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# Weighted Least Squares and Nonlinear Regression

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SUMMARY

The classical least squares approach for parameter estimation in nonlinear regression models does not use prior information available on the parameter. Here a weighted least square approach with the weights possibly depending on the unknown parameter is studied. Asymptotic properties of the estimator are investigated. An example is presented.

*Key words* : Weighted least squares, Nonlinear regression.

## 1. *Introduction*

The method of least squares is widely used in analysis of data especially in fitting linear and nonlinear models. In linear models, it provides estimators with some optimum properties and in nonlinear models, it gives estimators which are asymptotically well behaved. However, the classical least squares approach for parameter estimation does not use prior information available on the unknown parameter. Here we propose to study a weighted least squares approach where the weights possibly depend on the unknown parameter $\theta$. Examples of such problems occur in practice (cf. Kiefer [1]). It was shown in Prakasa Rao [3] that the ridge estimators for linear models can be considered as a special case of weighted least squares estimators for proper choice of the weight functions. Asymptotic theory for least squares estimators in nonlinear regression is surveyed in Prakasa Rao [2].

## 2. *Nonlinear Regression*

Consider the nonlinear regression model

$$Y_i = g_i(\theta) + \varepsilon_i, \ i \geq 1 \tag{2.1}$$

where $\varepsilon_i, i \geq 1$ are random variables with $E(\varepsilon_i) = 0$ and $0 < \text{Var}(\varepsilon_i) = \sigma_i^2 < \infty$, $\theta \in \Theta \subset R$. Suppose that $\sigma_i^2, i \geq 1$ are known. $\hat{\theta}_n$ is said to be a weighted least squares estimator (WLSE) corresponding to the sequence of weights $\{\lambda_{in}(\theta), 1 \leq i \leq n\}$ if $\hat{\theta}_n$ minimizes

$$Q_n(\theta) = \sum_{i=1}^{n} \frac{\lambda_{in}(\theta)(Y_i - g_i(\theta))^2}{\sigma_i^2} \tag{2.2}$$

over $\theta \in \overline{\Theta}$. Suppose $\lambda_{in}(\theta) = \lambda(\theta)$ for all i. Then $Q_n(\theta)$ reduces to

$$Q_n^*(\theta) = \lambda(\theta) \sum_{i=1}^{n} \frac{(Y_i - g_i(\theta))^2}{\sigma_i^2} \tag{2.3}$$

and $(\lambda(\theta))^{-1}$ can be interpreted as the prior "density" and the higher the prior density, the lower the weightage assigned to the sum of squares. This is appropriate for our discussion as the main purpose is to minimize the sum of squares of deviations. Let us denote the WLSE in this case by $\theta_n^*$.

## 3. *Consistency*

In order to give an idea of the type of regularity conditions needed for consistency of the estimator $\hat{\theta}_n$, we will state them in a general form. Suppose the following conditions hold:

(C 0) $\Theta$ is open

(C 1) There exists $0 < \gamma_n \to \infty$ such that, for every $\theta_0 \in \Theta$

$$p-\lim_{\substack{n\to\infty \\ |h|\to 0}} \frac{1}{\gamma_n} \sum_{i=1}^{n} \frac{\lambda_{in}(\theta_0 + h) - \lambda_{in}(\theta_0)}{\sigma_i^2} \varepsilon_i^2 = 0 \tag{3.1a}$$

$$p-\lim_{\substack{n\to\infty \\ |h|\to 0}} \frac{1}{\gamma_n} \sum_{i=1}^{n} \frac{\lambda_{in}(\theta_0 + h)[g_i(\theta_0 + h) - g_i(\theta_0)]}{\sigma_i^2} \varepsilon_i = 0 \tag{3.1b}$$

$$\lim_{\substack{n\to\infty \\ |h|\to 0}} \frac{1}{\gamma_n} \sum_{i=1}^{n} \frac{[\lambda_{in}(\theta_0 + h) - \lambda_{in}(\theta_0)][g_i(\theta_0 + h) - g_i(\theta_0)]^2}{\sigma_i^2} = 0 \tag{3.1c}$$

and $$\lim_{\substack{n\to\infty \\ |h|\to 0}} \frac{1}{\gamma_n} \sum_{i=1}^{n} \frac{\lambda_{in}(\theta_0)[g_i(\theta_0 + h) - g_i(\theta_0)]^2}{\sigma_i^2} = h^2 B(\lambda, \theta_0) + O(1) \tag{3.1d}$$

as $h \to 0$ where $B(\lambda, \theta_0) > 0$

*Theorem 3.1.* Under the conditions (C 0) and (C 1), there exists a local weakly consistent weighted least squares estimator $\hat{\theta}_n$ for $\theta$ corresponding to the weights $\{\lambda_{in}\}$, that is, for every $\varepsilon > 0$,

$$P_{\theta_0}(|\hat{\theta}_n - \theta_0| > \varepsilon) \to 0 \text{ as } n \to \infty \quad \text{for every } \theta_0 \in \Theta$$

*Proof.* Let $\theta_0 \in \Theta$. Note that

$$\frac{1}{\gamma_n}[Q_n(\theta_0 + h) - Q_n(\theta_0)]$$

$$= \frac{1}{\gamma_n}\sum_{i=1}^{n} \frac{\lambda_{in}(\theta_0 + h)(Y_i - g_i(\theta_0 + h))^2}{\sigma_i^2} - \frac{1}{\gamma_n}\sum_{i=1}^{n}\frac{\lambda_{in}(\theta_0)(Y_i - g_i(\theta_0))^2}{\sigma_i^2}$$

$$= \frac{1}{\gamma_n}\sum_{i=1}^{n}\frac{\lambda_{in}(\theta_0 + h)}{\sigma_i^2}[\varepsilon_i - g_i(\theta_0 + h) + g_i(\theta_0)]^2 - \frac{1}{\gamma_n}\sum_{i=1}^{n}\frac{\lambda_{in}(\theta_0)}{\sigma_i^2}\varepsilon_i^2$$

$$= \frac{1}{\gamma_n}\sum_{i=1}^{n}\left\{\frac{\lambda_{in}(\theta_0 + h) - \lambda_i(\theta_0)}{\sigma_i^2}\right\}\varepsilon_i^2$$

$$- \frac{2}{\gamma_n}\sum_{i=1}^{n}\frac{\lambda_{in}(\theta_0 + h)}{\sigma_i^2}\varepsilon_i[g_i(\theta_0 + h) - g_i(\theta_0)]$$

$$+ \frac{1}{\gamma_n}\sum_{i=1}^{n}\frac{\lambda_{in}(\theta_0 + h)}{\sigma_i^2}[g_i(\theta_0 + h) - g_i(\theta_0)]^2$$

$$= \frac{1}{\gamma_n}\sum_{i=1}^{n}\frac{\lambda_{in}(\theta_0 + h)}{\sigma_i^2}[g_i(\theta_0 + h) - g_i(\theta_0)]^2 + 0_p^{h,n}(1)$$

(by (3.1a) and (3.1b))

$$= \frac{1}{\gamma_n}\sum_{i=1}^{n}\frac{\lambda_{in}(\theta_0)}{\sigma_i^2}[g_i(\theta_0 + h) - g_i(\theta_0)]^2 + 0^{h,n}(1) + 0_p^{h,n}(1) \qquad \text{(by (3.1 c))}$$

$$= h^2\{B(\lambda, \theta_0) + 0^n(1)\} + 0^{h,n}(1) + 0_p^{h,n}(1) \qquad \text{(by (3.1d))}$$

where $0_p^{h,n}(1)$ denotes that the term tends to zero in probability as $n \to \infty$ and $|h| \to 0$ simultaneously, $0^{h,n}(1)$ denotes that the term tends to zero as $n \to \infty$ and $|h| \to 0$ simultaneously and $0^n(1)$ denotes that the term tends to zero as

$n \to \infty$. From the above expression it follows that $Q_n(\theta_0 + h) - Q_n(\theta_0) > 0$ with probability greater than $1 - \eta$ for n large for $|h| \leq \delta$, $\delta$ sufficiently small. Since $Q_n(\theta)$ is continuous in $\theta$, it attains its minimum over the interval $[\theta_0 - \delta, \theta_0 + \delta]$. The minimum is not attained at the end points since $Q_n(\theta_0 + h) > Q_n(\theta_0)$ for $h = -\delta$ or $h = +\delta$. Hence

$$P_{\theta_0}(|\hat{\theta}_n - \theta_0| < \delta) > 1 - \eta$$

for large *n*. This proves the theorem.

*Remarks 3.1.* Suppose $\lambda_{in}(\theta) = \lambda_n(\theta)$ for all i and $\lambda_n(\theta)$ is positive and continuous over $\Theta$ such that $\lambda_n(\theta) \to \lambda(\theta)$ as $n \to \infty$. Then the condition (C 1) holds provided

(C 1)* there exists $0 < \gamma_n \to \infty$ such that

$$p - \lim_{n \to \infty} \frac{1}{\gamma_n} \sum_{i=1}^{n} \frac{\varepsilon_i^2}{\sigma_i^2} < \infty \tag{3.2a}$$

$$p - \lim_{n \to \infty} \frac{1}{\gamma_n} \sum_{i=1}^{n} \frac{[g_i(\theta_0 + h) - g_i(\theta_0)]}{\sigma_i^2} \varepsilon_i = 0 \tag{3.2b}$$

and

$$\lim_{n \to \infty} \frac{1}{\gamma_n} \sum_{i=1}^{n} \frac{[g_i(\theta_0 + h) - g_i(\theta_0)]^2}{\sigma_i^2} = h^2 \gamma(\theta_0) < \infty \tag{3.2c}$$

where $\gamma(\theta_0) > 0$. Note that (3.1d) holds with $B(\lambda, \theta_0) = \lambda(\theta_0)\gamma(\theta_0)$.

*Corollary 3.1.* Under the conditions (C 0) and (C 1)*, there exist a locally weakly consistent weighted least squares estimator for $\theta$ corresponding to the weight function $\lambda(\theta)$.

## 4. *Asymptotic Normality*

Let us now obtain sufficient conditions for the asymptotic normality of WLSE corresponding to the sequence of weights $\{\lambda_{in}(\theta), 1 \leq i \leq n, n \geq 1\}$. Suppose the following conditions are satisfied.

(C 2) Suppose that $\lambda_{in}(.)$ and $g_i(.)$ are continuously differentiable twice in a neighbourhood $V_{\theta_0}$ of $\theta_0$ for every $\theta_0 \in \Theta$ and $E(\varepsilon_i^4) < \infty$, $i \geq 1$.

It is easy to see that

$$Q_n^{(1)}(\theta) = \sum_{i=1}^{n} \frac{\lambda_{in}^{(1)}(\theta)}{\sigma_i^2}(Y_i - g_i(\theta))^2$$

$$- 2\sum_{i=1}^{n} \frac{\lambda_{in}(\theta)}{\sigma_i^2}(Y_i - g_i(\theta))g_i^{(1)}(\theta) \tag{4.1}$$

and $$Q_n^{(2)}(\theta) = \sum_{i=1}^{n} \frac{\lambda_{in}^{(2)}(\theta)}{\sigma_i^2}(Y_i - g_i(\theta))^2 - 4\sum_{i=1}^{n} \frac{\lambda_{in}^{(1)}(\theta)g_i^{(1)}(\theta)}{\sigma_i^2}(Y_i - g_i(\theta))$$

$$+ 2\sum_{i=1}^{n} \frac{\lambda_{in}(\theta)[g_i^{(1)}(\theta)]^2}{\sigma_i^2} - 2\sum_{i=1}^{n} \frac{\lambda_{in}(\theta)g_i^{(2)}(\theta)}{\sigma_i^2}(Y_i - g_i(\theta)) \tag{4.2}$$

where $f^{(i)}(\theta)$ denotes the i-th derivative of $f(\theta)$ with respect to $\theta$. Expanding $Q_n^{(1)}(\theta)$ in a neighbourhood of the true parameter $\theta_0$, we have

$$Q_n^{(1)}(\theta) = Q_n^{(1)}(\theta_0) + [Q_n^{(2)}(\theta_0) + 0^n(1)](\theta - \theta_0) \tag{4.3}$$

(C 3) Suppose that WLSE $\hat{\theta}_n$ corresponding to $\{\lambda_{in}\}$ is consistent.

In particular, under the condition (C 2), $Q_n(\theta)$ is differentiable and it follows that $Q_n^{(1)}(\hat{\theta}_n) = 0$. Hence

$$0 = Q_n^{(1)}(\hat{\theta}_n) = Q_n^{(1)}(\theta_0) + [Q_n^{(2)}(\theta_0) + 0_p^n(1)](\hat{\theta}_n - \theta_0) \tag{4.4}$$

or equivalently $$\hat{\theta}_n - \theta_0 = \frac{Q_n^{(1)}(\theta_0)}{-[Q_n^{(2)}(\theta_0) + 0_p^n(1)]} \tag{4.5}$$

It is easy to see that

$$E_{\theta_0}[Q_n^{(1)}(\theta_0)] = \sum_{i=1}^{n} \lambda_{in}^{(1)}(\theta_0) \tag{4.6}$$

Note that

$$Q_n^{(1)}(\theta_0) = \sum_{i=1}^{n} f_{in}(\varepsilon_i) \tag{4.7}$$

where $$f_{in}(\varepsilon_i) \equiv \frac{1}{\sigma_i^2}[\lambda_{in}^{(1)}(\theta_0)\varepsilon_i^2 - 2\lambda_{in}(\theta_0)g_i^{(1)}(\theta_0)\varepsilon_i] \tag{4.8}$$

One can give sufficient conditions on the sequence $\{\varepsilon_i\}$ for the asymptotic normality of a normalized version of the sequence $Q_n^{(1)}(\theta_0)$ using standard limit theorems for double sequences of random variables.

Assume that the following conditions hold in addition to (C 2) and (C 3):

(C 4) Suppose there exists $0 < \gamma_n \to \infty$ such that

$$\frac{Q_n^{(2)}(\theta_0)}{\gamma_n} \xrightarrow{p} \tau(\lambda, \theta_0) > 0 \text{ as } n \to \infty \tag{4.9a}$$

$$\frac{E_{\theta_0}[Q_n^{(1)}(\theta_0)]}{\sqrt{\gamma_n}} \to 0 \text{ as } n \to \infty \tag{4.9b}$$

$$\frac{\mathrm{Var}_{\theta_0}[Q_n^{(1)}(\theta_0)]}{\gamma_n} \to \zeta^2(\lambda, \theta_0) > 0 \text{ as } n \to \infty \tag{4.9c}$$

and $$\frac{Q_n^{(1)}(\theta_0) - E[Q_n^{(1)}(\theta_0)]}{\sqrt{\mathrm{Var}(Q_n^{(1)}(\theta_0))}} \xrightarrow{\mathcal{L}} N(0,1) \text{ as } n \to \infty \tag{4.9d}$$

It is easy to see from (4.5) that

$$\gamma_n^{1/2}(\hat{\theta}_n - \theta_0) = \frac{\gamma_n^{-1/2}Q_n^{(1)}(\theta_0)}{-\gamma_n^{-1}[Q_n^{(2)}(\theta_0) + 0_p^n(1)]}$$

$$\cong \frac{\gamma_n^{-1/2}Q_n^{(1)}(\theta_0)}{\tau(\lambda, \theta_0)} \tag{4.10}$$

by (4.9a) where $X_n \cong Y_n$ indicates that $X_n$ and $Y_n$ have the same asymptotic distribution. Conditions (4.9c) and (4.9d) imply that

$$N(0,1) \cong \frac{Q_n^{(1)}(\theta_0) - EQ_n^{(1)}(\theta_0)}{\sqrt{\mathrm{Var}\,Q_n^{(1)}(\theta_0)}}$$

$$\cong \frac{Q_n^{(1)}(\theta_0) - EQ_n^{(1)}(\theta_0)}{\gamma_n^{1/2}\zeta(\lambda, \theta_0)} \tag{4.11}$$

$$\cong \frac{Q_n^{(1)}(\theta_0)}{\gamma_n^{1/2}\,\zeta(\lambda, \theta_0)} \qquad (\text{by } (4.9\text{ b}))$$

$$= \frac{\gamma_n^{-1/2}\, Q_n^{(1)}(\theta_0)}{\zeta(\lambda, \theta_0)} \tag{4.12}$$

Hence
$$\gamma_n^{1/2}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} N\left(0, \frac{\zeta^2(\lambda, \theta_0)}{\tau^2(\lambda, \theta_0)}\right) \text{ as } n \to \infty \tag{4.13}$$

and we have the following result.

*Theorem 4.1.* Under the conditions (C2) to (C4), the normalised WLSE $\gamma_n^{1/2}(\hat{\theta}_n - \theta_0)$ is asymptotically normal with mean 0 and variance $\zeta^2(\lambda, \theta_0)/\tau^2(\lambda, \theta_0)$ as $n \to \infty$.

*Remarks 4.1.* The conditions (4.9a), (4.9c) and (4.9d) will be satisfied under reasonable conditions on the functions $\lambda_{in}$, $g_i$ and random variables $\varepsilon_i$. The problem is with the condition (4.9b). Note that

$$\gamma_n^{-1/2} E_{\theta_0}[Q_n^{(1)}(\theta_0)] = \gamma_n^{-1/2} \sum_{i=1}^{n} \lambda_{in}^{(1)}(\theta_0) \tag{4.14}$$

For the classical least squares estimator, $\lambda_i(\theta) = 1$ for all i and for all $\theta$ and hence (4.9b) holds. Even if $\lambda_{in}(\theta) = \omega_i$ for all $\theta$ for some sequence $\omega_i$, the condition (4.9b), will hold. In general if $\lambda_{in}(\theta) = \lambda(\theta)$ for all i, then

$$\gamma_n^{-1/2} \sum_{i=1}^{n} \lambda_{in}^{(1)}(\theta_0) = n\gamma_n^{-1/2}\lambda^{(1)}(\theta_0) \tag{4.15}$$

and it is not likely to go to zero as $\gamma_n$ is of order n under standard conditions. If $\lambda^{(1)}(\theta_0) = 0$, then there is no problem. Hence, in order that the weight sequence of $\{\lambda_{in}\}$ lead to an asymptotically normal estimator, it is necessary that

$$\gamma_n^{-1/2} \sum_{i=1}^{n} \lambda_{in}^{(1)}(\theta_0) \to 0 \text{ as } n \to \infty \tag{4.16}$$

*Remarks 4.2.* Results obtained here can be generalized to the case when $\theta$ is a vector parameter and specialized to the case when $\{\varepsilon_i\}$ is either a mixing sequence of some type or a martingale difference sequence etc. But we will not pursue these cases here.

## 5. *Example*

*Example 5.1.* In order to illustrate the applicability of our results, let us consider a non-linear regression model

$$Y_i = g_i(\theta) + \varepsilon_i, \quad i \geq 1$$

where $\{\varepsilon_i, \ i \geq 1\}$ are i.i.d. random variables with mean zero and finite positive variance $\sigma^2$. Let us consider the weighted least squares estimator corresponding to the weights

$$\lambda_{in}(\theta) \equiv \lambda_n(\theta) = e^{\frac{1}{2}(\theta - \theta_{1n} - \alpha n^{-\gamma})^2}$$

where $\alpha > 0$, $\gamma > \frac{1}{2}$. Suppose that $\theta_{1n} \to \theta_1$ as $n \to \infty$. It is easy to see that $\lambda_n(\theta)$ is continuous in $\theta$ and

$$\lambda_n(\theta) \to \lambda(\theta) = e^{\frac{1}{2}(\theta - \theta_1)^2} \text{ as } n \to \infty$$

Let $\theta_0 \in \Theta$. It is easy to see that the conditions (3.1a)-(3.1d) for $\gamma_n = n$ hold provided

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} [g_i(\theta_0 + h) - g_i(\theta_0)]^2 = h^2 B(\theta_0) + 0(1) \text{ as } h \to 0 \quad \text{where } B(\theta_0) > 0$$

Suppose this condition holds. Then the weighted least squares estimator $\hat{\theta}_n$ corresponding to the weights

$$\lambda_n(\theta) = e^{\frac{1}{2}(\theta - \theta_{1n} - \alpha n^{-\gamma})^2}$$

is consistent. Observe that

$$\lambda_n^{(1)}(\theta) = (\theta - \theta_{1n} - \alpha n^{-\gamma}) e^{\frac{1}{2}(\theta - \theta_{1n} - \alpha n^{-\gamma})^2}$$

and
$$\lambda_n^{(2)}(\theta) = e^{\frac{1}{2}(\theta - \theta_{1n} - \alpha n^{-\gamma})^2} [1 + (\theta - \theta_{1n} - \alpha n^{-\gamma})^2]$$

Then $$\lambda_n(\theta_0) \rightarrow e^{\frac{1}{2}(\theta_0-\theta_1)^2}$$

$$\lambda_n^{(1)}(\theta_0) \rightarrow e^{\frac{1}{2}(\theta_0-\theta_1)^2}(\theta_0-\theta_1)$$

and $$\lambda_n^{(2)}(\theta_0) \rightarrow e^{\frac{1}{2}(\theta_0-\theta_1)^2}[1+(\theta_0-\theta_1)^2]$$

as $n \rightarrow \infty$. Suppose the regression functions $\{g_i(\theta)\}$ are twice differentiable and the random error sequence $\{\varepsilon_i\}$ are such that

$$p-\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i=1}^{n} g_i^{(2)}(\theta)\varepsilon_i = 0 \tag{5.0a}$$

$$\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i=1}^{n} g_i^{(1)}(\theta) = \alpha(\theta) < \infty \tag{5.0b}$$

and $$\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i=1}^{n} g_i^{(1)^2}(\theta) = \gamma(\theta) < \infty \tag{5.0c}$$

for all $\theta \in \Theta$. It can be checked that, with $\gamma_n = n$,

$$\frac{1}{n}Q_2^{(2)}(\theta_0) \xrightarrow{p} e^{\frac{1}{2}(\theta_0-\theta_1)^2}[1+2\gamma(\theta_0)+(\theta_0-\theta_1)^2] \equiv \tau(\lambda,\theta_0)$$

satisfying (4.9a). Further more

$$\frac{E_{\theta_0}[Q_n^{(1)}(\theta_0)]}{\sqrt{n}} = \sqrt{n}\,\lambda_n^{(1)}(\theta_0)$$

$$= \sqrt{n}\,e^{\frac{1}{2}(\theta_0-\theta_{1n}-\alpha n^{-\gamma})^2}(\theta_0-\theta_{1n}-\alpha n^{-\gamma})$$

and the last expression tends to zero provided

$$\theta_0 = \theta_{1n}+\alpha n^{-\gamma}+0\left(\frac{1}{\sqrt{n}}\right) \tag{5.1}$$

After a little algebra, it can be checked that

$$\frac{\mathrm{Var}\,(Q_n^{(1)}(\theta_0))}{n} \rightarrow \frac{e^{\frac{1}{2}(\theta_0-\theta_1)^2}}{\sigma^4}[(\theta_0-\theta_1)^2\beta+4\sigma^2\gamma(\theta_0)-4\delta(\theta_0-\theta_1)\alpha(\theta_0)]$$

where $\mathrm{Var}(\varepsilon_i) = \sigma^2$, $\mathrm{Var}(\varepsilon_i^2) = \beta$ and $\mathrm{Cov}(\varepsilon_i^2, \varepsilon_i) = \delta$. Hence (4.9b) holds provided

$$\zeta^2(\lambda, \theta_0) \equiv (\theta_0 - \theta_1)^2\beta + 4\sigma^2\gamma(\theta_0) - 4\delta(\theta_0 - \theta_1)\tau(\theta_0) > 0 \tag{5.2}$$

(B) Assume that the Lindeberg type of conditions hold on the sequence

$$f_{in}(\varepsilon_i) = \frac{1}{\sigma_i^2}[\varepsilon_i^2 - 2g_i^{(1)}(\theta_0)\varepsilon_i]\lambda_n^{(1)}(\theta_0)$$

Then it follows that

$$\frac{Q_n^{(1)}(\theta_0) - E_{\theta_0}[Q_n^{(1)}(\theta_o)]}{\sqrt{\mathrm{Var}[Q_n^{(1)}(\theta_0)]}} \xrightarrow{\mathcal{L}} N(0, 1) \text{ as } n \to \infty \tag{5.3}$$

Hence $$n^{1/2}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} N\left(0, \frac{\zeta^2(\lambda, \theta_0)}{\tau^2(\lambda, \theta_0)}\right) \text{ as } n \to \infty$$

provided (B) holds and the conditions (5.0) to (5.2) are satisfied. Observe that a necessary condition for (5.1) to hold is that $\theta_1 = \theta_0$.

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