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On a Class of Estimators in Linear Regression Model with Multivariate - t Distributed Error

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SUMMARY

The object of this paper is to consider a generalized estimator representing a class of estimators and to find better estimators in the proposed class than the existing ones for the estimation of regression coefficients in linear regression model when the error components have the joint multivariate Student-t distribution. Approximate expressions of the bias, the risk of the proposed generalized estimator and the efficiency (dominance) condition with respect to the risk criterion under a general quadratic loss function over the minimum variance unbiased estimator (MVUE) are obtained. A comparative study among some estimators is also made.

Key words: Linear regression model, Multivariate Student - t distribution, Bias, Risk, Generalized dominance condition.

1. *Introduction*

Consider the linear regression model

$$y = X\beta + u \tag{1.1}$$

where y is a $T \times 1$ vector of T observations on the variable to be explained, X is a $T \times p$ full column rank matrix of T observations on p explanatory non-stochastic variables, β is a non-null $p \times 1$ vector of regression coefficients and u is a $T \times 1$ vector of disturbances having a multivariate Student - t distribution with probability density function (pdf) given by

$$f(u/\gamma, \sigma^2) = p(\gamma)\sigma^{-T}\{\gamma + (u'u)/\sigma^2\}^{-(T+\gamma)/2} \tag{1.2}$$

where $\gamma > 0$, $\sigma > 0$, $u_i \in (-\infty, \infty)$ $(i = 1, 2, ..., T)$ are respectively the degrees of freedom, dispersion parameters of the distribution, the error components and

$$p(\gamma) = \frac{\gamma^{\gamma/2}\,\Gamma\left(\frac{\gamma+T}{2}\right)}{\pi^{T/2}\,\Gamma(\gamma/2)}$$

is the normalizing constant. Further, the error vector u has mean vector $E(u) = 0$ for $\gamma > 1$, variance-covariance matrix

$$E(u\,u') = \gamma(\gamma-2)^{-1}\sigma^2 I = \sigma_u^2 I \text{ for } \gamma > 2$$

and the common variance of u_i's is $\sigma_u^2 = \gamma\sigma^2/(\gamma-2)$

The ordinary least squares estimator of β is

$$b = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u \tag{1.3}$$

which is the minimum variance linear unbiased estimator (MVLUE) also.

Let D be a known $p \times p$ positive definite symmetric matrix, $\theta = \beta'X'X\beta$, $(y - Xb)' = \hat{u}'$ and

$$t = \frac{(y-Xb)'(y-Xb)}{b'X'Xb} = \frac{\hat{u}'\hat{u}}{b'X'Xb} = \frac{y'My}{b'X'Xb}$$

where $M = I - X(X'X)^{-1}X'$.

Seeing the forms of the existing estimators {for example, see Singh [2]}, a generalized estimator β_g representing a class of estimators of β is proposed as

$$\hat{\beta}_g = [I - f(t)D]b \tag{1.4}$$

where t has at least first m (≥ 6) moments finite and f(t) satisfying the validity conditions of Taylor's (Maclaurin's) series expansion with its first three derivatives with respect to t being bounded, is a bounded function of t such that $f(0) = 0$, and

$$f(t) = O\left(\frac{1}{\theta}\right) \text{ as } \theta \to \infty$$

For choices of D and f(t) satisfying the above conditions, several existing particular estimators can be obtained. For example, for $D = I$, the following estimators {considered by Singh [2]}

(i) $$\beta^{**} = \left[I - \frac{\gamma(\gamma-2)^{-1}T^{-1}\hat{u}'\hat{u}.I}{y'y - \hat{u}'\hat{u}\{1 - \gamma(\gamma-2)^{-1}T^{-1}\}}\right]b$$

$$= \left[I - \frac{\gamma(\gamma-2)^{-1}T^{-1}\hat{u}'\hat{u}.I}{b'X'Xb + \gamma(\gamma-2)^{-1}T^{-1}\hat{u}'\hat{u}}\right]b$$

$$= \left[I - \frac{\gamma(\gamma-2)^{-1}T^{-1}t.I}{1+\gamma(\gamma-2)^{-1}T^{-1}t} \right] b \tag{1.5}$$

(ii) $$\beta^{***} = \left[I - \frac{(T-p)^{-1}\hat{u}'\hat{u}.I}{y'y - \hat{u}'\hat{u}\{1-(T-p)^{-1}\}} \right] b$$

$$= \left[I - \frac{(T-p)^{-1}\hat{u}'\hat{u}.I}{b'X'Xb + (T-p)^{-1}\hat{u}'\hat{u}} \right] b$$

$$= \left[I - \frac{(T-p)^{-1}t.I}{1+(T-p)^{-1}t} \right] b \tag{1.6}$$

and for $k_1 (> 0)$, $k_2 (\leq 1)$ being the characterising scalars,

(iii) $$\tilde{\beta} = \left[I - \frac{k_1 \hat{u}'\hat{u}.I}{y'y - k_2 \hat{u}'\hat{u}} \right] b$$

$$= \left[I - \frac{k_1 t.I}{1+(1-k_2)t} \right] b \tag{1.7}$$

with the particular forms of f(t) given by (i) $\gamma(\gamma-2)^{-1}T^{-1}t$, (ii) $\frac{(T-p)^{-1}t}{1+(T-p)^{-1}t}$ and (iii) $\frac{k_1 t}{1+(1-k_2)t}$ respectively (each having the value zero at $t = 0$), belong to the proposed class $\hat{\beta}_g$ of estimators.

Some more estimators belonging to the class $\hat{\beta}_g$ are

$$\hat{\beta}_{g_1} = \left[I - k_1 \gamma(\gamma-2)^{-1}T^{-1}\{(1+t)^{k_2}-1\}D \right] b \tag{1.8}$$

$$\hat{\beta}_{g_2} = \left[I - \frac{k_1 \{(1+t)^{k_3}-1\}D}{1+(1-k_2)t} \right] b \tag{1.9}$$

$$\hat{\beta}_{g_3} = \left[I - k_1 \{(1+k_3 t)^{k_2}-1\}D \right] b \tag{1.10}$$

where k_1, k_2 and k_3 are the characterising scalars to be chosen suitably satisfying the regularity conditions of $\hat{\beta}_g$.

The approximate expressions for the bias and the risk $\rho(\hat{\beta}_g) = E(\hat{\beta}_g - \beta)' Q(\hat{\beta}_g - \beta)$ where Q is a positive definite symmetric matrix are given in the Appendix.

2. *Some Remarks*

(a) It may be easily seen that the results obtained by various authors are the special cases of the present study. For example, for the estimator

$$\tilde{\beta} = \left[I - \frac{k_1 t I}{1 + (1 - k_2)t} \right] b \qquad \text{by Singh [2]}$$

we have D = I, $f'(0) = k_1$ and $f''(0) = -2\,k_1(1 - k_2)$ which when substituted in the expression for bias $(\hat{\beta}_g)$ of the Theorem 1 of the Appendix, gives the same expression

$$-\frac{k_1(T-p)\gamma^2}{\theta(\gamma-2)}\left[1 + \frac{\{k_2(T-p+2) - T\}\gamma\sigma^2}{\theta(\gamma-4)}\right]\beta$$

of bias $(\tilde{\beta})$ obtained by Singh [2]. Also, the risk $\rho(\tilde{\beta})$ obtained by Singh [2] for the estimator $\tilde{\beta}$ may be easily seen to be the special case of the risk expression $\rho(\hat{\beta}_g)$ of the proposed generalized estimator $\hat{\beta}_g$ for D = I, $f'(0) = k_1$ and $f''(0) = -2k_1(1 - k_2)$.

(b) We know that the risk associated with OLS estimator b is

$$\rho(b) = \frac{\gamma\sigma^2}{(\gamma-2)} \operatorname{tr} Q^*$$

$$= \frac{\gamma\sigma^2}{(\gamma-2)} \operatorname{tr} \{(X'X)^{-1} Q\} \tag{2.1}$$

and ignoring terms that are $O\{\theta^{-2} \operatorname{tr}(X'X)^{-1}\}$, $O(\theta^{-3}\beta' Q D \beta)$ and $O(\theta^{-3}\beta' D' Q D \beta)$, for $\gamma > 4$ from Theorem 2 of the Appendix, we have

$$\rho(\hat{\beta}_g) = \frac{\gamma\sigma^2}{(\gamma-2)} \operatorname{tr} Q^* + \left[\frac{(T-p)\gamma^2\sigma^4 f'(0)}{(\gamma-2)(\gamma-4)\theta}\right] \times \left[-2 \operatorname{tr} Q^* + \frac{\{(T-p+2)f'(0)\beta' D' Q D \beta + 4\beta' Q D \beta\}}{\theta}\right] \tag{2.2}$$

From (2.1) and (2.2), in the sense of having $\rho(\hat{\beta}_g) < \rho(b)$, the dominance condition of $\hat{\beta}_g$ over b is

$$0 < + f'(0) \leq 2(d - \frac{2\beta' QD\,\beta}{B' D' QD\,\beta})\,(T - p + 2)^{-1} \tag{2.3}$$

where $d = \frac{tr\,Q^*}{\beta' D' QD\beta/\theta} = \frac{tr\{(X'X)^{-1}Q\}}{\beta' D' QD\beta/\theta} > 2\,\frac{\beta' QD\beta}{\beta' D' QD\beta}$

For D = I, the dominance condition (2.3) of $\hat{\beta}_g$ over b, reduces to

$$0 < f'(0) \leq 2(d^* - 2)(T - p + 2)^{-1} \tag{2.4}$$

where $d^* = \frac{tr\,Q^*}{\beta' Q\beta/\theta} = \frac{tr\{(X'X)^{-1}Q\}}{\beta' Q\beta/\theta} > 2$

It may be mentioned here that the dominance conditions of various estimators over b may be easily seen to be special cases of the dominance condition given by (2.3). For example, for the estimator $\tilde{\beta}$ (belonging to the class $\hat{\beta}_g$ of estimators), we have D = I, $f'(0) = k_1$ which when substituted in (2.3) or (2.4) gives, as the special case, the same dominance condition

$$0 < k_1 \leq 2(d^* - 2)(t - p + 2)^{-1} \tag{2.5}$$

for $\tilde{\beta}$ to be better than b as obtained by Singh [2].

(c) We can see that the class $\hat{\beta}_g$ of estimators contains some better estimators than the existing ones. For example, for the estimator

$$\hat{\beta}_{g_3} = \left[I - k_1\, \{(1 + k_3 t)^{k_2} - 1\} D \right] b$$

we have $f'(0) = k_1 k_2 k_3$ which after substituting in (2.3) or (2.4) and D = I, gives the dominance condition

$$0 < k_1 k_2 k_3 \leq 2(d^* - 2)(T - p + 2)^{-1} \tag{2.6}$$

for the estimator $\hat{\beta}_{g_3}$ to be better than b in the sense of having $\rho(\hat{\beta}_{g_3}) < \rho(b)$.

From (2.5) and (2.6), for D = I, $0 < k_2 < 1$ and $0 < k_3 < 1$ the range of the dominance condition (2.6) for $\hat{\beta}_{g_3}$ to be better than b is wider than that of the condition (2.5) by Singh [2] for $\tilde{\beta}$ to be better than b; hence for

$0 < k_2 < 1$ and $0 < k_3 < 1$, in the extended range of the dominance condition (2.6) over the dominance condition (2.5), the estimator $\hat{\beta}_{g_3}$ is better than both $\tilde{\beta}$ and b.

Appendix

From Zellner [4], it is known that the multivariate Student - t distribution is a member of the class of p.d.f.'s.

$$p(u/.) = \int p_{MN}(u/\tau)\, p(\tau/.)d\tau \tag{1}$$

where $p_{MN}(u/\tau) = (2\pi\tau^2)^{-T/2} \exp\,(-u'u/2\tau^2)$, $-\infty < u_i < \infty$ and $p(\tau/.)$ with $\tau > 0$, is a proper p.d.f. of τ. In particular, considering $p(\tau/.)$ to be Inverted Gamma (IG) p.d.f.

$$p_{IG}(\tau/\gamma, \sigma^2) = \frac{2}{\tau\Gamma\frac{\gamma}{2}}\left[\frac{\gamma\sigma^2}{2\tau^2}\right]^{\gamma/2} \exp\,(-\gamma\sigma^2/2\tau^2) \tag{2}$$

with γ, σ, τ in $(0, \infty)$, we see that the p.d.f. $p(u/.)$ becomes the p.d.f. $f(u/\gamma, \sigma^2)$ given in (1.2). The error vector u may be regarded as following a multivariate normal distribution with random standard deviation generated from the distribution (2) and given τ with p.d.f. (2), the conditional distribution of u under our regression model is multivariate normal with mean vector 0 and variance - covariance matrix $\tau^2 I$. Thus given τ,

$$\hat{\beta} = \beta + (X'X)^{-1}X'u \tag{3}$$

has a multivariate normal distribution with mean vector β and variance - covariance matrix $(X'X)^{-1}\tau^2 I$ and further it is also independent of $\hat{u}'\hat{u}$ which given τ is distributed as $\tau^2\chi^2_{(T-p)}$.

Let

$$z = \tau^{-1}(X'X)^{1/2}b, \quad \delta = \tau^{-1}(X'X)^{1/2}\beta \text{ and } v = \hat{u}'\hat{u}/\tau^2$$

so that given τ, $z \sim MN(\delta, I)$, $v \sim \chi^2_{(T-p)}$ and z and v are independent.

Also, let $\epsilon = (z - \delta)$, then given τ, ϵ is distributed as multivariate normal with mean vector 0 and variance - covariance matrix I.

Theorem 1: If $\gamma > 4$, then ignoring terms that are $O\left(\frac{1}{2+j}\right)$, θ, $j > 0$,

$$E(\hat{\beta}_g) = \beta - \frac{(T-p)\gamma\sigma^2 f'(0)}{\theta(\gamma-2)}\left[1 - \frac{[(p-2) - \{f''(0)/2f'(0)\}(T-p+2)]\gamma\sigma^2}{\theta(\gamma-4)}\right].D\beta$$

where f'(0) and f''(0) are respectively the first and second derivatives of f(t) with respect to t at the point t = 0.

Proof. We have

$$\hat{\beta}_g = b - f(t) \,.\, Db$$

Expanding f(t) in third order Taylor's series about the point t = 0 and noting that f(0) = 0, we get

$$(\hat{\beta}_g - \beta) = (b-\beta) - \left\{ f(0) + tf'(0) + \frac{t^2}{2!} f''(0) + \frac{t^3}{3!} f'''(t^*) \right\} Db \tag{4}$$

where $f'''(t^*)$ is the third derivative with respect to t at $t^* = \omega t$, $0 < \omega < 1$.

Utilizing $(b - \beta) = \tau(X'X)^{1/2}\epsilon$ with $E(\epsilon) = 0$,

$$t = \frac{\hat{u}'\hat{u}}{b'X'Xb} = \frac{v}{z'z} = \frac{v}{(\epsilon+\delta)'(\epsilon+\delta)} = v(\delta'\delta)^{-1}\left[1 + \frac{\epsilon'\epsilon + 2\delta'\epsilon}{\delta'\delta}\right]^{-1}$$

and taking expectation on both sides, we have

$$\begin{aligned} E(\hat{\beta}_g) = \beta - E\,E_\tau \tau D(X'X)^{-1/2}\Bigg[& v(\delta'\delta)^{-1}\Bigg\{(1 + \frac{\epsilon'\epsilon + 2\delta'\epsilon}{\delta'\delta})^{-1} f'(0) \\ & + \frac{1}{2} v(\delta'\delta)^{-1} (1 + \frac{\epsilon'\epsilon + 2\delta'\epsilon}{\delta'\delta})^{-2} f''(0) \\ & + \frac{v^2}{6} (\delta'\delta)^{-2} (1 + \frac{\epsilon'\epsilon + 2\delta'\epsilon}{\delta'\delta})^{-3} f'''(t^*)\,(\epsilon + \delta)\Bigg] \end{aligned} \tag{5}$$

since $E(.) = EE_\tau(.)$, where E_τ denotes the conditional expectation given τ.

Now, $\delta'\delta = \tau^{-2}\theta$ and considering

$$A_\theta = v(\delta'\delta)^{-1}\left[f'(0)\,\{1 - (\delta'\delta)^{-1}(\epsilon'\epsilon + 2\delta'\epsilon) + 4(\delta'\delta)^{-2}(\delta'\epsilon)^2\} + \frac{f''(0)}{2} v(\delta'\delta)^{-1}\right] \times$$

$$(\epsilon + \delta)$$

$$= (\tau^2/\theta)\, v\left[f'(0)\,\{ 1 - (\tau^2/\theta)(\epsilon'\epsilon + 2\delta'\epsilon) + 4(\tau^4/\theta^2)(\delta'\epsilon)^2 \} + \frac{f''(0)}{2}(\tau^2/\theta)v \right] \times (\epsilon + \delta)$$

to be the approximating polynomial of

$$A = v(\delta'\delta)^{-1}\left\{ f'(0)(1 + \frac{\epsilon'\epsilon + 2\delta'\epsilon}{\delta'\delta})^{-1} + \frac{v}{2}(\delta'\delta)^{-1} f''(0) \right.$$

$$.(1 + \frac{\epsilon'\epsilon + 2\delta'\epsilon}{\delta'\delta})^{-2} + \frac{v^2}{6} f'''(t^*)(\delta'\delta)^{-2}$$

$$\left. .(1 + \frac{\epsilon'\epsilon + 2\delta'\epsilon}{\delta'\delta})^{-3} \right\} . (\epsilon + \delta)$$

we have $\theta^2 \,|\, A - A_\theta \,|\, \xrightarrow{p} 0$ as θ goes to ∞, and $|\, E_\tau(A) - E_\tau(A_\theta) \,| = O\left(\frac{\tau^6}{\theta^{2+j}}\right)$, $j > 0$, and hence, $E(\hat{\beta}_g)$ in (5) can be expressed as

$$E(\hat{\beta}_g) = \beta + EE_\tau \tau D(X'X)^{-1/2} A_\theta + O\left(\frac{1}{\theta^{2+j}}\right),\ j > 0 \qquad (6)$$

Further, we have

$$E_\tau(vz) = (T - p)\delta$$

$$E_\tau(v\epsilon'\epsilon z) = (T - p).p.\delta$$

$$E_\tau(\delta'\epsilon vz) = (T - p)\,\delta$$

$$E_\tau(v(\delta'\epsilon)^2 z) = (T - p)(\delta'\delta)\delta$$

$$E_\tau(v^2 z) = (T - p)(T - p + 2)\delta$$

Substituting these results in (6), we get

$$E(\hat{\beta}_g) = \beta - E\, E_\tau \tau D(X'X)^{-1/2} A_\theta$$

$$= \beta - E\left\{ \tau D(X'X)^{-1/2}\left(\frac{\tau^2}{\theta}\right)\left[f'(0)\left[(T - p)\delta - \left(\frac{\tau^2}{\theta}\right)\{(T - p).p.\delta + 2(T - p)\delta\} \right.\right.\right.$$

$$\left.\left.\left. + 4(\tau^4/\theta^2)(T - p)(\delta'\delta)\delta \right] + \frac{f''(0)}{2}(\tau^2/\theta)(T - p)(T - p + 2)\delta \right]\right\}$$

$$= \beta - E\left\{ \frac{(T-p)}{\theta} f'(0) \left[\tau^2 - \frac{\tau^4}{\theta} \left\{ (p-2) - \frac{f''(0)}{2f'(0)} (T-p+2) \right\} \right] . D\beta \right\}$$

Using $E(\tau^2) = \dfrac{\gamma\sigma^2}{(\gamma-2)}$ for $\gamma > 2$ and

$$E(\tau^4) = \frac{\gamma^2 \sigma^4}{(\gamma-2)(\gamma-4)} \text{ for } r > 4$$

{from Zellner ([3], pp. 371-372)} for $\gamma > \gamma_0$ (large so that error bound becomes uniform in τ), we get

$$E(\hat{\beta}_g) = \beta - \frac{(T-p)\gamma\sigma^2 f'(0)}{\theta(\gamma-2)} \left[1 - \frac{\{(p-2) - (f''(0)/2f'(0))(T-p+2)\}}{\theta(\gamma-4)} . \gamma\sigma^2 \right] D\beta \tag{7}$$

which gives the required expression in Theorem 1.

Next we proceed to find the risk $\rho(\hat{\beta}_g)$ for $\hat{\beta}_g$.

Theorem 2. For $\gamma > 6$ and positive definite symmetric matrix Q, ignoring terms that are $O\left(\dfrac{1}{\theta^3}\right)$, the risk $\rho(\hat{\beta}_g)$ is given by

$$\rho(\hat{\beta}_g) = E(\hat{\beta}_g - \beta)' Q(\hat{\beta}_g - \beta)$$

$$= \frac{\gamma\sigma^2}{(\gamma-2)} \operatorname{tr} Q^* + \frac{\gamma^2\sigma^4}{(\gamma-2)(\gamma-4)\theta} f'(0)(T-p) \Bigg[-2\operatorname{tr} Q^*$$

$$+ \frac{\gamma\sigma^2}{(\gamma-6)\theta} \Bigg\{ (T-p+2)f'(0) \operatorname{tr} Q^{***} + 2(p-2) \operatorname{tr} Q^{**}$$

$$- (T-p+2)\frac{f''(0)}{f'(0)} \operatorname{tr} Q^{**} \Bigg\} + \frac{1}{\theta}\Big\{(T-p+2)f'(0)\beta'D'QD\beta + 4\beta'QD\beta\Big\}$$

$$- \frac{8\gamma\sigma^2}{(\gamma-6)\theta^2} \Bigg\{ (T-p+2)f'(0)\beta'D'QD\beta + (p-2)\beta'QD\beta$$

$$- \frac{(T-p+2)}{2} \frac{f''(0)}{f'(0)} \beta'QD\beta \Bigg\} \Bigg]$$

where
$$Q^* = (X'X)^{-1/2} Q (X'X)^{-1/2}$$
$$Q^{**} = (X'X)^{-1/2} Q D (X'X)^{-1/2}$$
$$Q^{***} = (X'X)^{-1/2} D'QD(X'X)^{-1/2}$$

Proof. We have

$$\rho(\hat{\beta}_g) = E(\hat{\beta}_g - \beta)'Q(\hat{\beta}_g - \beta)$$
$$= EE_T\tau^2\left[(z-\delta)'(X'X)^{-1/2} - f(t)z'(X'X)^{-1/2}D'\right]Q$$
$$\cdot\left[(X'X)^{-1/2}(z-\delta) - f(t)D(X'X)^{-1/2}z\right]$$
$$= EE_T\tau^2\left[(z-\delta)'Q^*(z-\delta) - 2f(t)(z-\delta)'Q^{**}z + \{f(t)\}^2 z'Q^{***}z\right] \quad (8)$$

Expanding f(t) in third order Taylor's series about the point t = 0 and noting that f(0) = 0, we have

$$E_T(\hat{\beta}_g - \beta)'Q(\hat{\beta}_g - \beta)$$
$$= E_T\tau^2\Bigg[\in'Q^*\in - 2v(\delta'\delta)^{-1}\Bigg\{(1+\frac{\in'\in + 2\delta'\in}{\delta'\delta})^{-1} f'(0)$$
$$+\frac{1}{2}v(\delta'\delta)^{-1}(1+\frac{\in'\in + 2\delta'\in}{\delta'\delta})^{-2} f''(0)$$
$$+\frac{v^2}{6}(\delta'\delta)^{-2}(1+\frac{\in'\in + 2\delta'\in}{\delta'\delta})^{-3} f'''(t^*)\Bigg\} . \in'Q^{**}(\in + \delta)$$
$$+\Bigg\{v(\delta'\delta)^{-1}(1+\frac{\in'\in + 2\delta'\in}{\delta'\delta})^{-1} f'(0)$$
$$+\frac{1}{2}v^2(\delta'\delta)^{-2}(1+\frac{\in'\in + 2\delta'\in}{\delta'\delta})^{-2} f''(0)$$
$$+\frac{v^3}{6}(\delta'\delta)^{-3}(1+\frac{\in'\in + 2\delta'\in}{\delta'\delta})^{-3} f'''(t^*)\Bigg\}^2$$
$$.(\in' + \delta')Q^{***}(\in + \delta)\Bigg] \quad (9)$$

Considering

$$B_\theta = -2f'(0)\,\tau^2\Bigg[\, v(\delta'\delta)^{-1} - v(\delta'\delta)^{-2}\left\{(2\delta'\epsilon + \epsilon'\epsilon) + \frac{v}{2}\frac{f''(0)}{f'(0)}\right\}$$

$$+\, 4v(\delta'\delta)^{-3}\left\{((\delta'\epsilon)^2 + (\delta'\epsilon)(\epsilon'\epsilon)) - \frac{v}{2}(\delta'\epsilon)\frac{f''(0)}{f'(0)}\right\}$$

$$-\, 8v(\delta'\delta)^{-4}(\delta'\epsilon)^3\Bigg]\{\epsilon' Q^{**}(\epsilon + \delta)\}$$

$$= -2f'(0)\frac{\tau^4}{\theta}\Bigg[\, v - v\frac{\tau^2}{\theta}\left\{(2\delta'\epsilon + \epsilon'\epsilon) + \frac{v}{2}\frac{f''(0)}{f'(0)}\right\}$$

$$+\, 4v\frac{\tau^4}{\theta^2}\left\{((\delta'\epsilon)^2 + (\delta'\epsilon)(\epsilon'\epsilon)) - \frac{v}{2}(\delta'\epsilon)\frac{f''(0)}{f'(0)}\right\}$$

$$-\, 8v\frac{\tau^6}{\theta^3}(\delta'\epsilon)^3\Bigg](\epsilon' Q^{**}\epsilon + \epsilon' Q^{**}\delta)$$

and

$$B^*_\theta = \tau^2 v^2(\delta'\delta)^{-2}\{f'(0)\}^2\left[1 - 4(\delta'\delta)^{-1}(\delta'\epsilon)\right](\epsilon' + \delta')Q^{***}(\epsilon + \delta)$$

$$= \frac{\tau^6}{\theta^2}(f'(0))^2\left[\, v^2 - \frac{4\tau^2}{\theta}v^2(\delta'\epsilon)\right](\epsilon' Q^{***}\epsilon + 2\epsilon' Q^{***}\delta + \delta' Q^{***}\delta)$$

to be the approximating polynomials for second and third terms of (8) respectively, we see that

$$\theta^{5/2}\,|-2f(t)\epsilon' Q^{**}z - B_\theta| \xrightarrow{\ p\ } 0$$

and $$\theta^{5/2}\,|\{f(t)\}^2 z' Q^{***}z - B^*_\theta| \xrightarrow{\ p\ } 0 \text{ as } \theta \text{ goes to } \infty$$

Also

$$|\,E_\tau(-2f(t)\epsilon' Q^{**}z) - E_\tau(B_\theta)\,| = O\left(\frac{1}{\theta^3}\right)$$

and $$|\,E_\tau(f(t))^2 z' Q^{***}z - E_\tau(B^*_\theta)\,| = O\left(\frac{1}{\theta^3}\right)$$

and hence, the risk $\rho(\hat{\beta}_g)$ of the generalized estimator $\hat{\beta}_g$ may be expressed as

$$\rho(\hat{\beta}_g) = EE_\tau \tau^2(\epsilon' Q^* \epsilon) + EE_\tau(\beta_\theta + \beta_\theta^*) + O\left(\frac{1}{\theta^3}\right) \tag{10}$$

Also, we have the following results

$$\begin{aligned}
E(\epsilon' Q^{**} \epsilon) &= \mathrm{Tr}\, Q^{**} \\
E(\delta'\epsilon \,.\, \epsilon' Q^{**} \epsilon) &= 0 \\
E(\epsilon'\epsilon \,.\, \epsilon' Q^{**} \epsilon) &= (p+2)\, \mathrm{tr}\, Q^{**} \\
E((\delta'\epsilon)^2\, \epsilon' Q^{**} \epsilon) &= (\mathrm{tr} Q^{**})\delta'\delta + 2\delta' Q^{**}\delta \\
E(\delta'\epsilon)(\epsilon'\epsilon)(\epsilon' Q^{**} \epsilon) &= 0 \\
E(\delta'\epsilon)^2\, (\epsilon' Q^{**} \delta) &= 0 \\
E(\delta'\epsilon)\, (\delta' Q^{**} \delta) &= 0 \\
E(\delta'\epsilon)\, (\epsilon'\epsilon)(\epsilon' Q^{**} \delta) &= (p+2)\delta' Q^{**} \delta \\
E(\delta'\epsilon)^3 (\epsilon' Q^{**} \delta) &= 3(\delta'\delta)\delta' Q^{**} \delta \\
E(\delta'\epsilon)^3\, (\epsilon' Q^{**} \epsilon) &= 0
\end{aligned}$$

Substituting these expected values in $E_\tau(B_\theta)$ and $E_\tau(B_\theta^*)$ and simplifying, we have

$$\begin{aligned}
E_\tau(B_\theta + B_\theta^*) = \frac{\tau^4}{\theta} f'(0)(T-p)\Bigg[-2\, \mathrm{tr}\, Q^* + \frac{\tau^2}{\theta}\Bigg\{ &(T-p+2) f'(0)\, \mathrm{tr}\, Q^{***} \\
&+ 2(p-2)\, \mathrm{tr}\, Q^{**} - (T-p+2)\frac{f''(0)}{f'(0)}\, \mathrm{tr}\, Q^{**} \Bigg\} \\
&+ \frac{\tau^2}{\theta}\Big\{ (T-p+2) f'(0)\, \delta' Q^{***} \delta + 4\delta' Q^{**} \delta \Big\} \\
&- \frac{8\tau^4}{\theta^2}\Bigg\{ (T-p+2) f'(0)\, \delta' Q^{***} \delta + (p-2)\delta' Q^{**} \delta \\
&- \frac{(T-p+2)}{2}\frac{f''(0)}{f'(0)}\, \delta' Q^{**} \delta \Bigg\} \Bigg]
\end{aligned}$$

$$= \frac{\tau^4}{\theta} f'(0)(T-p)\Bigg[-2\,\mathrm{tr}\,Q^* + \frac{\tau^2}{\theta}\Bigg\{(T-p+2)f'(0)\,\mathrm{tr}\,Q^{***} + 2(p-2)\,\mathrm{tr}\,Q^{**}$$

$$-(T-p+2)\frac{f''(0)}{f'(0)}\,\mathrm{tr}\,Q^{**}\Bigg\} + \frac{\{(T-p+2)f'(0)\,\beta'D'QD\beta + 4\beta'QD\beta\}}{\theta}$$

$$-\frac{8\tau^2}{\theta^2}\Bigg\{(T-p+2)f'(0)\beta'D'QD\beta + (p-2)\beta'QD\beta - \frac{(T-p+2)}{2}\frac{f''(0)}{f'(0)}\beta'QD\beta\Bigg\}\Bigg] \tag{11}$$

(since $\delta'Q^{**}\delta = \tau^{-2}\beta'QD\beta$ and $\delta'Q^{***}\delta = \tau^{-2}\beta'D'QD\beta$)

Substituting the expressions for $E_\tau(B_\theta + B_\theta^*)$ from (11) in (10) and noting that

$$E(\in'Q^*\in) = \mathrm{tr}\,Q^*$$

$$E(\tau^2) = \frac{\gamma\sigma^2}{(\gamma-2)}$$

$$E(\tau^4) = \frac{\gamma^2\sigma^4}{(\gamma-2)(\gamma-4)}$$

$$E(\tau^6) = \frac{\gamma^3\sigma^6}{(\gamma-2)(\gamma-4)(\gamma-6)},$$ we have

$$\rho(\hat{\beta}_g) = \frac{\gamma\sigma^2}{(\gamma-2)}\,\mathrm{tr}\,Q^* + \frac{\gamma^2\sigma^4}{(\gamma-2)(\gamma-4)\,\theta} f'(0)(T-p)\Bigg[-2\mathrm{tr}\,Q^*$$

$$+\frac{\gamma\sigma^2}{(\gamma-6)\theta}\Bigg\{(T-p+2)f'(0)\,\mathrm{tr}Q^{***} + 2(p-2)\,\mathrm{tr}\,Q^{**}$$

$$-(T-p+2)\frac{f''(0)}{f'(0)}\,\mathrm{tr}\,Q^{**}\Bigg\} + \frac{1}{\theta}\Bigg\{(T-p+2)f'(0)$$

$$.\,\beta'D'QD\beta + 4\beta'QD\beta\Bigg\} - \frac{8\gamma\sigma^2}{(\gamma-6)\theta^2}\Bigg\{(T-p+2)f'(0)\beta'D'QD\beta$$

$$+(p-2)\beta'QD\beta - \frac{(T-p+2)}{2}\frac{f''(0)}{f'(0)}\beta QD\beta\Bigg\}\Bigg]$$

which gives the required result for the risk $\rho(\hat{\beta}_g)$ in Theorem 2.

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