).

*Proof.* The function (2.2) is a p.d.f. if

$$f(Z; \alpha, \lambda, p, A, C) \geq 0$$

and 
$$\frac{1}{K} \int_0^\infty f(Z; \alpha, \lambda, p, A, C) dZ = 1 \tag{2.4}$$

Using the results of [Prudnikov *et al.* [15]; p. 19(15)], we get the desired result.

*Theorem 2.* If  $Z$  is a G.P.D., with p.d.f. as given in (2.2) then the  $r$ -th moment about origin is

$$\mu'_r = \Gamma(\lambda + 1) \alpha^{\lambda+1} (p + \alpha)^{-\lambda-1} F(C - A, \lambda + 1; C; \frac{\alpha}{\alpha + p}) \quad (2.5)$$

and its moment generating function is

$$M_z(t) = \frac{\alpha^{\lambda+1} \Gamma(\lambda + 1)}{(p - t)^{\lambda+1}} F(A, \lambda + 1; C; -\frac{\alpha}{p - t}); \quad p - t > 0$$

$$\lambda > -1; \quad (p + \alpha - t) > 0 \quad (2.6)$$

*Corollary 1.* When  $p \rightarrow 0$ , (2.2) reduces to

$$K^{-1} \alpha^{\lambda+1} (z)^\lambda M(A, C, -\alpha z) \quad (2.7)$$

and

$$K = \frac{\Gamma(C) \Gamma(A - \lambda - 1) \Gamma(\lambda + 1)}{\Gamma(A) \Gamma(C - \lambda - 1)} \quad (2.8)$$

(2.5) reduces to

$$\mu'_r = \frac{\Gamma(C - \lambda - 1) \Gamma(\lambda + r + 1)}{\Gamma(C - \lambda - r - 1) \Gamma(\lambda + 1)} \frac{1}{\alpha^r}$$

Now we will derive some of the well known widely used probability distributions both from non-exponential as well as from exponential family [Stuart and Ord [14] p. 192] under Corollary 1.

*Case I.*  $A \neq C$  (Non exponential family)

Some special cases of Case I, are given in Corollary 2.

*Corollary 2.* If random variable  $Z = X$  is a G.P.D., then  $X$  follows uniform distribution for  $A = 0$ ,  $\lambda = 0$ , and if  $A = a$ ,  $C = a + 1$ ,  $\lambda = 0$ ,  $\alpha = 1$ , then (2.7) reduces to incomplete gamma distribution [Abramowitz and Stegun [1] p. 509]. While for

$$A = \frac{1}{2}, \quad C = \frac{3}{2}, \quad \lambda = 0, \quad Z = \exp\left(\frac{x - \mu}{\sigma}\right)$$

(2.7) is

$$f(x; \alpha, \mu, \sigma) = \frac{\sqrt{\alpha\pi}}{2\sigma} \exp\left(\frac{x - \mu}{\sigma}\right) \phi\left\{\sqrt{\alpha} \exp\left(\frac{x - \mu}{\sigma}\right)\right\}$$

where  $\text{Erfc } z = \sqrt{\pi/2} \phi(z) = \int_0^z e^{-t^2} dt$

and for  $A = -n, C = p + 1, Z = \exp\left(\frac{x - \mu}{\sigma}\right)$

$$f(x; \alpha, \mu, \sigma, n, p) = \frac{\alpha \Gamma(n+1)}{\sigma (p+1)_n} \exp\left(\frac{x - \mu}{\sigma}\right) L_n^p\left(-\exp\left(\frac{x - \mu}{\sigma}\right)\right)$$

which is a p.d.f. of Laguerre polynomial [Abramowitz and Stegun [1] p. 509] and may find applicability in quantal response data. By giving different values to  $A, C, \alpha, \gamma$  and using  $Z = \Psi(x)$ , where  $\Psi$  is function of  $X$ , a family of new distributions can also be derived.

*Case II*  $A = C$  (Exponential family of distribution)

Some special cases of Case II are given in Corollary 3.

*Corollary 3.* If random variable:

(i)  $Z = X^\beta$  is a G.P.D., then  $X$  follows weibull distribution for  $\lambda = 0$ .

*Proof.* It is easy to see that (2.7) reduces to  $f(z) = \alpha e^{-\alpha z}$ , which is a p.d.f. of exponential distribution. It is well known that the random variable  $X$  follows weibull distribution if  $Z = X^\beta$  is an exponential distribution. Note that  $\beta = 2$  is rayleigh distribution.

Now in Corollary 3 :

- (ii) If  $Z = x, \lambda = \beta - 1$  Gamma distribution
- (iii) If  $Z = \log x^2, \lambda = -\frac{1}{2}, \alpha = \frac{1}{2}$  Log normal distribution
- (iv) If  $Z = \log(1 + e^x), \lambda = 0, \alpha = 1$  Logistic distribution
- (v) If  $Z = e^{-x}, \lambda = 0, \alpha = 1$  Gumbel or extreme value  
 and if  $Z = e^x$  Reduced log weibull distribution
- (vi) If  $Z = x^2, \lambda = \frac{1}{2}, \alpha = \frac{1}{2} \alpha$  Maxwell distribution
- (vii) If  $Z = \left| \frac{x - \mu}{\beta} \right|; \alpha = 1, \lambda = 0$  Generalized laplace distribution

### 3. Generalized Probability Distribution of Mixture (G.P.D.M.)

In this section, by assuming parameter  $\alpha$  in (2.7) as a random variable follows gamma distribution with shape parameter  $\varphi$ , a G.P.D.M. is obtained. It will serve as a general formula for mixing any probability distribution, which is a member of G.P.D., with gamma distribution.

*Theorem 3.* If  $Z$  is a G.P.D., as defined by (2.7) and  $\alpha$  follows gamma distribution with shape parameter  $\varphi$ , then the p.d.f. of G.P.D.M. is

$$K_1 Z^\lambda F(A, \lambda + \varphi + 1; C, -Z) \quad (3.1)$$

$$\text{where } K_1 = \frac{\Gamma(A)\Gamma(C - \lambda - 1)\Gamma(\lambda + \varphi + 1)}{\Gamma(C)\Gamma(A - \lambda - 1)\Gamma(\lambda + 1)\Gamma(\varphi)} \quad (3.2)$$

*Proof.* A mixture or compounded distribution is given by

$$\int_0^\infty f(z; \alpha, \lambda, A, C) h(\alpha) d\alpha \quad (3.3)$$

$$\text{where } h(\alpha) = \frac{\exp^{-\alpha} \alpha^{\varphi-1}}{\Gamma(\varphi)}$$

Now (3.3) is equal to

$$\frac{Z^\lambda}{K_1 \Gamma(\lambda + 1) \Gamma(\varphi)} \int_0^\infty \alpha^{\lambda + \varphi} e^{-\alpha} M(A, C, -\alpha Z) d\alpha$$

Using the result [Prudnikov *et al.* [12] p. 19], we get (3.1)

*Theorem 4.* If  $Z$  follows G.P.D.M., then its  $r$ -th moment about origin is:

$$\mu'_r = \frac{\Gamma(C - \lambda - 1) \Gamma(A - r - \lambda - 1) \Gamma(\varphi - r) \Gamma(r + \lambda + 1)}{\Gamma(\varphi) \Gamma(\lambda + 1) \Gamma(A - \lambda - 1) \Gamma(C - r - \lambda - 1)} \quad (3.4)$$

The moment will exist only if  $\varphi > r$ .

Particular cases:

*Case 1.*  $A = C$

*Corollary 4.* If  $Z = X^2$  is a G.P.D.M., as defined in (3.1), the  $X$  follows Cauchy's distribution for  $\lambda = 0$ ,  $\varphi = 1/2$ , while  $X$  follows t-distribution if  $\lambda = -1/2$ ,  $\varphi = v$ .

*Corollary 5.* If  $Z = \exp\left(\frac{X-\mu}{\sigma}\right)$  is G.P.D.M., then  $X$  follows reversed generalized logistic distribution for  $\lambda = 0$ .

*Case II.*  $A \neq C$

$$(i) \quad A = -n, \quad C = 1, \quad \varphi = n, \quad Z = \exp\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x) = \left(\frac{\alpha}{\sigma}\right) \exp\left(\frac{x-\mu}{\sigma}\right) P_n \left(1 + 2 \exp\left(\frac{x-\mu}{\sigma}\right)\right)$$

$X$  is p.d.f. of Legendre's Polynomial [Abramowitz and Stegun [1], p. 509]

By giving different values to  $A$ ,  $C$ ,  $\lambda$ ,  $\varphi$  and  $z = \xi(x)$ , a family of new distributions can be derived. However, the moments will exist only if  $\varphi > r$ .

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