Jour. Ind. Soc. Ag. Statistics
50(2), 1997 : 130–134

A Note on the Efficiencies of Three Product-Type Estimators Under a Linear Model

Arun K. Singh and Housila P. Singh[1]
Nagaland University, Medziphema-797106
(Received : May, 1995)

SUMMARY

The relative efficiencies of three product-type estimators of population mean are evaluated under Durbin's model. Results are exact for any sample size.

Key words: Product-type estimator, Linear model, Simple random sample, Bias, Mean square error, Relative efficiencies.

1. *Introduction*

In sample surveys the product method of estimation is often used for estimating population mean of the study variate y utilizing an auxiliary variate x that is negatively correlated with y. Suppose (y_i, x_i) (i = 1, 2, ..., n) denotes simple random sample of size n from a bivariate infinite population with mean $(\bar{Y}, \bar{X})$. Let $(\bar{y}, \bar{x})$ be respectively the sample mean estimators of $(\bar{Y}, \bar{X})$. When the population mean $\bar{X}$ of x is known, the classical product estimator for $\bar{Y}$ is given by

$$\bar{y}_p = \bar{y}\,(\bar{x}/\bar{X}) \tag{1.1}$$

which is due to Robson [4] and Murthy [3]. It is well known result that $\bar{y}_p$ will estimate $\bar{Y}$ in large samples more precisely than sample mean $\bar{y}$ if $\rho < -C_x/(2C_y)$ where ρ is the correlation coefficient between y, x ; C_y and C_x are coefficients of variation of y and x respectively.

Robson [4] attempted to make $\bar{y}_p$ unbiased and proposed an estimator for $\bar{Y}$ as

$$t_1 = \bar{y}_p - (1/n)\, s_{yx}/\bar{X}\,) \tag{1.2}$$

1 Vikram University, Ujjain-456010

where $s_{yx} = \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})/(n-1)$

Recently, Dubey [1] suggested another product-type estimator for $\bar{Y}$ as

$$t_2 = \bar{y}_p - (1/n)(s_{yx}/\bar{x}) \tag{1.3}$$

In this paper, an attempt has been made to investigate the exact efficiencies of the estimators $\bar{y}_p$, t_1 and t_2 under Durbin's [2] model where the relation between y and x is of the form

$$y_i = \alpha + \beta x_i + e_i \ ; \ \beta < 0$$

with

$$E(e_i/x_i) = 0$$

$$E(e_i e_j/x_i x_j) = 0 \quad \text{for } i \neq j \tag{1.4}$$

$$V(e_i/x_i) = n\delta \qquad (\delta \text{ is a constant of order } n^{-1})$$

where the variate x_i/n have the gamma distribution with the parameter $m = nh$.

2. *Biases and Mean Square Errors (MSE's)*

Under model (1.4) we find the exact biases of $\bar{y}, \bar{y}_p, t_1$ and t_2 respectively as

$$B(\bar{y}) = 0,\ B(\bar{y}_p) = \beta,\ B(t_1) = 0 \text{ and } B(t_2) = \beta/(m+1) \tag{2.1}$$

It is obvious from (2.1) that the estimators $\bar{y}$ and Robson's estimator t_1 are model unbiased. The estimator t_2 suggested by Dubey [1] is less biased than that of Robson [4] and Murthy [3] estimator $\bar{y}_p$.

It follows from (2.1) that

$$0 = B(\bar{y}) = B(t_1) < B(t_2) < B(\bar{y}_p) \tag{2.2}$$

The MSEs or $\bar{y}_p$, t_2 and the variances of $\bar{y}$ and t_1 under model (1.4) are respectively given by

$$MSE(\bar{y}_p) = (1/m)\,[\alpha^2 + (4m^2 + 11m + 6)\beta^2 + 4(m+1)\alpha\beta + (m+1)\delta] \tag{2.3}$$

$$\mathrm{MSE}(t_2) = (\alpha^2/m) + \left\{ \frac{(m+1)(m+2)(m+3)}{m} + \frac{m(m(n+1)+6(n-1))}{(n-1)(m+2)(m+3)} \right.$$

$$\left. + \frac{2m^2}{(m+1)} - (m+2)^2 \right\} \beta^2 + 2\left\{ \frac{(m+1)(m+2)}{m} - \frac{(m^2+2m+2)}{(m+1)} \right\} \alpha\beta$$

$$+ \left\{ \frac{(m+1)}{m} + \frac{1}{(n-1)(m+1)} \right\} \delta \qquad (2.4)$$

$$V(t_1) = \left[(\alpha^2/m) + \frac{\{4m(n-1)+2n\}}{(n-1)} \beta^2 + 4\alpha\beta + \frac{\{m(n-1)+n}{m(n-1)} \delta \right] \qquad (2.5)$$

$$V(\bar{y}) = (\beta^2 m + \delta) \qquad (2.6)$$

We note that in terms of the model (1.4)

$$\alpha = \bar{Y}\,[(K-\rho)/K]$$

$$\beta = \bar{Y}[\rho/(Km)]$$

$$\delta = \bar{Y}^2[(1-\rho^2)/(K^2 m)] \qquad (2.7)$$

and

$$K = C_x/C_y$$

The exact efficiencies of $\bar{y}_p, t_1, t_2$ and $\bar{y}$, relative to that of $\bar{y}$ are given by

$$E_p = V(\bar{y})/\mathrm{MSE}(\bar{y}_p)$$

$$E_1 = V(\bar{y})/V(t_1)$$

$$E_2 = V(\bar{y})/\mathrm{MSE}(t_2) \qquad (2.8)$$

Now, using (2.3) to (2.6) and substituting the values of α, β and δ given by (2.7) efficiencies E_p, E_1 and E_2 can be expressed explicitly as functions of $K = C_x/C_y$, $m = nh$, ρ.

Since the expressions for the relative efficiencies are complex, we evaluated these quantities E_p, E_1 and E_2 (percentage) for fixed $h = 1$ and selected values of n, ρ, K and presented in Table 1.1. We conclude from Table 1.1 that Dubey's [1] estimator t_2 is superior to conventional unbiased estimator $\bar{y}$, classical product estimator $\bar{y}_p$, Robson's [4] unbiased estimator t_1 in the situations where $5 \le n \le 20$, $\rho = -0.7$ and $K = 1$ (i.e., $C_x = C_y$). For $n > 20$, $\rho = -0.7$ and $K = 1, t_1$ and t_2 are almost equally efficient. However,

Table 1.1 : Percentage relative efficiencies of the estimators $\bar{y}_p$, t_1 and t_2 with respect to sample mean $\bar{y}$

n = 5, m = 5

K	$\rho = -0.4$			$\rho = -0.5$			$\rho = -0.7$			$\rho = -0.9$		
	E_p	E_1	E_2	E_p	E_1	E_2	E_p	E_1	E_2	E_p	E_1	E_2
0.25	82	87	87	79	89	83	68	92	78	55	94	70
0.50	89	88	88	90	94	92	85	108	96	74	125	94
1.00	76	68	72	86	76	82	106	103	108	118	153	142
2.00	31	27	29	36	30	34	50	39	46	76	54	68

n = 10, m = 10

K	$\rho = -0.4$			$\rho = -0.5$			$\rho = -0.7$			$\rho = -0.9$		
	E_p	E_1	E_2	E_p	E_1	E_2	E_p	E_1	E_2	E_p	E_1	E_2
0.25	97	101	99	97	105	101	94	114	104	87	123	103
0.50	103	103	103	109	113	112	120	140	131	124	182	152
1.00	80	75	78	94	88	86	135	131	135	111	250	244
2.00	30	28	29	35	32	34	49	42	45	77	63	71

n = 20, m = 20

K	$\rho = -0.4$			$\rho = -0.5$			$\rho = -0.7$			$\rho = -0.9$		
	E_p	E_1	E_2	E_p	E_1	E_2	E_p	E_1	E_2	E_p	E_1	E_2
0.25	106	108	107	109	114	112	113	126	121	115	141	129
0.50	110	56	110	121	123	122	146	159	154	180	225	203
1.00	82	79	81	97	94	96	160	157	162	354	338	338
2.00	30	29	29	14	14	14	48	44	46	77	67	71

n = 32, m = 32

K	$\rho = -0.4$			$\rho = -0.5$			$\rho = -0.7$			$\rho = -0.9$		
	E_p	E_1	E_2	E_p	E_1	E_2	E_p	E_1	E_2	E_p	E_1	E_2
0.25	111	111	110	115	119	117	123	133	128	130	151	140
0.50	113	114	113	128	129	128	160	163	158	212	255	231
1.00	70	82	83	100	97	98	161	163	163	408	444	422
2.00	30	29	30	35	34	34	48	44	45	78	72	74

the estimators $\bar{y}_p, t_1, t_2$ are exactly equally efficient for (n = 10, K = 1/2, $\rho \neq -0.4$) and more efficient than $\bar{y}$. It is further observed that the estimator t_1 is preferable over $\bar{y}_p$ and t_2 for $n \geq 10$, $K \leq 0.5$ and $\rho \in [-0.9, -0.5]$. When K = 2 (i.e. the coefficient of variation of C_x is twice the C_y), the performance of all the estimators $\bar{y}_p$, t_1 and t_2 are poor than the sample mean $\bar{y}$ for all values of n and ρ.

ACKNOWLEDGEMENT

Authors are thankful to the referee for his valuable suggestions regarding the improvement of the paper.

REFERENCES

[1] Dubey, V., 1993. An almost unbiased product estimator. *J. Indian Soc. Agric. Statist.*, 45(2), 226-229.

[2] Durbin, J., 1959. A note on the application of Quenouille's method of bias reduction to the estimation of ratios. *Biometrika*, **46**, 477-480.

[3] Murthy, M.N., 1964. Product method of estimation. *Sankhya*, **A26**, 69-74.

[4] Robson, D.S., 1957. Application of multivariate pokykays to the theory of unbiased ratio-type estimators. *J. Amer. Statist. Assoc.*, **52**, 511-522.