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Analysis of Data from Incompletely Repeated Measurement Designs

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SUMMARY

The analysis of data from incompletely repeated measurement designs is considered. Approximate methods for analysis not requiring the value of uniform correlation coefficient among repeated observations are developed. These methods are illustrated for some practical example. The numerical results reveal that the uniform correlation among repeated observations affects the significance level of the group (main) effect and its interaction with repeated measure effect. The Cochran-Cox approximation provides better results than the Welch-Satterthwaite approximation for the size of tests.

Key words: ANOVA, Incompletely repeated measurement, Size of test, Uniform correlation.

1. *Introduction*

The multiple observations on an individual over several treatment conditions or time points are commonly referred to as repeated measure data. In biological sciences all the individuals/animals may not provide observations at all the treatment conditions (time points) due to death or ill conditions at certain stage of the experiment. Such data, known as incompletely repeated data, are more common than the single observation on the individual/animal. The analysis of completely repeated data has been given by Gill [5] and Crowder and Hand [4]. The analysis of incompletely repeated observations has been considered by Gill [6] by using the method for complete time profile with the estimated value of uniform correlation from the available data. His approach may not be valid due to non-orthogonality caused by incomplete observations.

Here we consider data from incompletely repeated measurement designs under the following linear model (see Berk [1]);

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \tag{1.1}$$

$$(i = 1, 2, ..., a; \;\; j = 1, 2, ..., b; \;\; k = 1, 2, ..., n_{ij})$$

where n_{ij} is the number of animals at j-th time point in i-th group, μ is the general mean, α_i is the i-th group effect, β_j is the j-th time point effect, $(\alpha\beta)_{ij}$ is their interaction effect and e_{ijk} is the error variable associated with y_{ijk}.

Assume that each animal has equal chance of survival at any stage of experiment. Define that $\delta_{ijk} = 1$, if the observation on k-th individual in i-th group is available at j-th time point and zero otherwise, and also assume that e_{ijk} are normally distributed with

$$\text{Var}(e_{ijk}) = \delta_{n_{ijk}}\,\sigma^2 \text{ and Cov }(e_{ijk}, e_{ij'k}) = \delta_{ijk}\,\delta_{ij'k} \in \sigma^2, j \neq j' \quad (1.2)$$

that is, the observations at different repeated points on the same individual are uniformly correlated ($\in$) with constant variance (σ^2).

The null hypotheses of interest are

$$H_{01} : \alpha_1 = \alpha_2 = \ldots = \alpha_a,\ H_{02} : \beta_1 = \beta_2 = \ldots = \beta_b$$

and $\quad H_{03}$: no interaction effect (1.3)

We define sums of squares, test statistics and their distributions for analysis of data under model (1.1 - 1.2). Approximate methods for analysis of such data not requiring the value of uniform correlation are developed for practical applications. The numerical results are presented for some apriori parametric values. The methodology is illustrated for some practical data.

2. *Sums of Squares*

The actual sums of squares under (1.1 and 1.2) are the adjusted sums of squares obtained through the fitting constant method. The resulting sums of squares are expressed as follows:

Define that

$$N = \text{number of individuals},\quad N_0 = \sum_i\sum_j n_{ij},\quad n_{i\cdot} = \sum_j n_{ij}$$

$$= \sum_i n_{i1}$$

$$n_{i.k} = \sum_j \delta_{ijk} \text{ and } n_{.j} = \sum_i n_{ij},$$

$$y_{\cdot j} = \sum_i\sum_k y_{ijk},\ \bar{y}_{i\cdot} = \sum_j\sum_k y_{ijk}/n_{i\cdot},\ \bar{y}_{\cdot j} = y_{\cdot j}/n_{\cdot j},\ y_{ij} = \sum_k y_{ijk}$$

$\bar{y}_{ij} = y_{ij}/n_{ij},\ \bar{y}_{i.k} = \sum_j y_{ijk}/n_{i.k}$ and $\bar{y} = \sum_i \sum_j \sum_k y_{ijk}/N_0$

Now we write that

$$SS(\beta/\alpha) = Q'_\beta\, C_\beta^-\, Q_\beta$$

where $$Q_{\beta j} = y_{\cdot j} - \sum_i n_{ij}\,\bar{y}_{i\cdot},\ j = 1, 2, .., b$$

$$C_\beta = (c_{\beta jj'})$$

with $$c_{\beta jj'} = n_{\cdot j} - \sum_i n_{ij}^2/n_{i\cdot}\ ,\ j' = j$$

$$= -\sum_i n_{ij}\, n_{ij'}/n_{i\cdot}\ ,\ j' \neq j$$

$$SS(\alpha/\beta) = SS(\beta/\alpha) + SS\alpha - SS\beta$$

where $$SS\alpha = \sum_i n_i\,.\,(\bar{y}_i\,. - \bar{y})^2 \text{ and } SS\beta = \sum_j n_{\cdot j}(\bar{y}_{\cdot j} - \bar{y})^2$$

Int. SS = Between cell SS $-$ SSα $-$ SS(β/α)

where

$$\text{Between cell SS} = \sum_i \sum_j n_{ij}\,(\bar{y}_{ij} - \bar{y})^2$$

$$SSE_1 = \sum_i SS(I/\beta)$$

$$= \sum_i [SS_i(\beta/I) + SS_i(I) - SS_i(\beta)]$$

where $SS_i(I) = \sum_k n_{i.k}\,(\bar{y}_{i.k} = \bar{y}_{i\cdot})^2$, $SS_i(\beta) = \sum_j n_{ij}(\bar{y}_{ij} - \bar{y}_{i\cdot})^2$

and $$SS_i(\beta/I) = Q'_{i\beta}\, C_{i\beta}^-\, Q_{i\beta}$$

with $$Q_{i\beta j} = y_{ij} - \sum_k n_{ij}\,\bar{y}_{i.k},\quad j = 1, 2, ..., b$$

$$C_{i\beta} = (c_{i\beta jj'})$$

$$c_{i\beta jj'} = n_{ij} - \sum_k \delta_{ijk}^2/n_{i.k}\ ,\ j' = j$$

$$= -\sum_k \delta_{ijk}\,\delta_{ij'k}/n_{i.k}\ ,\ j' \neq j$$

$$SSE_2 = \sum_i [TSS_i - SS_i(\beta/I) - SS_i(I)]$$

$$\text{where } TSS_i = \sum_j \sum_k \delta_{ijk} (y_{ijk} - \bar{y}_{i\cdot})^2 \tag{2.1}$$

3. *Distributions*

The expected values of sums of squares (2.1) under (1.1 - 1.3) are obtained as follows. The details of few results are given in Appendix.

$$E_0[SS(\alpha/\beta)] = (a-1)(1-\epsilon)\sigma^2 + \epsilon[\sum_i \sum_k (1/n_{i\cdot}) n_{i.k}^2 - b]\,\sigma^2$$

$$+ [tr(C_\beta C_{\beta*}^-) - (b-1)(1-\epsilon)]\sigma^2$$

$$E_0[SS(\beta/\alpha)] = [tr(C_\beta^- C_{\beta*})]\sigma^2$$

$$E_0[SS(\alpha*\beta)] = (a-1)(b-1)(1-\epsilon)\sigma^2 + \epsilon\{ab - \sum_i \sum_k (1/n_{i\cdot}) n_{i.k}^2\}\sigma^2$$

$$- [tr(C_\beta^- C_{\beta*}) - (b-1)(1-\epsilon)]\sigma^2$$

$$E[SSE_1] = (N-a)(1-\epsilon)\sigma^2 + \epsilon(N_0 - ab)\sigma^2 \text{ and}$$

$$E[SSE_2] \doteq (N_0 - N - ab + a)(1-\epsilon)\sigma^2 \tag{3.1}$$

$$C_{\beta*} = (C_{\beta jj'*}) \text{ with}$$

$$C_{\beta jj'*} = C_{\beta jj}(1-\epsilon)\sigma^2 + [n_{\cdot j} + \sum_i \sum_k (n_{ij\cdot} n_{i.k}/n_{i\cdot})^2$$

$$- 2\sum_i \sum_k n_{ij} \,.\, n_{i\cdot k}\delta_{ijk}/n_{i\cdot}]\, \epsilon\, \sigma^2 \qquad \text{for } j = j'$$

$$= C_{\beta jj'}(1-\epsilon)\sigma^2 + \sum_i \sum_k [\delta_{ijk}\delta_{ij'k} - (n_{ij'\cdot} n_{i\cdot k}\delta_{ijk}/n_{i\cdot})$$

$$- (n_{ij\cdot} n_{i\cdot k}\delta_{ij'k}/n_{i\cdot}) + (n_{ij\cdot} n_{i\cdot k}^2 n_{ij'\cdot}/n_{i\cdot}^2)]\epsilon\sigma^2 \quad \text{for } j = j'$$

Note that the expressions (3.1) are simplified when

$$(\delta_{ijk}/n_{ij\cdot}) = (n_{i\cdot k}/n_{i\cdot}) \tag{3.2}$$

which may not be unjustified with incompletely repeated observations, since each animal has equal chance of survival to a particular stage of experiment. The simplified expressions are given by

$$E_0[SS(\alpha/\beta)] = (a-1)(1-\epsilon)\sigma^2 + \epsilon[\sum_i \sum_k (1/n_{i\cdot}) n_{i.k}^2 - b]\sigma^2$$

$$E_0[SS(\beta/\alpha)] = (b-1)(1-\epsilon)\sigma^2$$

$$E_0[SS(\alpha*\beta)] = (a-1)(b-1)(1-\epsilon)\sigma^2 + \epsilon\{ab - \sum_i\sum_k (1/n_{i.})n_{i.k}^2\}\sigma^2$$

$$E[SSE_1] = (N-a)(1-\epsilon)\sigma^2 + \epsilon(N_0 - ab)\,\sigma^2 \quad \text{and}$$

$$E[SSE_2] = (N_0 - N - ab + a)(1-\epsilon)\sigma^2 \qquad (3.3)$$

The expressions (3.3) reduce to the following under balanced (n_{ijk} = 1, for all i, j and k) situations:

$$E_0[SS(\alpha)] = (a-1)[1+(b-1)\epsilon]\sigma^2, \; E_0[SS(\beta)] = (b-1)(1-\epsilon)\sigma^2$$

$$E_0[SS(\alpha*\beta)] = (a-1)(b-1)(1-\epsilon)\sigma^2, \; E[SSE_1] = (N-a)[1+(b-1)\epsilon]\sigma^2$$

and $$E[SSE_2] = (N-a)(b-1)(1-\epsilon)\sigma^2 \qquad (3.4)$$

We here note that the sums of squares (2.1) are independently distributed under (1.1 - 1.3). The exact distributions of these sums of squares can be derived by using the standard theory for distributions of quadratic forms in normal variables (see, Johnson and Kotz [7]) but these are very tedious for practical situations. Here we write only the one moment approximations to their null distributions under (3.2) as follows:

$$SS(\alpha/\beta) \sim a_1\,\sigma^2\,\chi^2_{(a-1)}, \; SS(\beta/a) \sim b_1\,\sigma^2\,\chi^2_{(b-1)}$$

$$SS[\alpha*\beta] \sim b_2\,\sigma^2\,\chi^2_{(a-1)(b-1)}, \; SSE_1 \sim a_2\,\sigma^2\,\chi^2_{(N-a)}$$

and $$SSE_2 \sim b_3\,\sigma^2\,\chi^2_{(N_0 - N - ab + a)} \qquad (3.5)$$

where $$a_1 = (1-\epsilon) + \epsilon[\sum_i\sum_k(1/n_{i.})n_{i.k}^2 - b]/(a-1)$$

$$a_2 = (1-\epsilon) + \epsilon(N_0 - ab)/(N-a), \; b_1 = (1-\epsilon)$$

$$b_2 = (1-\epsilon) + \epsilon\{ab - \sum_i\sum_k(1/n_{i.})n_{i.k}^2\}/(a-1)(b-1) \text{ and } b_3 = (1-\epsilon)$$

These expressions under balanced situations are given by

$$a_1 = a_2 = [1+(b-1)\epsilon] \quad \text{and} \quad b_1 = b_2 = b_3 = (1-\epsilon)$$

Also the sums of squares (2.1) under balanced situations have exact chi-square distributions with respective degrees of freedom.

4. *Testing Procedures*

The test statistics for testing the null hypothesis H_{01} is defined as

$$F_1 = MS(\alpha/\beta)/MSE_1 \tag{4.1}$$

where $MS(\alpha/\beta)$ and MSE_1 are the usual mean squares.

The approximate null distribution of (4.1) is (a_1/a_2) times Snedecor F with $[a-1,\ (N-a)]$ degrees of freedom. The probability of critical region is expressed as

$$P[F_1 \geq F_{01}] = I_{w1}[(N-a)/2\ ,\ (a-1)/2] \tag{4.2}$$

where $$w1 = [1 + \{(a-1)/(N-a)\}\,(a_2/a_1)\,F_{01}]^{-1}$$

The test statistics for testing the null hypothesis H_{02} is

$$F_2 = MS\,(\beta/\alpha)\,/MSE_2 \tag{4.3}$$

with null distribution as Snedecor F on $(b-1, N_0 - N - ab + a)$ degrees of freedom. The probability of critical region is expressed as

$$P[F_2 \geq F_{02}] = I_{w2}[(N_0 - N - ab + a)/2,\ (b-1)/2] \tag{4.4}$$

where $$w2 = [1 + \{(b-1)/N_0 - N - ab + a)\}F_{02}]^{-1}$$

The test statistics for testing the null hypothesis H_{03} is

$$F_3 = \text{Int. SS}/MSE_2 \tag{4.5}$$

with approximate null distribution as (b_2/b_3) times as Snedecor F on $[(a-1)(b-1),\ N_0 - N - ab + a]$ degrees of freedom. The probability of critical region is expressed as

$$P[F_3 \geq F_{03}] = I_{w3}\,[(N_0 - N - ab + a)/2,\ (a-1)(b-1)/2] \tag{4.6}$$

where $w3 = [1 + \{(a-1)(b-1)/(N_0 - N - ab + a)\}(b_3/b_2)F_{03}]^{-1}$

Under balanced situations the null distributions of the above test-statistics are exactly Snedecor F with corresponding degrees of freedom as mentioned in (3.4).

The sums of squares, null expectations, testing statistics and approximate distributions are presented in ANOVA form in Table-0.

Table-0 : Analysis of Variance table for repeated measurement design

Source	df	SS	$E_0(MS)$	F-statistic	Null distribution	
					Incomplete	Balanced
Group (Main)	$a-1$	$SS(\alpha/\beta)$	$a_1\sigma^2$	$MS(\alpha/\beta)/MSE_1$	$(a_1/a_2)\,F_{[a-1,\,N-a]}$	$F_{[a-1,\,N-a]}$
Error 1	$N-a$	SSE1	$a_2\sigma^2$			
Time (Sub)	$b-1$	$SS(\beta/\alpha)$	$b_1\sigma^2$	$MS(\beta/\alpha)/MSE_2$	$F_{[b-1,\,N_0-N-ab+a]}$	$F_{[b-1,\,(N-a)(b-1)]}$
Int. $(\alpha*\beta)$	$(a-1)*(b-1)$	$SS(\alpha*\beta)$	$b_2\sigma^2$	$MS(\alpha*\beta)/MSE_2$	$F_{[(a-1)(b-1),\,N_0-N-ab+a]}$	$F_{[(a-1)(b-1),\,(N-a)(b-1)]}$
Error 2	$(N_0-N-ab+a)$	SSE2	$b_3\sigma^2$ $(b_1 = b_3)$	$df = (N-a)(b-1)$	under balanced case	

5. *Approximate Methods*

The methods of analysis discussed in Section 4 depend upon the value of uniform correlation which is not known under practical situations. We here consider two approximate methods, based on Welch-Satterthwaite and Cochran - Cox approximations (see, Cochran [3]. These methods do not require the value of uniform correlation for analysis of such data.

5.1 Welch-Satterthwaite Approximation

Brown and Forsythe [2] have suggested a modified F-ratio by changing the denominator so that the expectation of both numerator and dominator have same expectation under the null hypothesis. The modified F is assumed to be distributed approximately as Snedecor F with modified degrees of freedom for denominator, obtained by using the Satterthwaite [8] method of combining the degrees of freedom.

We have exact testing for null hypothesis H_{02}. For testing null hypothesis H_{01} and H_{03}, approximate tests are developed as follows:

One can write that

$$E_0 (SS\alpha/\beta) = E(k_1\, MSE_1 + k_2\, MSE_2)$$

and

$$E_0(SS\alpha*\beta) = E(k_3\, MSE_1 + k_4\, MSE_2)$$

where

$$k_1 = [a(n-1)/(N-ab)][\sum_i \sum_k (1/n_{i}.)n_{i.k}^2 - b]$$

$$k_2 = (a-1) - k_1$$

$$k_3 = [a(n-1)/(N-ab)][ab - \sum_i \sum_k (1/n_{i}.)n_{i.k}^2]$$

and

$$k_4 = (a-1)(b-1) - k_3$$

These expressions give that

$$SSE(\alpha/\beta) = k_1\, MSE_1 + k_2\, MSE_2$$

and

$$SSE(\alpha*\beta) = k_3\, MSE_1 + k_4\, MSE_2$$

and the modified test statistics for testing H_{01} and H_{03} are

$$F_\alpha = SS(\alpha/\beta)/SSE(\alpha) \text{ and } F_{\alpha\beta} = SS(\alpha*\beta)/SSE(\alpha*\beta), \text{ respectively}$$

Their approximate null distributions are Snedecor F with degrees of freedom as $(a-1, f_\alpha)$ and $[(a-1)(b-1), f_{\alpha\beta}]$, respectively, where

$$f_\alpha = [k_1 MSE_1 + k_2 MSE_2]^2 / [\{(k_1 MSE_1)^2/(N-a)\} + \{(k_2 MSE_2)^2/(N_0 - N - ab + a)\}]$$

and $$f_{\alpha\beta} = [k_3 MSE_1 + k_4 MSE_2]^2 / [\{(k_3 MSE_1)^2/(N-a)\} + \{(k_4 MSE_2)^2/(N_0 - N - ab + a)\}] \quad (5.1)$$

5.2 Cochran-Cox Approximation

In this method we use the same values of modified F ratios as in subsection (5.1) but compare them with the corresponding F-value obtained as the weighted average of tabulated F at two error degrees of freedom (see Cochran, [3]) as follows:

For H_{01}, the modified tabulated value is defined as

$$F_1 = [k_1 MSE_1 F_{01} + k_2 MSE_2 F_{02}]/[k_1 MSE_1 + k_2 MSE_2]$$

where F_{01} and F_{02} are the tabulated values of Snedecor F with $[a-1, (N-a)]$ and $[a-1, N_0 - N - ab + a]$ degrees of freedom, respectively.

Similarly, the modified tabulated value for H_{03} is defined as

$$F_3 = [k_3 MSE_1 F_{03} + k_4 MSE_2 F_{04}]/[k_3 MSE_1 + k_4 MSE_2]$$

where F_{03} is tabulated value of Snedecor F with $[(a-1)(b-1), (N-a)]$ degrees of freedom and F_{04} is the corresponding value at $[(a-1)(b-1), N_0 - N - ab + a]$ degrees of freedom.

6. *Numerical Results*

Here, we present numerical values for size of tests under (1.1 - 1.3, 3.2) for some apriori parametric values as $\epsilon = -0.3(0.2)0.7$, $a = 3$, $b = 5$, $n = 4$, two incompletely repeated measure designs D_1 and D_2. The size for these tests are also obtained by using two approximations to see their comparative validity. The two apriori designs are defined in Table 1 and the size of tests in Table 2. The corresponding values for approximation are given in Table 3.

Table 1. Values of $n_{i.k}$ for two designs

	Design 1				Design 2			
i\k	1	2	3	4	1	2	3	4
1	5	4	3	2	5	3	3	2
2	5	5	4	2	5	3	2	2
3	5	4	2	2	5	4	2	2

Table 2. Size of test for different values of correlation

Design	Effect	$\epsilon = -0.3$	-0.1	0.1	0.3	0.5	0.7
1	α	0.0195	0.0442	0.0531	0.0574	0.0603	0.0620
	$\alpha\beta$	0.0354	0.0437	0.0577	0.0833	0.1387	0.2906
2	α	0.0280	0.0446	0.0532	0.0584	0.0617	0.0642
	$\alpha\beta$	0.0326	0.0428	0.0602	0.0932	0.1652	0.3534

Table 3. Size of test for two approximations

Method	Design	Effect	$\epsilon=-0.3$	-0.1	0	0.1	0.3	0.5	0.7
WS	1	α	.1315	.0684	.0614	.0586	.0547	.0533	.0515
		$\alpha\beta$	.0471	.0437	.0421	.0419	.0374	.0358	.0419
	2	α	.1454	.0716	.0645	.0606	.0555	.0537	.0536
		$\alpha\beta$	.0429	.0371	.0338	.0323	.0306	.0313	.0398
CC	1	α	.0391	.0466	.0476	.0482	.0490	.0494	.0497
		$\alpha\beta$	.0478	.0451	.0436	.0420	.0383	.0337	.0281
	2	α	.0417	.0471	.0479	.0485	.0491	.0495	.0497
		$\alpha\beta$	.0476	.0452	.0439	.0425	.0396	.0363	.0328

The numerical results reveal that the size of test is affected by all values of correlation. The effect is more for interaction than the group effect. Both approximations provide good results for group effects. For interaction effect, Cochran-Cox method approximates the size of test more closely to the actual value (0.05) than the Welch-Satterthwaite method. It is, therefore, suggested that Cochran-Cox approximation may be used for analysis of data from incompletely repeated measure designs with uniform correlation among repeated observations.

7. *Example*

The effect of a vitamin E diet supplement on the growth of guinea pigs was investigated as follows. For each animal the body weight was recorded at the end of weeks 1, 3, 4, 5 and 6. Three groups of animals, numbering four in each, received respectively zero, low and high doses of vitamin E. The body weights (in grams) are given in Table 4. These data belong to Crowder and Hand ([4], example 3.1). Some observations are deliberately ignored to make the design incompletely repeated. The parameters of this design are same as taken for design 2 in Section 6. The resulting ANOVA is shown in Table 5.

Table 4. Effect of diet supplement on growth rates

Group	Animal No. 1	Weeks				
		1	3	4	5	6
1	1	455	460	510	504	436
	2	467	565	610	—	—
	3	445	530	580	—	—
	4	485	642	—	—	—
2	5	514	560	565	524	552
	6	440	480	536	—	—
	7	495	570	—	—	—
	8	520	590	—	—	—
3	9	496	560	622	622	632
	10	498	540	589	557	—
	11	478	510	—	—	—
	12	545	565	—	—	—

Table 5. ANOVA for factors affecting growth rates

Source	df	SS	MS	F	Pr.	F*	Pr(WS)	Pr(CC)
Group (Adj)	2	11764.8	5882.4	1.638	0.2474	1.459	0.2858	0.2828
Error 1	9	32317.1	3590.8					
Time (Adj)	4	42239.4	10559.8	15.714				
Int. (Adj)	8	20499.3	2562.4	3.813	0.0141	2.034	0.0945	0.1172
Error 2	14	9408.0	672.0					

* Modified F-statistic for two approximations.

The results of analysis indicate that the probability of critical region for group effect is slightly under estimated by the usual method of analysis, however, the same is seriously underestimated in case of interaction effect. The two approximations provide almost same inference for group effect and slightly different for interaction effect.

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Appendix

1. $E(SS\alpha) = \sum_i n_{i\cdot}[\alpha_i + \sum_j (n_{ij}/n_{i\cdot})\beta_j]^2 + \sum_i n_{i\cdot} V(\bar{e}_i - \bar{e})$

$$V(\bar{e}_i) = (1/n_{1\cdot}^2) V(\sum_j \sum_k e_{ijk})$$

$$= (1/n_{i\cdot}^2)[\sum_j \sum_k V(e_{ijk}) + \sum_k \sum_j \sum_{j' \neq j} Cov(e_{ijk}, e_{ij'k})]$$

$$= (1/n_{i\cdot}^2)[\sum_j \sum_k \delta_{ijk} + \epsilon \sum_k \sum_j \sum_{j' \neq j} \delta_{ijk} \delta_{ij'k}] \sigma^2$$

$$= (1/n_{i\cdot})[(1 - \epsilon) + \epsilon \sum_k n_{i.k}^2 / n_{i\cdot}] \sigma^2$$

$$V(\bar{e}) = V[(1/N_0) \sum_i \sum_j \sum_k e_{ijk}]$$

$$= [(1 - \epsilon) + \epsilon \sum_i \sum_k n_{i.k}^2 / N_0] \sigma^2 / N_0$$

$$Cov(\bar{e}_i, \bar{e}) = (n_{i\cdot} / N_0) V(\bar{e}_i)$$

$$= (1/N_0)[(1 - \epsilon) + \epsilon \sum_k n_{i.k}^2 / n_{i\cdot}] \sigma^2$$

This gives that

$$\sum_i n_{i\cdot} V(\bar{e}_i - \bar{e}) = [(a-1)(1 - \epsilon) + \epsilon \sum_i \sum_k \{(1/n_{i\cdot}) - (1/N_0)\} n_{i.k}^2] \sigma^2$$

and $E(SS\alpha) = \sum_i n_{i\cdot}[\alpha_i + \sum_j (n_{ij\cdot}/n_{i\cdot})\beta_j]^2 + (a-1)(1 - \epsilon)\sigma^2$

$$+ \epsilon \sum_i \sum_k [\{(1/n_{i\cdot}) - (1/N_0)\} n_{i.k}^2] \sigma^2 \qquad (A.1)$$

Similarly,

2. $E(SS\beta) = \sum_j n_{\cdot j} [\beta_j + \sum_i (n_{ij}/n_{\cdot j})\alpha_i]^2 + (b-1)(1 - \epsilon)\sigma^2$

$$+ \epsilon [b - \sum_i \sum_k (n_{i.k}^2 / N_0)] \sigma^2 \qquad (A.2)$$

3. $E(SS(\beta/\alpha)] = \sum_j n_{\cdot j} \beta_j^2 - \sum_i n_{i\cdot}.\{\sum_j n_{ij}.\beta_j / n_{i\cdot}.\}^2 + tr(C_\beta^- C_{\beta *})\sigma^2 \qquad (A.3)$

where $C_{\beta *} = (C_{\beta * jj'})$

$$C_{\beta*jj}\sigma^2 = V(Q_{\beta j}) = V[e_{.j} - \sum_i (n_{ij.}/n_{i.})e_i]$$

$$= V(e_{.j}) + V[\sum_i (n_{ij.}/n_{i.})e_i] - 2\,\mathrm{Cov}\,[e_{.j}, \sum_i (n_{ij.}/n_{i.})e_i]$$

$$V(e_{.j}) = n_{.j}\,\sigma^2$$

$$V[\sum_i (n_{ij.}/n_{i.})e_i] = [\sum_i (n_{ij.}{}^2/n_{i.})(1-\epsilon) + \epsilon \sum_i \sum_k (n_{ij.}n_{i.k}/n_{i.})^2]\sigma^2$$

$$\mathrm{Cov}\,[e_{.j}, \sum_i (n_{ij.}/n_{i.})e_i] = [\sum_i (n_{ij.}{}^2/n_{i.})(1-\epsilon) + \epsilon \sum_i \sum_k (n_{ij.}n_{i.k}\delta_{ijk}/n_{i.})]\sigma^2$$

These expressions give that

$$C_{\beta*jj} = C_{\beta jj}(1-\epsilon) + \epsilon[n_{.j} + \sum_i \sum_k (n_{ij.}n_{i.k}/n_{i.})^2 - 2\sum_i\sum_k (n_{ij.}n_{i.k}\delta_{ijk}/n_{i.})]$$

Similarly,

$$C_{\beta*jj'} = C_{\beta jj'}(1-\epsilon) + \epsilon[\sum_i \sum_k \delta_{ijk}\delta_{ij'k} - \sum_i \sum_k (n_{ij'.}n_{i.k}\delta_{ijk}/n_{i.})$$

$$- \sum_i \sum_k (n_{ij.}n_{i.k}\delta_{ij'k}/n_{i.}) + \sum_i \sum_k (n_{ij.}n_{ij'.}n_{i.k}^2/n_{i.}{}^2)]$$

Under H_{02}, The expression (A.3) reduces to

$$E_0(SS(\beta/\alpha)] = \mathrm{tr}(C_\beta^- C_{\beta*})\,\sigma^2 \qquad (A.4)$$

From (A.1) to (A.3) we obtain that

$$4.\ E[SS(\alpha/\beta)] = E[SS(\beta/\alpha) + SS(\alpha) - SS(\beta)]$$

$$= \sum_i n_{i.}\,\alpha_i^2 - \sum_j n_{.j}\,\{\sum_i n_{ij}\,.\,\alpha_i/n_{.j}\}^2 + (a-1)(1-\epsilon)\,\sigma^2$$

$$+ \epsilon[\sum_i \sum_k (n_{i.k}^2/n_{i.}) - b + \mathrm{tr}(C_\beta^- C_{\beta*}) - (b-1)(1-\epsilon)]\sigma^2 \qquad (A.5)$$

and

$$E_0[SS(\alpha/\beta)] = (a-1)(1-\epsilon)\sigma^2 + \epsilon[\sum_i \sum_k (n_{i.k}^2/n_{i.}) - b$$

$$+ \mathrm{tr}(C_\beta^- C_{\beta*}) - (b-1)(1-\epsilon)]\sigma^2 \qquad (A.6)$$