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Some Methods of Construction of Generalized Cyclic Row - Column Designs

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SUMMARY

Factorial experiments can be set out in row-column designs using the generalized cyclic methods of construction. Various methods for the construction of such designs are given in a systematic manner.

Key words: Block designs, Confounding, Efficiency factor, Factorial experiments, Generalized cyclic designs, Fractional sets, Generators, Basic contrasts.

1. *Introduction*

Factorial experiments can be set out in $p \times q$ row-column designs. The simplest type of design would be to use a Latin - Square for the v-treatment combinations. The main effects and interaction would then be estimated independently of both row and column parameters. As for large v, the fully orthogonal designs are impractical, so there is a need of row-column design in which the number of rows, columns or both are less than v. Orthogonality must necessarily be sacrificed so that certain treatment comparisons will be confounded with rows or with columns or some with rows and other with columns. A general procedure for constructing factorial designs in various blocking structures, including row-column designs have been given by Patterson [7] and Patterson and Bailey [8]. John and Lewis [5] have considered the use of n-cyclic designs in row-column factorial experiments, but no systematic method of construction was given.

Let us consider a class of incomplete block designs with $v = m_1 m_2 \dots m_n = \prod_{j=1}^{n} m_j$ treatments in b blocks of k units per block. Each treatment is replicated r times. A treatment is denoted by the n-tuple $a_1 a_2 \dots a_n$; where a_i is an integer between 0 and $m_i - 1$ ($i = 1, 2, \dots, n$). A set of generalized cyclic [GC(n)] is generated by the cyclic development of an initial block consisting of k treatment combinations in such a way that the j-th block of the set is given by adding the j-th treatment combination in the initial block. If we can arrange the v treatment combinations in p rows and q columns such that

(a) a single treatment combination is applied to each of the pq cells of the design;

(b) the treatment combinations are replicated the same number of times (hence pq must be a multiple of v); and

(c) the rows and columns form a GC(n) set respectively; then the design is called generalized cyclic row column design.

Orthogonality of these designs may be obtained by using the method of Mukherjee [6].

The information matrix A for the $p \times q$ generalized cyclic row-column designs is given by

$$A = rI_v - \frac{1}{q} N_1 N'_1 - \frac{1}{p} N_2 N'_2 + \frac{r^2}{pq} E_{vv}$$

where N_1 and N_2 are the row-treatment and column-treatment incidence matrices respectively; and E_{vv} is the matrix with every entry one.

The reduced normal equations for estimating the treatment parameter τ is

$$A\hat{\tau} = Q$$

where Q is the vector of adjusted treatment totals. The information available on the factorial effects is provided by the canonical efficiency factors (c.e.f.) of the design given by

$$e_{u_1 u_2 \ldots u_n} = 1 - \frac{1}{pq} \sum_{i_1 = 0}^{m_1 - 1} \cdots \sum_{i_n = 0}^{m_n - 1} \lambda_{i_1 \ldots i_n} \cos\left[2\pi \sum_{j=1}^{n} \frac{u_j i_j}{m_j} \right]$$

for $u_j = 0, \ldots, m_j - 1$; $j = 1, 2, \ldots, n$. Where $\lambda_{i_1, \ldots, i_n}$ is an element from the first row of the respective concurrence matrix (NN′).

Again any contrast which is basic for the row and column components, with c.e.f.'s e_1 and e_2 respectively, gives rise to a c.e.f. for the row-column design is given by $e = e_1 + e_2 - 1$.

2. *Methods of Construction of GC(n) Row-column Designs*

Lemma 2.1: (Dean and Lewis [3]) : Cyclic development of an initial block generates a binary GC(n) with exactly v/d distinct blocks if and only if the initial block can be expressed as $S \oplus R$ where

(i) S is a subgroup of G whose order $d \geq 1$ is a common factor of k and v,

(ii) R is a subgroup of G of size k/d,

(iii) There is no subgroup S^* of G of order greater than d for which the initial block may be expressed as $S^* \oplus R^*$, $R \subset G$.

Theorem 2.1: If rv = pq is not a prime power, p is the number of rows and q is the number of columns where p and q are such that either of them is not divisible by other, r is the number of replications, then a GC(1) row-column design with parameters v, p, q and r can always be constructed as :

$$X = \begin{bmatrix} 0 & p & 2p & \cdots & (q-1)p \\ q & & & & \\ 2q & & & & \\ \cdot & & & & \\ \cdot & & & & \\ (p-1)q & & & & \end{bmatrix}$$

In this design if np = (0, 1, 2, ..., p − 1) mod v then we will consider the elements 1, 2, ... p − 1 if they are not occurred previously in the initial row. Similarly in the initial column if mq = (0, 1, 2, ... q-1) mod v we will consider the elements out of 1, 2, ..., q-1 if they are not occurred previously in the initial column, subject to the condition that the number of treatments common between initial row and initial column should be equal to r.

Proof: By using Lemma 2.1 the initial row of the X matrix provides the GC(1) design of size q and initial column provides GC(1) design of size p. If the p and q are divisible of each other then the condition that the number of common treatments between initial row and initial column will be greater than r and design will not be constructed. If in the initial row and initial column np = 0 mod (v), 1 mod (v), 2 mod (v), ... and mq = 0 mod (v), 1 mod (v), 2 mod (v), ... respectively, then the treatment 0, 1, 2, ..., will not be taken into account since they have already been taken in the first row and first column, the treatments will be considered on that place between 1 to p-1, 2 to p-1, 3 to p-1, ... in the rows and 1 to q-1, 2 to q-1, 3 to q-1, ... in the columns respectively if they have not occurred previously.

As an example consider v = 12 treatments arranged in 4 rows and 6 columns in two replications, the design so obtained is given by:

$$X = \begin{bmatrix} 0 & 4 & 8 & 1 & 5 & 9 \\ 6 & 10 & 2 & 7 & 11 & 3 \\ 1 & 5 & 9 & 2 & 6 & 10 \\ 7 & 11 & 3 & 8 & 0 & 4 \end{bmatrix}$$

The efficiency e_1, considering each row as a block is 0.845. Similarly the efficiency e_2, considering each column as a block is 0.565. Therefore, overall efficiency of the design comes to be 0.41.

Theorem 2.2 : If rv = pq is a prime power and p, q < v then we can not construct GC(1) row-column design.

Proof: By using the method of David and Wolock [2] for finding the fractional sets for prime powers, the fractional sets for the initial row and the initial column do not exist simultaneously. Hence the GC(1) row-column design does not exist.

Lemma 2.2: In single replicate s^2-factorial experiment where s is a prime number then there will be (s-1) generators which will provide (s-1) independent solutions of GC(2) design of size s in which all main effects will be unconfounded.

Proof: In s^2 factorial experiment there will be s^2 generators of type $a_i a_j$; $i, j = 0, 1, 2, \ldots, s-1$. The total number of non-zero generators are $s^2 - 1$. The generators in which $a_i = 0$ or $a_j = 0$ are $2(s-1)$. These $2(s-1)$ generators will give solutions in which main effects are confounded. So the generators which provide main effect unconfounded are $(s-1)^2$.

A generator $a_i a_j$ is said to be independent of $u_i u_j$ if $n(a_i a_j) \pmod{s} \neq (u_i u_j)$ where n is any integer. Out of $(s-1)^2$ generators $(s-1)(s-2)$ generators are such that which give solutions just repetition of the solutions given by remaining $(s-1)$ generators. Hence there are only $(s-1)$ generators which provide $(s-1)$ independent solutions of GC(2) design of size s in which all main effects will be unconfounded.

Theorem 2.3: For $v = s^2$ where s is a prime we can obtain ${}^{s-1}C_2$ solutions for GC(2) row-column (s × s) design in which all main effects are unconfounded in single replication.

Proof: By using Lemma 2.2 we take 2 solutions out of (s − 1) independent solutions. The total number of GC(2) row-column designs will be ${}^{s-1}C_2$ in single replication in which all main effects are unconfounded.

As an example we take $v = 5^2$ in 5×5 factorial experiment. Here s = 5, therefore, the four generators are 11, 12, 13, and 14. Taking generator 11 for the initial column and 12 for the initial row we obtain the GC(2) row-column design as:

00	12	24	31	43
11	23	30	42	04
22	34	41	03	10
33	40	02	14	21
44	01	13	20	32

In this design canonical efficiency factors e_{12}, e_{24}, e_{31} and e_{43} have zero value in the rows while all other components of main effects and interactions have value one. Similarly, in columns, the components e_{14}, e_{23}, e_{32} and e_{41} have value zero each while the components of all main effects and other interactions have efficiency one. So we observe that all main effects in this row-column design are unconfounded while 4 + 4 d.f. of interaction AB are completely confounded.

By taking other combinations of generators the remaining five GC(2) row-column design can be constructed.

Remark: Theorem 2.3 can easily be constructed by taking the elements in the i-th row and j-th column in the (s × s) GC(2) row-column design as

$[(i + j) \bmod s, (t_1 i + t_2 j) \bmod s]$ where $t_1 = 1, 2, ..., s\text{-}2$; $t_2 = 2, 3, ..., s - 1$, $t_1 < t_2$ and $(i, j = 0, 1, 2, ..., s\text{-}1)$.

Lemma 2.3: In s^2 factorial experiment where s is even there will be $(s - d)$ generators which will provide $(\frac{s}{2} - d)$ independent solutions of main effects unconfounded, d being the number of odd integers having common divisors with s other than one.

Proof: Proceeding as in Lemma 2.2, out of $s^2 - 1$ non-zero generators, the generators in which either a_i or a_j equal to zero are $2(s - 1)$. These $2(s - 1)$ generators will give the main effects confounded. In the same way $(\frac{3s^2}{4} + sd - 2s - d^2 + 1)$ generators are such that in which either a_i or a_j or both have a common divisor with s other than 1 and the solutions obtained from these generators will give the main effects partially confounded. In remaining $(\frac{s}{2}-d)^2$ generators $(\frac{s}{2}-d)$ $(\frac{s}{2}-d-1)$ are the generators which give solutions just repetition of the solutions given by $(\frac{s}{2}-d)$ generators. Hence there are only $(\frac{s}{2}-d)$ generators which will provide $(\frac{s}{2}-d)$ independent solutions of main effects unconfounded.

Theorem 2.4: For $v = s^2$ where s is even, we can not construct single replicate GC(2) row-column design in which all main effects are unconfounded.

Proof: By using the method of generators (John and Dean [4]) for $v = s^2$ when s is even, two generators which provide main effects unconfounded in rows and columns simultaneously in single replication do not exist. Hence we can not construct the GC(2) row-column design in single replication in which all main effects are unconfounded.

Theorem 2.5: For $v = (2 \times t^n)^2$ where t is a prime number greater than 2 and n is any positive integer we can obtain $2t^{3n-2}(t-1)(t-2)$ solution for GC(2) row-column $(2t^n \times 4t^n)$ design in two replications in which all main effects are unconfounded.

Proof: By using Lemma 2.3, there are $t^{n-1}(t-1)$ independent solutions from which we can construct GC(2) design in which all main effects are unconfounded in single replications. In the same procedure as in Lemma 2.3, the total number of generators which will provide GC(2) row-column design with each of $t^{n-1}(t-1)$ generators in two replications are $2t^{n-1}(t-2)$. Total number of GC(2) row-column design will be $2t^{n-1}(t-2)$ $t^{n-1}(t-1)$. By interchanging the rows and taking t^n combinations of these $2t^{2n-2}(t-1)(t-2)$ we get $2t^{3n-2}(t-1)(t-2)$ solutions.

Following example will clear for t = 3 and n = 1, the construction of the design for $v = (2 \times 3^1)^2 = 6^2$ treatments in 6×12 row-column factorial experiment. Following Lemma 2.3, here s = 6 and d = 1. Total generators which give independent solutions in single replication are (11, 15) and those which give independent solutions in double replication are (12, 14, 21, 41). The combination which will provide GC(2) row-column design in double replication are (11, 12), (11, 21), (15, 14) and (15, 41). Each combination will give three solutions. One solution with combination (11, 12) is given below:

00	12	24	30	42	54	11	23	35	41	53	05
11	23	35	41	53	05	22	34	40	52	04	10
22	34	40	52	04	10	33	45	51	03	15	21
33	45	51	03	15	21	44	50	02	14	20	32
44	50	02	14	20	32	55	01	13	25	31	43
55	01	13	25	31	43	00	12	24	30	42	54

The other two solutions for this combination are obtained by adjusting 4th and 6th rows in 2nd replication.

Canonical efficiency factors $e_{42}, e_{24}, e_{15}, e_{51}$ and e_{33} have zero value in columns while all main effects and other interaction factors have full efficiency i.e. 1. Similarly in rows the components e_{22} and e_{44} have c.e.f.'s 0.75 each while e_{25} and e_{41} have 0.25 each. All main effects and other interaction factors have c.e.f. equal to 1 each. Therefore, in the row-column design out of 24 degrees of freedom of interaction AB, 5 d.f. are completely confounded while 4 d.f. are partially confounded.

Theorem 2.6: For $v = (t \times 2^n)^2$ where t is a prime number greater than 2, $n > 1$ is any positive integer, we can obtain $2^{3n-3}t(t-1)(t-2)$ solutions for GC(2) row-column $(t2^n \times t2^{n+1})$ design in two replication in which all main effects are unconfounded.

Proof: The theorem can easily be proved on the similar pattern as the Theorem 2.5.

Example: We take $v = (3 \times 2^2)^2 = 12^2$ treatments in 12×24 row-column factorial experiments. Here $s = 12$ and $d = 2$. Total generators which give independent solutions in single replication are (11, 15, 17, 1.11) and those which give independent solutions in double replications are (12, 21, 1.10, 10.1). The combination which will provide GC(2) row-column design in double replication are (11, 12), (11, 21), (15, 1.10), (15, 10.1), (17, 12), (17, 21), (1.11, 1.10), (1.11, 10.1). Each combination will give 6 solutions i.e. a total of $8 \times 6 = 48$ solutions.

Corollary: In Theorem 2.6 when $t = 1$ then the value of d is zero and the total number of solutions will be $(2^{n-1})^3$.

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