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Estimation of Location and Scale Parameters of U-shaped Distribution

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SUMMARY

In this paper, we obtain explicit expressions for the single and product moments of order statistics arising from the standard U-shaped distribution. Some recurrence relations for moments are also derived. Best linear unbiased estimators of location and scale parameters of the U-shaped distribution based on order statistics are obtained.

Key words: Order statistics, Single and product moments, Recurrence relations, Best linear unbiased estimation.

1. Introduction

The probability density function of U-shaped distribution (see, Sarhan and Greenberg [7], p. 391) is given by

$$g(z) = \begin{cases} [3(z-\theta_1)^2]/2\theta_2^3 & \theta_1 - \theta_2 \le z \le \theta_1 + \theta_2 \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

where $-\infty < \theta_1 < \infty$ and $\theta_2 > 0$. Some applications of the U-shaped distribution have been pointed out in some Meteriological studies (for example, see, Bailey [1], p. 14). Irwin ([3], p. 162) has discussed certain applications of the U-shaped distribution in some epidemics studies.

If Z is a random variable whose distribution is as given in (1.1), then the probability density function of the random variable $x = (z - \theta_1)/\theta_2$ is given by

$$f(x) = \begin{cases} (3/2)x^2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

Clearly, f(x) is symmetric about zero and is the density function of the standard form of the distribution defined in (1.1). The distribution function corresponding to the density function (1.2) is given by

$$F(x) = \begin{cases} 0 & x < -1 \\ (x^3+1)/2 & -1 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Obviously, in this case, f(x) and F(x) are functionally connected by the following relation

$$xf(x) = (3/2) - 3[1 - F(x)] \quad -1 \le x \le 1 \tag{1.3}$$

Moments of order statistics of a random sample drawn from (1.2) are necessary to obtain Llyod's [4] Best Linear Unbiased Estimators (BLUE) of θ_1 and θ_2 involved in (1.1). Sarhan [6] has given the coefficients of order statistics in the BLUE of the parameters involved in (1.1) based on random samples of size $n \le 5$. However, he has not given the exact expressions or the recurrence relations for the moments of order statistics arising from (1.2) so as to obtain the BLUE of θ_1 and θ_2 for $n > 5$. Moreover, estimation of θ_1 and θ_2 for $n > 5$ is important due to the application of the U-shaped distribution in practical problems as mentioned by Bailey [1] and Irwin [3] and in similar other problems which may arise elsewhere. Hence, the aim of this paper is to obtain the BLUE of the location parameter θ_1 and the scale parameter θ_2 involved in (1.1) based on random samples of size $n \le 20$.

To obtain the BLUE of θ_1 and θ_2, first we derive explicit expressions for the single and product moments of order statistics arising from (1.2). These results are given in Section 2. Recurrence relations on the moments of order statistics arising from a distribution facilitate quick evaluation of the values of the moments. So, we devote Section 3 to derive some recurrence relations for the single and product moments of order statistics arising from (1.2). Finally, in Section 4, we derive the BLUE of θ_1 and θ_2 and their variances based on order statistics of random samples of size n, for n = 2 (1) 20.

2. *Explicit Expressions for the Moments*

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the order statistics of a random sample of size n drawn from (1.2). Then the moment of order k of the order statistic $X_{r:n}$ for $1 \le r \le n$ is denoted by $\mu_{r:n}^{(k)}$ and is given by

$$\mu_{r:n}^{(k)} = r\binom{n}{r}\int_{-1}^{1} x^k\,[F(x)]^{r-1}\,[1 - F(x)]^{n-r}\,f(x)dx \tag{2.1}$$

We may write $\mu_{r,s:n}$ to denote the product moment of the order statistics $X_{r:n}$ adn $X_{s:n}$ for $1 \le r < s \le n$, $n \ge 2$ and is given by

$$\mu_{r,s:n} = C(r,s:n) \int_{-1}^{1} \int_{-1}^{y} xy\,[F(x)]^{r-1}\,[F(y)-F(x)]^{s-r-1}$$

$$[1-F(Y)]^{n-s}\,f(x)\,f(y)\,dx\,dy \tag{2.2}$$

where, $C(r,s:n) = n!\,/\,[(r-1)!\,(s-r-1)!\,(n-s)!\,]$

Further, we may write $\sigma_{r,s:n}$ to denote the covariance between $X_{r:n}$ and $X_{s:n}$ for $1 \le r \le s \le n$.

We can easily see that the distribution obtained by folding the distribution given in (1.2) about zero has the following density function

$$h(y) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Let $Y_{1:n} \le Y_{2:n} \le \ldots \le Y_{n:n}$ be the order statistics of a random sample of size n drawn from (2.3). Let

$$v_{r:n}^{(k)} = E(Y_{r:n}^{k}) \qquad 1 \le r \le n$$

and

$$v_{r,s:n} = E(Y_{r:n}\,Y_{s:n}) \quad 1 \le r < s \le n$$

Then, using the results of Govindarajulu [2], we get the following results connecting the moments of order statistics arising from (1.2) and (2.3).

For positive integers n and k and for $1 \le r \le n$,

$$2^n \mu_{r:n}^{(k)} = \sum_{j=0}^{r-1} \binom{n}{j} v_{r-j:n-j}^{(k)} + (-1)^k \sum_{j=r}^{n} \binom{n}{j} v_{j-r+1:j}^{(k)} \tag{2.4}$$

For $n \ge 2$ and for integers r and s such that $1 \le r < s \le n$

$$2^n \mu_{r,s:n} = \sum_{j=0}^{r-1} \binom{n}{j} v_{r-j,s-j:n-j} + \sum_{j=s}^{n} \binom{n}{j} v_{j+1-s,j+1-r:j}$$

$$-\sum_{j=r}^{s-1} \binom{n}{j} v_{j-r+1:j}\, v_{s-j:n-j} \tag{2.5}$$

Malik [5] has derived the exact expressions for the moments of order statistics arising from (2.3) and those are given below:

$$v_{r:n}^{(k)} = \frac{\Gamma(n+1)\Gamma(r+(k/3))}{\Gamma(r)\Gamma(n+1+(k/3))} \tag{2.6}$$

$$v_{r,s:n} = \frac{\Gamma(n+1)\Gamma(r+(1/3))\Gamma(s+(2/3))}{\Gamma(r)\Gamma(s+(1/3))\Gamma(n+1+(2/3))} \tag{2.7}$$

If we use (2.6) in (2.4) and apply (2.6) and (2.7) in (2.5) and simplify the resulting expressions, we get the explicit expressions for the single and product moments of order statistics arising from the standard U-shaped distribution defined in (1.2). Explicitly for $1 \le r \le n$, we have

$$\mu_{r:n}^{(k)} = r\binom{n}{r}2^{-n}\left\{(n-r)!\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{\Gamma(r+(k/3)-j)}{\Gamma(n-j+(k/3)+1)}\right.$$
$$\left.+(-1)^k(r-1)!\sum_{j=r}^{n}\binom{n-r}{j-r}\frac{\Gamma(j+1+(k/3)-r)}{\Gamma(j+(k/3)+1}\right\} \tag{2.8}$$

Further, for $1 \le r < s \le n$

$$\mu_{r,s:n} = \frac{n!}{2^n}\left\{\frac{3}{(r-1)!}\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{\Gamma(r-j+\frac{1}{3})\Gamma(s-j+\frac{2}{3})}{[3(n-j)+2]\Gamma(s-j+\frac{1}{3})\Gamma(n-j+\frac{2}{3})}\right.$$
$$-\frac{1}{(s-r-1)!}\sum_{j=r}^{s-1}\binom{s-r-1}{j-r}\frac{\Gamma(j-r+\frac{4}{3})\Gamma(s-j+\frac{1}{3})}{\Gamma(j+\frac{4}{3})\Gamma(n+1-j+\frac{1}{3})}$$
$$\left.+\frac{3}{(n-s)!}\sum_{j=s}^{n}\binom{n-s}{j-s}\frac{\Gamma(j-s+\frac{4}{3})\Gamma(j+1-r+\frac{2}{3})}{(3j+2)\Gamma(j-r+\frac{4}{3})\Gamma(j+\frac{2}{3})}\right\} \tag{2.9}$$

Exact expressions for $\sigma_{r,r:n}$ and $\sigma_{r,s:n}$ can be obtained from (2.8) and (2.9). Numerical values of $\mu_{r:n}$ accurate to eight decimal places for $r = 1+\left[\frac{n+1}{2}\right], \ldots, n$ and $n = 2\,(1)\,20$ are obtained and presented in Table 1, where [.] is the usual integer function. For other values of $\mu_{r:n}$, we can use the relations

$$\mu_{r:n} = -\mu_{n-r+1:n} \text{ and } \mu_{\left[\frac{n}{2}\right]+1:n} = 0 \text{ when n is odd}$$

Also, numerical values of $\sigma_{r,s:n}$ accurate to eight decimal places for $1 \le r \le s \le n$, $2 \le r+s \le n+1$ and $n = 2(1)\,20$ are given in Table 2. The other values of $\sigma_{r,s:n}$ can be obtained by using the relations

$$\sigma_{r,s:n} = \sigma_{n-s+1,n-r+1:n} \text{ and } \sigma_{r,s:n} = \sigma_{s,r:n} \tag{2.10}$$

3. *Recurrence Relation on the Moments*

Recurrence relations on the moments of order statistics facilitate quick evaluation of the values of the moments of higher orders. Moreover, they are useful to check the accuracy in the computed values of the moments. Thus, we derive some useful recurrence relations for the single and product moments of order statistics arising from (1.2).

Relation 3.1

For $n \ge 2$ and $k \ge 1$,

$$(k+3n)\,\mu_{1:n}^{(k)} = \frac{3n}{2}\left\{\mu_{1:n-1}^{(k)} + (-1)^k\right\} \tag{3.1}$$

$$(k+3n)\,\mu_{n:n}^{(k)} = \frac{3n}{2}\left\{\mu_{n-1:n-1}^{(k)} + 1\right\} \tag{3.2}$$

and for $n \ge 3$ and $2 \le r \le n-1$,

$$(k+3n)\,\mu_{r:n}^{(k)} = \frac{3n}{2}\left\{\mu_{r:n-1}^{(k)} + \mu_{r-1:n-1}^{(k)}\right\} \tag{3.3}$$

Proof.

On using the functional relation given by (1.3) in the expression (2.1) for $\mu_{r:n}^{(k)}$, we have, for $1 \le r \le n$,

$$\mu_{r:n}^{(k)} = (3/2)r\binom{n}{r}\int_{-1}^{1} x^{k-1}\,[F(x)]^{r-1}\,[1-F(x)]^{n-r}\,dx$$

$$-3r\binom{n}{r}\int_{-1}^{1} x^{k-1}\,[F(x)]^{r-1}\,[1-F(x)]^{n-r+1}\,dx \tag{3.4}$$

Table 1 : Expected values of the standard U-shaped order statistics

n	*r*	*Mean*	*n*	*r*	*Mean*	*n*	*r*	*Mean*
2	2	0.42857143	12	11	0.87193710	17	14	0.80324776
3	3	0.64285714	12	12	0.94184906	17	15	0.86458129
4	3	0.29670330	13	8	0.28385655	17	16	0.91527384
4	4	0.75824176	13	9	0.52040367	17	17	0.95974681
5	4	0.49450549	13	10	0.69036716	18	10	0.11805209
5	5	0.82417582	13	11	0.80437703	18	11	0.33841599
6	4	0.23423944	13	12	0.88422075	18	12	0.51919070
6	5	0.62463852	13	13	0.94665141	18	13	0.65348654
6	6	0.86408329	14	8	0.13862762	18	14	0.74898137
7	5	0.40991903	14	9	0.39277824	18	15	0.81875245
7	6	0.71052632	14	10	0.59130668	18	16	0.87374706
7	7	0.88967611	14	11	0.72999135	18	17	0.92046468
8	5	0.19676113	14	12	0.82466403	18	18	0.96205753
8	6	0.53781377	14	13	0.89414687	19	11	0.22429897
8	7	0.76809717	14	14	0.95069023	19	12	0.42141018
8	8	0.90704453	15	9	0.25992678	19	13	0.57622933
9	6	0.35417004	15	10	0.48134589	19	14	0.68914372
9	7	0.62963563	15	11	0.64628708	19	15	0.77035196
9	8	0.80765761	15	12	0.76042926	19	16	0.83165924
9	9	0.91946790	15	13	0.84072273	19	17	0.88163853
10	6	0.17137260	15	14	0.90236597	19	18	0.92503246
10	7	0.47603500	15	15	0.95414196	19	19	0.90411447
10	8	0.69546447	16	9	0.12731108	20	11	0.11031097
10	9	0.83570589	16	10	0.36307233	20	12	0.31756188
10	10	0.92877479	16	11	0.55231002	20	13	0.49064239
11	7	0.31418310	16	12	0.68900392	20	14	0.62231462
11	8	0.56852180	16	13	0.78423771	20	15	0.71778476
11	9	0.74306797	16	14	0.85375773	20	16	0.78787436
11	10	0.85629210	16	15	0.90931001	20	17	0.84260546
11	11	0.93602306	16	16	0.95713076	20	18	0.88852672
12	7	0.15284583	17	10	0.24047648	20	19	0.92908866
12	8	0.42942401	17	11	0.44888942	20	20	0.96595794
12	9	0.63807070	17	12	0.60872126			
12	10	0.77806706	17	13	0.72245503			

Table 2 : Covariances of the standard U-shaped order statistics

n	r	s	Cov	n	r	s	Cov	n	r	s	Cov
2	1	1	0.41632653	7	1	3	0.03532546	9	1	2	0.01350549
2	1	2	0.18367347	7	1	4	0.03010366	9	1	3	0.01723830
3	1	1	0.24128015	7	1	5	0.01872770	9	1	4	0.01925428
3	1	2	0.17532468	7	1	6	0.00834847	9	1	5	0.01726317
3	1	3	0.06261596	7	1	7	0.00257798	9	1	6	0.01195569
3	2	2	0.49090909	7	2	2	0.11914677	9	1	7	0.00640046
4	1	1	0.13416035	7	2	3	0.12446093	9	1	8	0.00275163
4	1	2	0.12567652	7	2	4	0.10089016	9	1	9	0.00093148
4	1	3	0.07469454	7	2	5	0.06076305	9	2	2	0.04762104
4	1	4	0.02391016	7	2	6	0.02678351	9	2	3	0.05739012
4	2	2	0.40287624	7	3	3	0.29337071	9	2	4	0.06018019
4	2	3	0.23831114	7	3	4	0.23670383	9	2	5	0.05148574
5	1	1	0.07474491	7	3	5	0.14009480	9	2	6	0.03466902
5	1	2	0.08165719	7	4	4	0.39548942	9	2	7	0.01838579
5	1	3	0.06459927	8	1	1	0.01581939	9	2	8	0.00797539
5	1	4	0.03356372	8	1	2	0.02051647	9	3	3	0.14633411
5	1	5	0.01022894	8	1	3	0.02473909	9	3	4	0.14922196
5	2	2	0.28487608	8	1	4	0.02461629	9	3	5	0.12329664
5	2	3	0.22099749	8	1	5	0.01879382	9	3	6	0.08089744
5	2	4	0.11193719	8	1	6	0.01072464	9	3	7	0.04234787
5	3	3	0.43315508	8	1	7	0.00463437	9	4	4	0.29187311
6	1	1	0.04266489	8	1	8	0.00149030	9	4	5	0.24188935
6	1	2	0.05128692	8	2	2	0.07503488	9	4	6	0.15678421
6	1	3	0.04919475	8	2	3	0.08537223	9	5	5	0.36821429
6	1	4	0.03410430	8	2	4	0.08005792	10	1	1	0.00731942
6	1	5	0.01614565	8	2	5	0.05864326	10	1	2	0.00924961
6	1	6	0.00488168	8	2	6	0.03277095	10	1	3	0.01211341
6	2	2	0.18736683	8	2	7	0.01416726	10	1	4	0.01468533
6	2	3	0.17328139	8	3	3	0.21170962	10	1	5	0.01494277
6	2	4	0.11571533	8	3	4	0.19529747	10	1	6	0.01212329
6	2	5	0.05357710	8	3	5	0.13949070	10	1	7	0.00773900
6	3	3	0.37828696	8	3	6	0.07647884	10	1	8	0.00397710
6	3	4	0.25376586	8	4	4	0.35677448	10	1	9	0.00173850
7	1	1	0.02533296	8	4	5	0.25721079	10	1	10	0.00062145
7	1	2	0.03215101	9	1	1	0.01045512	10	2	2	0.03088076

n	r	s	Cov	n	r	s	Cov	n	r	s	Cov
10	2	3	0.03850290	11	2	9	0.00665506	12	2	9	0.00845668
10	2	4	0.04392304	11	2	10	0.00304603	12	2	10	0.00430327
10	2	5	0.04258978	11	3	3	0.06627188	12	2	11	0.00205026
10	2	6	0.03347524	11	3	4	0.07700930	12	3	3	0.04466898
10	2	7	0.02102689	11	3	5	0.07855000	12	3	4	0.05390348
10	2	8	0.01079842	11	3	6	0.06757040	12	3	5	0.05926581
10	2	9	0.00478220	11	3	7	0.04792540	12	3	6	0.05658542
10	3	3	0.09884808	11	3	8	0.02819059	12	3	7	0.04541357
10	3	4	0.10868777	11	3	9	0.01433927	12	3	8	0.03038788
10	3	5	0.10090627	11	4	4	0.16356058	12	3	9	0.01733016
10	3	6	0.07670782	11	4	5	0.16439949	12	3	10	0.00889850
10	3	7	0.04725030	11	4	6	0.13793029	12	4	4	0.11638134
10	3	8	0.02416062	11	4	7	0.09579226	12	4	5	0.12506616
10	4	4	0.22343036	11	4	8	0.05561992	12	4	6	0.11560288
10	4	5	0.20628442	11	5	5	0.28703728	12	4	7	0.09029931
10	4	6	0.15406893	11	5	6	0.24261366	12	4	8	0.05936182
10	4	7	0.09338998	11	5	7	0.16705834	12	4	9	0.03362498
10	5	5	0.33884572	11	6	6	0.34717347	12	5	5	0.22889678
10	5	6	0.25613984	12	1	1	0.00417809	12	5	6	0.21176948
11	1	1	0.00540493	12	1	2	0.00494280	12	5	7	0.16334611
11	1	2	0.00661699	12	1	3	0.00635686	12	5	8	0.10593665
11	1	3	0.00866728	12	1	4	0.00832673	12	6	6	0.32381162
11	1	4	0.01106778	12	1	5	0.01002981	12	6	7	0.25320637
11	1	5	0.01241398	12	1	6	0.01029881	13	1	1	0.00335173
11	1	6	0.01146954	12	1	7	0.00868584	13	1	2	0.00384111
11	1	7	0.00850868	12	1	8	0.00597311	13	1	3	0.00479769
11	1	8	0.00509814	12	1	9	0.00342375	13	1	4	0.00630494
11	1	9	0.00257187	12	1	10	0.00173014	13	1	5	0.00796366
11	1	10	0.00116154	12	1	11	0.00081491	13	1	6	0.00889095
11	1	11	0.00043722	12	1	12	0.00032096	13	1	7	0.00835835
11	2	2	0.02068521	12	2	2	0.01441985	13	1	8	0.00649750
11	2	3	0.02613936	12	2	3	0.01815018	13	1	9	0.00420556
11	2	4	0.03165245	12	2	4	0.02280158	13	1	10	0.00235010
11	2	5	0.03391764	12	2	5	0.02638543	13	1	11	0.00120945
11	2	6	0.03034466	12	2	6	0.02628444	13	1	12	0.00059595
11	2	7	0.02207859	12	2	7	0.02172673	13	1	13	0.00024370
11	2	8	0.01313135	12	2	8	0.01477968	13	2	2	0.01049664

n	r	s	Cov	n	r	s	Cov	n	r	s	Cov
13	2	3	0.01298868	14	1	3	0.00373053	14	4	7	0.06782586
13	2	4	0.01657730	14	1	4	0.00483617	14	4	8	0.05531976
13	2	5	0.02026605	14	1	5	0.00626882	14	4	9	0.03872751
13	2	6	0.02202660	14	1	6	0.00745693	14	4	10	0.02372730
13	2	7	0.02031391	14	1	7	0.00767428	14	4	11	0.01334414
13	2	8	0.01560168	14	1	8	0.00664352	14	5	5	0.12940562
13	2	9	0.01004401	14	1	9	0.00481660	14	5	6	0.13682162
13	2	10	0.00562078	14	1	10	0.00298124	14	5	7	0.12626421
13	2	11	0.00291381	14	1	11	0.00165315	14	5	8	0.10071835
13	2	12	0.00144851	14	1	12	0.00087667	14	5	9	0.06939808
13	3	3	0.03060331	14	1	13	0.00045112	14	5	10	0.04218172
13	3	4	0.03774181	14	1	14	0.00019010	14	6	6	0.23100313
13	3	5	0.04395141	14	2	2	0.00797218	14	6	7	0.21425016
13	3	6	0.04577853	14	2	3	0.00962094	14	6	8	0.16936884
13	3	7	0.04092415	14	2	4	0.01225345	14	6	9	0.11536977
13	3	8	0.03081741	14	2	5	0.01549987	14	7	7	0.31102056
13	3	9	0.01966753	14	2	6	0.01802771	14	7	8	0.24950447
13	3	10	0.01103211	14	2	7	0.01823471	15	1	1	0.00233752
13	3	11	0.00578415	14	2	8	0.01559735	15	1	2	0.00255278
13	4	4	0.08155590	14	2	9	0.01122890	15	1	3	0.00298527
13	4	5	0.09229062	14	2	10	0.00693590	15	1	4	0.00377565
13	4	6	0.09249960	14	2	11	0.00385767	15	1	5	0.00492947
13	4	7	0.08003684	14	2	12	0.00205909	15	1	6	0.00613016
13	4	8	0.05891743	14	2	13	0.00106634	15	1	7	0.00679083
13	4	9	0.03716668	14	3	3	0.02150520	15	1	8	0.00645281
13	4	10	0.02084745	14	3	4	0.02671034	15	1	9	0.00518827
13	5	5	0.17473947	14	3	5	0.03239099	15	1	10	0.00355324
13	5	6	0.17401773	14	3	6	0.03620363	15	1	11	0.00213672
13	5	7	0.14783843	14	3	7	0.03551503	15	1	12	0.00119404
13	5	8	0.10696239	14	3	8	0.02975021	15	1	13	0.00065678
13	5	9	0.06664922	14	3	9	0.02116484	15	1	14	0.00035129
13	6	6	0.28111490	14	3	10	0.01303360	15	1	15	0.00015155
13	6	7	0.24125289	14	3	11	0.00728981	15	2	2	0.00629182
13	6	8	0.17351862	14	3	12	0.00393468	15	2	3	0.00738318
13	7	7	0.33023818	14	4	4	0.05692074	15	2	4	0.00925767
14	1	1	0.00276794	14	4	5	0.06691869	15	2	5	0.01188808
14	1	2	0.00308804	14	4	6	0.07171521	15	2	6	0.01452363

n	r	s	Cov
15	2	7	0.01584984
15	2	8	0.01489314
15	2	9	0.01188480
15	2	10	0.00810778
15	2	11	0.00487521
15	2	12	0.00273403
15	2	13	0.00151187
15	2	14	0.00081221
15	3	3	0.01560131
15	3	4	0.01926946
15	3	5	0.02393578
15	3	6	0.02824484
15	3	7	0.02996240
15	3	8	0.02758269
15	3	9	0.02172251
15	3	10	0.01472552
15	3	11	0.00885982
15	3	12	0.00500265
15	3	13	0.00279289
15	4	4	0.03996312
15	4	5	0.04819751
15	4	6	0.05451000
15	4	7	0.05565412
15	4	8	0.04976763
15	4	9	0.03844775
15	4	10	0.02581671
15	4	11	0.01553801
15	4	12	0.00885158
15	5	5	0.09400003
15	5	6	0.10427265
15	5	7	0.10329354
15	5	8	0.08988977
15	5	9	0.06806266
15	5	10	0.04517637
15	5	11	0.02712389
15	6	6	0.18207974
15	6	7	0.18020269
15	6	8	0.15470166
15	6	9	0.11546455
15	6	10	0.07576952
15	7	7	0.27497235
15	7	8	0.23887672
15	7	9	0.17759069
15	8	8	0.31618549
16	1	1	0.00200843
16	1	2	0.00215781
16	1	3	0.00245185
16	1	4	0.00300877
16	1	5	0.00389716
16	1	6	0.00497807
16	1	7	0.00584131
16	1	8	0.00600886
16	1	9	0.00530033
16	1	10	0.00399769
16	1	11	0.00262069
16	1	12	0.00155460
16	1	13	0.00088620
16	1	14	0.00050658
16	1	15	0.00027995
16	1	16	0.00012300
16	2	2	0.00512998
16	2	3	0.00585873
16	2	4	0.00717149
16	2	5	0.00919799
16	2	6	0.01159634
16	2	7	0.01343937
16	2	8	0.01368645
16	2	9	0.01198373
16	2	10	0.00899595
16	2	11	0.00588600
16	2	12	0.00349527
16	2	13	0.00199945
16	2	14	0.00114764
16	2	15	0.00063610
16	3	3	0.01172438
16	3	4	0.01426014
16	3	5	0.01786662
16	3	6	0.02190048
16	3	7	0.02475821
16	3	8	0.02473836
16	3	9	0.02137568
16	3	10	0.01591956
16	3	11	0.01038817
16	3	12	0.00618493
16	3	13	0.00356183
16	3	14	0.00205966
16	4	4	0.02847448
16	4	5	0.03480886
16	4	6	0.04101293
16	4	7	0.04465698
16	4	8	0.04332092
16	4	9	0.03666191
16	4	10	0.02696703
16	4	11	0.01752451
16	4	12	0.01047589
16	4	13	0.00609358
16	5	5	0.06762694
16	5	6	0.07786265
16	5	7	0.08191551
16	5	8	0.07701808
16	5	9	0.06364245
16	5	10	0.04610092
16	5	11	0.02977135
16	5	12	0.01784552
16	6	6	0.13917474
16	6	7	0.14539679
16	6	8	0.13415643
16	6	9	0.10883020
16	6	10	0.07772096
16	6	11	0.04977155
16	7	7	0.23120627

n	r	s	Cov	n	r	s	Cov	n	r	s	Cov
16	7	8	0.21499004	17	3	3	0.00912812	17	6	10	0.07571307
16	7	9	0.17329373	17	3	4	0.01086582	17	6	11	0.05224911
16	7	10	0.12255738	17	3	5	0.01355042	17	6	12	0.03300584
16	8	8	0.29997738	17	3	6	0.01698395	17	7	7	0.18689162
16	8	9	0.24552976	17	3	7	0.02016389	17	7	8	0.18416643
17	1	1	0.00174920	17	3	8	0.02159616	17	7	9	0.15951559
17	1	2	0.00185635	17	3	9	0.02027228	17	7	10	0.12202089
17	1	3	0.00205955	17	3	10	0.01649896	17	7	11	0.08332844
17	1	4	0.00244978	17	3	11	0.01171171	17	8	8	0.26896236
17	1	5	0.00311371	17	3	12	0.00743978	17	8	9	0.23601203
17	1	6	0.00402096	17	3	13	0.00443702	17	8	10	0.18010442
17	1	7	0.00492142	17	3	14	0.00262511	17	9	9	0.30425396
17	1	8	0.00540782	17	3	15	0.00157019	18	1	1	0.00154005
17	1	9	0.00517605	17	4	4	0.02074233	18	1	2	0.00161942
17	1	10	0.00426941	17	4	5	0.02541606	18	1	3	0.00176289
17	1	11	0.00305310	17	4	6	0.03079583	18	1	4	0.00203679
17	1	12	0.00194154	17	4	7	0.03532117	18	1	5	0.00252354
17	1	13	0.00115200	17	4	8	0.03676942	18	1	6	0.00325024
17	1	14	0.00067548	17	4	9	0.03379998	18	1	7	0.00408899
17	1	15	0.00040061	17	4	10	0.02713067	18	1	8	0.00473742
17	1	16	0.00022737	17	4	11	0.01912246	18	1	9	0.00486671
17	1	17	0.00010133	17	4	12	0.01214357	18	1	10	0.00435447
17	2	2	0.00429457	17	4	13	0.00728560	18	1	11	0.00338678
17	2	3	0.00478885	17	4	14	0.00435139	18	1	12	0.00232192
17	2	4	0.00570185	17	5	5	0.04861238	18	1	13	0.00145119
17	2	5	0.00721327	17	5	6	0.05750424	18	1	14	0.00087140
17	2	6	0.00923308	17	5	7	0.06360024	18	1	15	0.00052785
17	2	7	0.01119022	17	5	8	0.06400734	18	1	16	0.00032356
17	2	8	0.01218907	17	5	9	0.05729691	18	1	17	0.00018758
17	2	9	0.01158600	17	5	10	0.04515687	18	1	18	0.00008454
17	2	10	0.00950918	17	5	11	0.03151411	18	2	2	0.00367094
17	2	11	0.00678036	17	5	12	0.01998661	18	2	3	0.00401337
17	2	12	0.00430900	17	5	13	0.01206929	18	2	4	0.00464889
17	2	13	0.00256091	17	6	6	0.10413103	18	2	5	0.00575314
17	2	14	0.00150632	17	6	7	0.11374125	18	2	6	0.00737118
17	2	15	0.00089610	17	6	8	0.11176605	18	2	7	0.00920558
17	2	16	0.00050960	17	6	9	0.09779054	18	2	8	0.01058714

n	r	s	Cov	n	r	s	Cov	n	r	s	Cov
18	2	9	0.01080757	18	5	7	0.04875400	19	1	10	0.00426710
18	2	10	0.00962263	18	5	8	0.05203357	19	1	11	0.00359112
18	2	11	0.00745898	18	5	9	0.05006636	19	1	12	0.00265742
18	2	12	0.00510513	18	5	10	0.04272296	19	1	13	0.00176650
18	2	13	0.00319127	18	5	11	0.03225057	19	1	14	0.00109849
18	2	14	0.00191990	18	5	12	0.02184058	19	1	15	0.00067352
18	2	15	0.00116607	18	5	13	0.01372737	19	1	16	0.00042185
18	2	16	0.00071638	18	5	14	0.00841348	19	1	17	0.00026598
18	2	17	0.00041587	18	6	6	0.07690371	19	1	18	0.00015681
18	3	3	0.00734360	18	6	7	0.08705925	19	1	19	0.00007131
18	3	4	0.00853388	18	6	8	0.09033899	19	2	2	0.00318944
18	3	5	0.01048608	18	6	9	0.08462858	19	2	3	0.00343274
18	3	6	0.01324522	18	6	10	0.07069767	19	2	4	0.00387843
18	3	7	0.01627678	18	6	11	0.05260472	19	2	5	0.00467573
18	3	8	0.01845242	18	6	12	0.03537790	19	2	6	0.00592853
18	3	9	0.01862387	18	6	13	0.02224994	19	2	7	0.00752439
18	3	10	0.01644560	18	7	7	0.14656210	19	2	8	0.00902124
18	3	11	0.01268063	18	7	8	0.15172250	19	2	9	0.00978443
18	3	12	0.00865935	18	7	9	0.14008255	19	2	10	0.00937093
18	3	13	0.00541790	18	7	10	0.11522602	19	2	11	0.00785852
18	3	14	0.00327132	18	7	11	0.08465296	19	2	12	0.00580213
18	3	15	0.00199627	18	7	12	0.05645239	19	2	13	0.00385375
18	3	16	0.00123113	18	8	8	0.23029578	19	2	14	0.00239816
18	4	4	0.01553036	18	8	9	0.21468137	19	2	15	0.00147321
18	4	5	0.01889042	18	8	10	0.17581427	19	2	16	0.00092473
18	4	6	0.02323980	18	8	11	0.12809629	19	2	17	0.00058399
18	4	7	0.02772235	18	9	9	0.29031767	19	2	18	0.00034461
18	4	8	0.03062316	18	9	10	0.24151933	19	3	3	0.00607885
18	4	9	0.03029644	19	1	1	0.00136802	19	3	4	0.00689952
18	4	10	0.02637866	19	1	2	0.00142861	19	3	5	0.00830018
18	4	11	0.02016453	19	1	3	0.00153244	19	3	6	0.01043536
18	4	12	0.01372432	19	1	4	0.00172627	19	3	7	0.01308956
18	4	13	0.00860444	19	1	5	0.00207923	19	3	8	0.01551046
18	4	14	0.00522906	19	1	6	0.00264272	19	3	9	0.01665568
18	4	15	0.00321654	19	1	7	0.00337108	19	3	10	0.01582875
18	5	5	0.03519797	19	1	8	0.00406586	19	3	11	0.01320128
18	5	6	0.04235646	19	1	9	0.00443460	19	3	12	0.00971527

n	r	s	Cov
19	3	13	0.00644742
19	3	14	0.00401861
19	3	15	0.00247702
19	3	16	0.00156047
19	3	17	0.00098810
19	4	4	0.01198543
19	4	5	0.01436458
19	4	6	0.01773016
19	4	7	0.02171572
19	4	8	0.02516043
19	4	9	0.02653065
19	4	10	0.02487459
19	4	11	0.02055609
19	4	12	0.01505188
19	4	13	0.00998059
19	4	14	0.00624102
19	4	15	0.00387020
19	4	16	0.00245341
19	5	5	0.02585657
19	5	6	0.03135245
19	5	7	0.03717174
19	5	8	0.04166902
19	5	9	0.04274198
19	5	10	0.03925079
19	5	11	0.03198298
19	5	12	0.02324040
19	5	13	0.01539165
19	5	14	0.00967189
19	5	15	0.00605109
19	6	6	0.05649456
19	6	7	0.06572990
19	6	8	0.07144481
19	6	9	0.07114330
19	6	10	0.06378078
19	6	11	0.05108472
19	6	12	0.03675110
19	6	13	0.02427734
19	6	14	0.01532335
19	7	7	0.11240008
19	7	8	0.12127951
19	7	9	0.11849144
19	7	10	0.10419817
19	7	11	0.08215152
19	7	12	0.05847151
19	7	13	0.03843592
19	8	8	0.18998725
19	8	9	0.18664127
19	8	10	0.16290293
19	8	11	0.12713040
19	8	12	0.08962349
19	9	9	0.26322626
19	9	10	0.23293442
19	9	11	0.18156804
19	10	10	0.29394027
20	1	1	0.00122433
20	1	2	0.00127183
20	1	3	0.00134897
20	1	4	0.00148801
20	1	5	0.00174293
20	1	6	0.00217017
20	1	7	0.00277308
20	1	8	0.00343889
20	1	9	0.00393955
20	1	10	0.00404073
20	1	11	0.00365635
20	1	12	0.00291471
20	1	13	0.00207149
20	1	14	0.00135025
20	1	15	0.00084456
20	1	16	0.00053181
20	1	17	0.00034382
20	1	18	0.00022190
20	1	19	0.00013256
20	1	20	0.00006073
20	2	2	0.00280680
20	2	3	0.00298449
20	2	4	0.00330118
20	2	5	0.00387420
20	2	6	0.00482184
20	2	7	0.00614264
20	2	8	0.00758269
20	2	9	0.00864482
20	2	10	0.00882846
20	2	11	0.00796050
20	2	12	0.00632956
20	2	13	0.00449189
20	2	14	0.00292738
20	2	15	0.00183295
20	2	16	0.00115625
20	2	17	0.00074882
20	2	18	0.00048384
20	2	19	0.00028924
20	3	3	0.00515246
20	3	4	0.00572545
20	3	5	0.00672470
20	3	6	0.00833583
20	3	7	0.01053638
20	3	8	0.01288846
20	3	9	0.01456955
20	3	10	0.01477463
20	3	11	0.01325050
20	3	12	0.01049697
20	3	13	0.00743538
20	3	14	0.00484600
20	3	15	0.00304008
20	3	16	0.00192336
20	3	17	0.00124901
20	3	18	0.00080850
20	4	4	0.00953597
20	4	5	0.01120885
20	4	6	0.01373986

n	r	s	Cov	n	r	s	Cov	n	r	s	Cov
20	4	7	0.01706326	20	5	12	0.02404822	20	7	10	0.09098850
20	4	8	0.02049123	20	5	13	0.01692317	20	7	11	0.07672134
20	4	9	0.02279783	20	5	14	0.01105225	20	7	12	0.05834264
20	4	10	0.02283245	20	5	15	0.00700029	20	7	13	0.03052809
20	4	11	0.02029275	20	5	16	0.00448782	20	7	14	0.02647469
20	4	12	0.01598238	20	6	6	0.04158133	20	8	8	0.15218011
20	4	13	0.01129116	20	6	7	0.04931385	20	8	9	0.15641886
20	4	14	0.00736394	20	6	8	0.05569282	20	8	10	0.14457451
20	4	15	0.00463654	20	6	9	0.05852055	20	8	11	0.12032003
20	4	16	0.00294854	20	6	10	0.05597139	20	8	12	0.09045810
20	4	17	0.00192362	20	6	11	0.04809629	20	8	13	0.06231956
20	5	5	0.01938685	20	6	12	0.03706983	20	9	9	0.22872385
20	5	6	0.02347768	20	6	13	0.02593954	20	9	10	0.21372633
20	5	7	0.02837203	20	6	14	0.01696047	20	9	11	0.17736650
20	5	8	0.03307029	20	6	15	0.01081691	20	9	12	0.13240219
20	5	9	0.03584690	20	7	7	0.08491192	20	10	10	0.28177176
20	5	10	0.03517999	20	7	8	0.09479985	20	10	11	0.23758892
20	5	11	0.03081526	20	7	9	0.09734244				

If we consider (3.4) with $r = 1$ and $r = n$, apply integration by parts to each integral on the right side by treating x^{k-1} for integration and the rest of the integrand for differentiation and simplify the resulting equation in both cases, then we obtain the required results (3.1) and (3.2) respectively.

For $n \geq 3$ and $1 \leq r \leq n-1$, if we apply integration by parts to both integrals on the right side of (3.4) by considering x^{k-1} for integration and rest of the integrand for differentiation and simplify the resulting equation, then we get the result (3.3).

Relation 3.2

For $n \geq 3$ and $1 \leq r \leq n-2$, we have

$$[1+3(n-s+1)]\,\mu_{r,s:n} = \frac{3}{2}\left\{2(n-r)\,\mu^{(2)}_{r:n} - n\mu^{(2)}_{r:n-1} + n\mu_{r,r+1:n-1}\right\} \quad \text{if } s = r+1$$

$$= \frac{3}{2}\left\{2\mu_{r,n-1:n} - n\mu_{r,n-1:n-1} + n\mu_{r:n-1}\right\} \quad \text{if } s = n \tag{3.5}$$

and, for $n \geq 4$, $1 \leq r < s \leq n-1$ and $(s-r) \geq 2$,

$$[1+3(n-s+1)]\mu_{r,s:n} = \frac{3}{2}\left\{2(n-s+1)\,\mu_{r,s-1:n} + n\mu_{r,s:n-1} - n\mu_{r,s-1:n-1}\right\} \tag{3.6}$$

Proof.

From (2.2), we have, for $1 \leq r < s \leq n$,

$$\mu_{r,s:n} = C(r,s:n)\int_{-1}^{1} x\,[F(x)]^{r-1}\, f(x)\, I_{r,s:n}(x)\,dx \tag{3.7}$$

where

$$I_{r,s:n}(x) = \int_{x}^{1} y[F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s}\, f(y)\,dy \tag{3.8}$$

On using the functional relation given by (1.3) in (3.8), we get

$$I_{r,s:n}(x) = \frac{3}{2}\int_x^1 [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}\,dy$$

$$-3\int_x^1 [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}\,dy \qquad (3.9)$$

By applying integration by parts, treating dy for integration and the rest of the integrand for differentiation to both integrals on the right side of (3.9), we may obtain, for $n \geq 3$, $1 \leq r \leq n-2$ and $s = r+1$,

$$I_{r,r+1:n}(x) = 3x[1-F(x)]^{n-r} - (3/2)x[1-F(x)]^{n-r-1}$$

$$+(3/2)(n-r-1)\int_x^1 y[1-F(y)]^{n-r-2} f(y)\,dy$$

$$-3(n-r)\int_x^1 y[1-F(y)]^{n-r-1} f(y)\,dy \qquad (3.10)$$

and for $n \geq 4$, $1 \leq r < s \leq n-1$ and $s-r \geq 2$,

$$I_{r,s:n}(x) = 3(s-r-1)\int_x^1 y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1} f(y)dy$$

$$-(3/2)(s-r-1)\int_x^1 y[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s} f(y)dy$$

$$+(3/2)(n-s)\int_x^1 y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s-1} f(y)dy$$

$$-3(n-s+1)\int_x^1 y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(y)dy$$

(3.11)

If we use (3.10) in (3.7) for $s = r+1$ and simplify, the result (3.5) with $s = r+1$ follows. Also, the result (3.6) follows if we use (3.11) in (3.7) and simplify the resulting equation. The proof of the result (3.5) with $s = n$ is similar to that of the result (3.5) with $s = r+1$ and hence omitted.

Remark 3.1 Since (1.2) is symmetric about zero, we have $\mu_{1,2:2} = 0$ and $\mu_{n-1,n:n} = \mu_{1,2:n}$. Hence, all product moments of order statistics arising from the standard U-shaped distribution can be evaluated in a systematic and recursive manner using the results (3.5) and (3.6) without evaluating any integral for the moments of order statistics.

4. *Best Linear Unbiased Estimators of θ_1 and θ_2 Using Quasi-midranges and Quasi-ranges*

Let $Z_{1:n} \le Z_{2:n} \le \ldots \le Z_{n:n}$ be the order statistics of a random sample of size n. Then

$$M_{i:n} = (Z_{n-i+1:n} + Z_{i:n})/2 \qquad i = 1, 2, \ldots, [(n+1)/2]$$

is known as the ith quasi-midrange and

$$R_{i:n} = Z_{n-i+1:n} - Z_{i:n} \qquad i = 1, 2, \ldots, [n/2]$$

is the ith quasi-range of the sample. Thomas [8] has shown that if the parent distribution belongs to the location-scale family of distributions which are symmetric about the location parameter, then Lloyd's BLUE of the location and scale parameters based on order statistics reduce to those based on the quasi-midranges and quasi-ranges of the sample respectively.

We have used the results of Thomas [8] and obtained the coefficients c_i in the BLUE of θ_1 viz. $\theta_1^* = \sum_{i=1}^{\left[\frac{n+1}{2}\right]} c_i M_{i:n}$ along with Var (θ_1^*) for $n = 2\,(1)\,20$ and those are presented in Table 3. Also, the results of Thomas [8] are used to obtain the coefficients d_i in the BLUE, $\theta_2^* = \sum_{i=1}^{[n/2]} d_i R_{i:n}$ of θ_2 along with Var (θ_2^*) for $n = 2\,(1)\,20$ and those are given in Table 4. It may be noted that the coefficients of $Z_{r:n}$ in the BLUE of θ_1 and θ_2 obtained from Sarhan [6] are the same as those obtained from our estimators θ_1^* and θ_2^* for $n \le 5$.

The estimators of θ_1 and θ_2 obtained by the method of moments are

$$\hat{\theta}_1 = \bar{Z} \text{ and } \qquad \hat{\theta}_2 = \sqrt{(5/3n) \sum_{i=1}^{n} (Z_i - \bar{Z})^2}$$

Table 3 : BLUE of the location parameter θ_1 of the U-shaped distribution of Quasi-midranges

n	Coefficient in the BLUE, $\theta_1^* = \sum_{i=1}^{\left[\frac{n+1}{2}\right]} c_i M_{i:n}$										$v(\theta_1^*)/\theta_2^2$
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	
2	1.000000										0.300000
3	1.080000	-0.080000									0.150078
4	1.106145	-0.106145									0.076790
5	1.116959	-0.089718	-0.027241								0.040528
6	1.120941	-0.080352	-0.040589								0.022249
7	1.120628	-0.072140	-0.034926	-0.013563							0.012826
8	1.116825	-0.063189	-0.031812	-0.021824							0.007834
9	1.109769	-0.052700	-0.029524	-0.019276	-0.008268						0.005097
10	1.099632	-0.040477	-0.027288	-0.017867	-0.014001						0.003534
11	1.086867	-0.026959	-0.024602	-0.016939	-0.012666	-0.005701					0.002597
12	1.072389	-0.013260	-0.021076	-0.016129	-0.011917	-0.010006					0.002004
13	1.057502	-0.000939	-0.016467	-0.015166	-0.011446	-0.009219	-0.004264				0.001608
14	1.043548	0.008551	-0.010805	-0.013806	-0.011065	-0.008758	-0.007664				0.001329
15	1.031499	0.014440	-0.004503	-0.011843	-0.010625	-0.008465	-0.007142	-0.003361			0.001124
16	1.021768	0.016875	0.001684	-0.009147	-0.009991	-0.008237	-0.006819	-0.006132			0.000967
17	1.014291	0.016689	0.006888	-0.005739	-0.009032	-0.007992	-0.006608	-0.005755	-0.002740		0.000844
18	1.008734	0.014953	0.010458	-0.001855	-0.007630	-0.007651	-0.006448	-0.005511	-0.005050		0.000744
19	1.004687	0.012593	0.012192	0.002064	-0.005716	-0.007133	-0.006290	-0.005346	-0.004764	-0.002287	0.000662
20	1.001763	0.010221	0.012331	0.005480	-0.003316	-0.006351	-0.006086	-0.005224	-0.004571	-0.004248	0.000593

Table 4 : BLUE of the scale parameter θ_2 of the U-shaped distribution by Quasi-ranges

n	Coefficient in the BLUE, $\theta_2^* = \sum_{i=1}^{[n/2]} d_i R_{i:n}$										$v(\theta_2^*)/\theta_2^2$
	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	
2	1.166667										0.633333
3	0.777778										0.216162
4	0.673713	-0.036527									0.095504
5	0.631337	-0.041118									0.047021
6	0.610167	-0.040439	-0.008431								0.024888
7	0.597634	-0.038101	-0.011292								0.014055
8	0.588721	-0.034713	-0.012456	-0.003223							0.008479
9	0.581118	-0.030342	-0.012887	-0.004797							0.005474
10	0.573709	-0.025031	-0.012853	-0.005697	-0.001620						0.003775
11	0.566055	-0.018999	-0.012388	-0.006257	-0.002578						0.002762
12	0.558187	-0.012728	-0.011439	-0.006588	-0.003212	-0.000958					0.002124
13	0.550449	-0.006890	-0.009949	-0.006709	-0.003661	-0.001591					0.001699
14	0.543295	-0.002135	-0.007931	-0.006593	-0.003981	-0.002044	-0.000627				0.001401
15	0.537077	0.001153	-0.005534	-0.006190	-0.004182	-0.002384	-0.001069				0.001182
16	0.531939	0.002971	-0.003043	-0.005461	-0.004256	-0.002642	-0.001401	-0.000437			0.001015
17	0.527841	0.003614	-0.000802	-0.004409	-0.004176	-0.002829	-0.001659	-0.000760			0.000884
18	0.524632	0.003500	0.000908	-0.003105	-0.003913	-0.002941	-0.001863	-0.001010	-0.000319		0.000778
19	0.522129	0.003010	0.001964	-0.001699	-0.003442	-0.002967	-0.002020	-0.001209	-0.000562		0.000691
20	0.520162	0.002408	0.002415	-0.000381	-0.002765	-0.002887	-0.002131	-0.001370	-0.000755	-0.000241	0.000618

Clearly, Var $(\hat{\theta}_1) = (3/5n)\theta_2^2$ and hence one can easily verify from Table 3 that the BLUE θ_1^* is uniformly better than $\hat{\theta}_1$. The estimator $\hat{\theta}_2$ is indeed a biased estimator for θ_2. Therefore, we need not compare the efficiency of our BLUE θ_2^* with $\hat{\theta}_2$ for small samples. Again, the likelihood equation for estimating θ_1 is a polynomial in θ_1 of degree $n-1$. Hence, the maximum likelihood estimators of θ_1 and θ_2 are not available explicitly. However, our estimators are given explicitly with least variance among the class of all linear estimators of the parameters.

REFERENCES

[1] Bailey, N.T.J., 1959.*Statistical Methods in Biology*, The English Universities Press Ltd., London.

[2] Govindarajulu, Z., 1963. Relationships among moments of order statistics in samples from two related populations. *Technometrics*, **5**, 514-518.

[3] Irwin, J.O., 1964. The contributions of G.U. Yule and A.G Mckendrick to stochastic process methods in biology and medicine — Stochastic Models in Medicine and Biology, edited by Gurland, J., The University of Wisconsin Press, Madison.

[4] Lloyd, E.H., 1952. Least-squares estimation of location and scale parameters using order statistics. *Biometrika*, **39**, 88-95.

[5] Malik, H.J., 1967. Exact moments of order statistics from a power function distribution. *Skand-Aktuarietidskr.*, 64-69.

[6] Sarhan, A.E., 1955. Estimation of the mean and standard deviation by order statistics, Part II. *Ann. Math. Statist.*, **26**, 505-511.

[7] Sarhan, A.E. and Greenberg, B.G. (eds.), 1962. *Contributions to Order Statistics*, John Wiley and Sons, New York.

[8] Thomas, P.Y., 1990. Estimating location and scale parameters of a symmetric distribution by systematic statistics. *J. Indian Soc. Agric. Stat.*, **42**, 250-256.