Available online at www.isas.org.in/jisas
JOURNAL OF THE INDIAN SOCIETY OF
AGRICULTURAL STATISTICS 75(2) 2021 157–168

New D-optimal Covariate Designs in CRD and RCBD set-ups

Hiranmoy Das[1], Ankita Dutta[2], Dikeshwar Nishad[3] and Anurup Majumder[2]

[1]*ICAR-Indian Institute of Soil Science, Bhopal*
[2]*Bidhan Chandra Krishi Viswavidyalaya, Nadia*
[3]*Pt. Shiv Kumar Shastri College of Agriculture and Research Station, Rajnandgaon*

Received 12 October 2020; Revised 11 November 2021; Accepted 04 December 2021

SUMMARY

One new series of D-optimal covariate designs in Completely Randomized Design (CRD) set-up and two new series of D-optimal covariate designs in Randomized Complete Block Design (RCBD) set-up have been developed in the present study. The methods of construction of D- optimal covariate designs are developed with the help of Special Array (Das *et al.*, 2020). The existence of $\mathbf{H}_v$ and $\mathbf{H}_{b-1}$ can develop c = (v – 1) number of D-optimal covariates in CRD and RCBD set-ups with the treatment number v for any odd number of replications or blocks, b. Again, if $\mathbf{H}_v$ and $\mathbf{H}_{b-r}$ exist, then c = (v – 1) D-optimal covariates will exist in a RCBD set-up with v number of treatments for b odd number of blocks provided r (> 1) is an odd number and $\mathbf{H}_{r-1}$ exists. In the Special Array, r is the number of rows (and columns) with all elements zero. The developed optimal covariate designs in this article are not yet available in the existing literature.

Keywords: Hadamard matrix, D-optimality, CRD, RCBD.

1. INTRODUCTION

Analysis of Covariance (ANCOVA) is an established method for minimizing the error affecting the treatment comparisons. ANCOVA models are nothing but blending of ‘regression models’ (in the absence of treatment parameters) and ‘varietal design models’ (in the absence of covariates). But the problem of determining the optimum designs for the estimation of regression parameters corresponding to controllable covariates was not a topic of research for many years. The topic was firstly considered by Troya (1982a, 1982b). Although her investigations were pioneer in history in the topic of Optimum Covariate Designs (OCDs) but she was restricted only to Completely Randomized Design (CRD) set-up. After a long gap, Das *et al.* (2003) extended the work on OCDs to the block design set-up, viz., Randomized Complete Block Design (RCBD) and some series of Balanced Incomplete Block Design (BIBD). They also constructed OCDs for the estimation of covariate parameters. Rao *et al.* (2003) also revisited the problem in CRD and RCBD set-ups. They identified that the solutions of construction of OCDs by using Mixed Orthogonal Arrays (MOAs) and thereby giving further insights and some new solutions. Dutta (2004, 2009) and Dutta *et al.* (2007, 2009a, 2009b, 2010a) developed OCDs to different design set-ups. Das *et al.* (2015) has published a book, viz., ‘Optimal Covariate Designs’ with a detail discussion on the topic. Mostly the designs developed by above mentioned authors are global optimal and the development of such designs are dependent on existence of Hadamard matrix of order v or b or k (v be the treatment numbers, b be the number of replications/ blocks in CRD/RCBD and k be the size of blocks in a variance balanced incomplete block design). However, Dutta *et al.* (2010b) developed some D-optimal covariate designs for estimation of regression coefficients in incomplete block design set-up, when global optimal designs do not exist. Das *et al.* (2015) described the D-optimal designs under non-regular situations in the case when Hadamard matrices and consequently global optimal covariate designs do not exist. Furthermore, Das *et al.* (2020) has

Corresponding author: Hiranmoy Das
E-mail address: hiranmoydas.stat@gmail.com

reported some new series of universal/global optimal covariate designs in CRD and RCBD set-ups without the existence of Hadamard matrix of order v or b.

In the present piece of investigation, an effort has been made to construct D-optimal covariate designs in CRD and RCBD set-ups under non-regular situations with b odd number of blocks or replications and v number of treatments. In the developed designs, the covariates are mutually orthogonal to each other. The methods of constructions of D-optimal covariate designs are developed by using Special Array (Das *et al.* 2020). The study contains five sections including the present introductory section. In section 2, the definition and properties of Special Array are presented. Section 3 and 4 describe the situations and conditions of the D-optimal covariate designs for CRD and RCBD set-ups. Construction of a new series of D-optimal covariate designs in CRD set-up has been presented in section 3. Similarly, construction of two new series of D-optimal covariate designs in RCBD set-up has been given in section 4. Conclusion of the study has been given in section 5.

2. SPECIAL ARRAY (SA); DEFINITION, PROPERTIES AND APPLICATIONS (DAS *ET AL.*, 2020)

2.1 Definition

A square matrix with elements 1, -1 and 0 of order h with r (≥ 1) number of rows (and columns) with all elements 0, whose all the distinct row or column vectors except r rows (or columns) are mutually orthogonal is referred to as **Special Array** (SA) of order h. In SA, each row or column sum is zero except the first row or column. The simplest examples, one for order 3 and two for order 5 are given below:

$$\underset{r=1}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}}, \underset{r=3}{\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}} \text{ and } \underset{r=1}{\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 \end{pmatrix}}$$

2.2 Properties:

Let the Special Array (SA) of order h be denoted as H_h^*, then

1) $\left|\det H_h^*\right| = 0$

2) $H_h^* H_h^{*T} = H_h^{*T} H_h^*$

3) Let H_1^* and H_2^* be two SA of order h_1 and h_2, respectively. Then the Kronecker product of H_1^* and H_2^* is also a SA of order h_1h_2. For example,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

2.3 Application

In this study, Special Array (constructed from Hadamard matrix with r rows and columns with all elements zero in the middle) is used to find out the number of D-optimal covariates in CRD and RCBD set-ups. The detailed discussion on this is given in the following sections.

3. COVARIATE DESIGNS UNDER NON-REGULAR CASES IN A EQUI-REPLICATED CRD SET-UP

Let there be v treatments and c covariates in a design with total n experimental units. In matrix notation the model can be represented as

$$(\mathbf{Y}, \mathbf{X}\boldsymbol{\tau} + \mathbf{Z}\boldsymbol{\gamma}, \sigma^2\mathbf{I}_n) \tag{3.1}$$

where, for $1 \le i \le v$, $1 \le j \le n_i$ (n_i is the number of times the i^{th} treatment is replicated; clearly $\sum_{i=1}^{v} n_i = n$) and $1 \le t \le c$, $\mathbf{Y}$ is an observation vector and $\mathbf{X}$ is the design matrix corresponding to vector of treatment effects τ^{vx1} and $\mathbf{Z}^{nxc} = ((z_{ij}^{(t)}))$ is the design matrix corresponding to vector of covariate effects $\gamma^{cx1} = (\gamma_1, \gamma_2, \ldots, \gamma_c)'$. This is referred to as one–way model with covariates without general mean. In the above, $\mathbf{Z}$ is called covariate matrix of c covariates $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_c$. Here $\mathbf{z}$'s are assumed to be controllable non-stochastic covariates.

The non-regular situations may arise where at least any one of the conditions $\mathbf{X'Z} = \mathbf{0}$ and $\mathbf{Z'Z} = n\mathbf{I}_c$

is violated. In that case, it is not possible to estimate simultaneously ANOVA parameters and γ-parameters orthogonally and/or most efficiently. In that case, we may consider D-optimality criterion to give an efficient allocation of treatments and covariates in CRD set-up. Dey and Mukerjee (2006) and Dutta *et al.* (2014) had also considered this situation and they had given solutions for some D-optimal designs.

Let D* be a CRD with v treatments and the vector of parameters be

$\theta = (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \ldots, \tau_v, \gamma_1, \gamma_2, \ldots, \gamma_c)$,

According to Dutta *et al.* (2014), construction of D-optimal covariate designs in CRD set-up with equal replication, maximum value of the determinant of information matrix $\mathbf{I}(\theta)$ of the design D* be denoted as $\det(\mathbf{I}(\theta))$, with respect to the design variables $\{z_{ij}^{(t)}\}$ satisfying $z_{ij}^{(t)} \in [-1,\ 1]; \forall i,\ j,\ t$.

Then,

$$\begin{aligned}\det(I(\theta)) &= \Big(\prod_{i=1}^{v} n_i\Big)\det\Big(Z'Z - \sum_i n_i^{-1} T_i' T_i\Big)\\ &= (b^v)\det\Big(Z'Z - b^{-1}\sum_i T_i' T_i\Big) \qquad \text{as } n_i = b\ \forall\ i\\ &= \det(N)\det(C) \qquad (3.2)\end{aligned}$$

where,

$\mathbf{N} = \text{Diag}(b, b, \ldots, b)$, $\mathbf{T}_i = \mathbf{1}_b'\mathbf{Z}_i$, $\mathbf{Z}^{n\times c} = (\mathbf{Z}_1', \mathbf{Z}_2', \ldots, \mathbf{Z}_v')'$ and

$$Z_i^{b\times c} = \begin{pmatrix} z_{i1}^{(1)} & z_{i1}^{(2)} & \ldots & z_{i1}^{(c)} \\ z_{i2}^{(1)} & z_{i2}^{(2)} & \ldots & z_{i2}^{(c)} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ z_{ib}^{(1)} & z_{ib}^{(2)} & \ldots & z_{ib}^{(c)} \end{pmatrix}$$

Note that

$$C = Z'Z - b^{-1}\sum_i T_i' T_i \qquad (3.3)$$

C is the information matrix for the regression coefficients $\gamma_1, \gamma_2, \ldots, \gamma_c$. The maximization of $\det(\mathbf{I}(\theta))$ is done for varying z-values for fixed n_i's. This leads to an upper bound for $\det(\mathbf{I}(\theta))$ obtained through completely symmetric **C**-matrices.

According to Lemma 2.3.1 (Das *et al.* 2015), a necessary condition for maximization of $\det(\mathbf{C})$ can be achieved by taking $z_{ij}^{(t)} = \pm 1$ for fixed n_i's. Now we can develop the following theorem for construction of D-optimal covariate designs in a CRD set-up.

Theorem 3.1: When at least any one of the conditions $\mathbf{X'Z} = \mathbf{0}$ and $\mathbf{Z'Z} = n\mathbf{I}_c$ is violated and if there exists $\mathbf{H}_v$, then a set of c (= v – 1) orthogonal covariates will exist which ultimately produces a D-optimal covariate design in a CRD set-up with v number of treatments for b odd number of replications, provided $\mathbf{H}_{b-1}$ exists.

Proof (by construction)**:** For construction of the D-optimal covariate design in a Completely Randomized Design set-up with v number of treatments for b odd number of replications, we follow the steps given below.

Step 1. Let us consider a Hadamard matrix of order v, $\mathbf{H}_v$.

$\mathbf{H}_v = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{v-1})$

Step 2. Let us construct a Special Array $\mathbf{H}_b$* of order b from $\mathbf{H}_{b-1}$ with one row and column with all zero elements in middle, i.e. ($\mathbf{1}^*, \mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{(b-1)/2-1}^*, \mathbf{0}, \mathbf{h}_{(b-1)/2}^*, \ldots, \mathbf{h}_{b-2}^*$).

$$H_b^* = \begin{pmatrix} 1\ldots\ 1\ 0\ \ 1\ldots 1 \\ \ldots\ldots\ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ 1\ldots\ {-1}\ 0\ {-1}\ldots 1 \\ 0\ldots\ 0\ 0\ \ 0\ldots 0 \\ 1\ldots\ 1\ 0\ {-1}\ldots {-1} \\ \ldots\ldots\ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ 1\ldots\ {-1}\ 0\ \ 1\ldots {-1} \end{pmatrix}$$

Step 3. Using $\mathbf{H}_b$*and $\mathbf{H}_v$, by Kronecker product of these two matrices, we get (b-2) set of (v-1) $\mathbf{W}_{ij}$** matrices of order bxv (without considering the first column and the column with all zeros), where i=1, 2,…, (b-2) and j=1, 2,…, (v-1). In each of the $\mathbf{W}_{ij}$** matrix there is one row with all elements zero in the middle.

$W_{ij}^{**} = h_i^* \otimes h'_j$, ⊗ denotes the Kronecker product

Step 4. According to Das *et al.* (2015), maximization of the determinant of information matrix of the design, $\mathbf{I}(\theta)$, can be achieved by replacing the zero elements in $\mathbf{W}_{ij}$**either by -1 or +1. In each of (b-2) sets, let us replace the zero elements of $\mathbf{W}_{ij}$** matrix by (j+1) th row of $\mathbf{H}_v$. In that way, we get (b-2) sets of (v-1) mutually orthogonal $\mathbf{W}_{ij}$* matrices. The matrix of orthogonal optimal covariates, **Z** of order nxc can be constructed from the (v-1) $\mathbf{W}_{ij}$ (=$\mathbf{W}_{ij}$*′) matrices in any one of i sets. It has been verified that any set of c (=v – 1) orthogonal covariates ultimately produces a D-optimal covariate design with unique maximum determinant value of information matrix of the design, **I**(θ), in a CRD set-up. It is also verified that the determinant of the information matrix of the above design, **I**(θ), with c covariates has achieved the maximum value with the following result, as mentioned in theorem 2.3.1 (Das *et al.,* 2015).

$$\det(I(\theta)) \le (b^v)\left\{a+(c-1)b^*\right\}(a-b^*)^{c-1} \qquad (3.4)$$

Where $a = n-\delta,\ b^* = |\xi - \delta|$

$$\delta = \sum_{i=1}^{v} n_i^{-1}\delta_i,\ \delta_i = 1(0) \text{ if } n_i = \text{ odd(even)}$$

$$\xi = \xi(n,\delta) = \begin{cases} [\delta] \text{if both of } n,\ [\delta] \text{are odd or even} \\ [\delta]+1 \text{ if } n = \text{odd},\ [\delta] = \text{even or } n = \text{even},\ [\delta] = \text{odd} \end{cases}$$

$[\delta] =$ greatest integer less than equal to δ

Corollary 3.1: The off diagonal elements of the information matrix (**C**) of regression coefficients of the designs developed in theorem 3.1, will be zero.

Proof: The proof is straight forward because in each ith set, the $\mathbf{W}_{ij}$ matrices are mutually orthogonal to each other, where, i = 1, 2, …, b-2 and j = 1, 2, …, v-1.

For easy understanding of the constructional procedure we may see the following example.

Example 3.1: Let us consider a CRD with v = 8 and b = 5. The Special Array of order 5 is

$$H_5^* = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 \end{pmatrix}$$

The Hadamard matrix of order 8 is

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

The three sets of 7**W**** matrices have been constructed using $\mathbf{H}_5$*and $\mathbf{H}_8$, by Kronecker product of these two matrices.

First set of 7**W**** matrices are

$$W_{11}^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix},\ W_{12}^{**} = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix},$$

$$W_{13}^{**} = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix},\ W_{14}^{**} = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$W_{15}^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{pmatrix},\ W_{16}^{**} = \begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \end{pmatrix},$$

$$W_{17}^{**} = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

Second set of 7 **W**** matrices are

$$W_{21}^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix},\ W_{22}^{**} = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix},$$

$$W_{23}^{**}=\begin{pmatrix}1&-1&-1&1&1&-1&-1&1\\1&-1&-1&1&1&-1&-1&1\\0&0&0&0&0&0&0&0\\-1&1&1&-1&-1&1&1&-1\\-1&1&1&-1&-1&1&1&-1\end{pmatrix},\ W_{24}^{**}=\begin{pmatrix}1&1&1&1&-1&-1&-1&-1\\1&1&1&1&-1&-1&-1&-1\\0&0&0&0&0&0&0&0\\-1&-1&-1&-1&1&1&1&1\\-1&-1&-1&-1&1&1&1&1\end{pmatrix}$$

$$W_{25}^{**}=\begin{pmatrix}1&-1&1&-1&-1&1&-1&1\\1&-1&1&-1&-1&1&-1&1\\0&0&0&0&0&0&0&0\\-1&1&-1&1&1&-1&1&-1\\-1&1&-1&1&1&-1&1&-1\end{pmatrix},\ W_{26}^{**}=\begin{pmatrix}1&1&-1&-1&-1&-1&1&1\\1&1&-1&-1&-1&-1&1&1\\0&0&0&0&0&0&0&0\\-1&-1&1&1&1&1&-1&-1\\-1&-1&1&1&1&1&-1&-1\end{pmatrix},$$

$$W_{27}^{**}=\begin{pmatrix}1&-1&-1&1&-1&1&1&-1\\1&-1&-1&1&-1&1&1&-1\\0&0&0&0&0&0&0&0\\-1&1&1&-1&1&-1&-1&1\\-1&1&1&-1&1&-1&-1&1\end{pmatrix}$$

Third set of 7 **W**** matrices are

$$W_{31}^{**}=\begin{pmatrix}1&-1&1&-1&1&-1&1&-1\\-1&1&-1&1&-1&1&-1&1\\0&0&0&0&0&0&0&0\\-1&1&-1&1&-1&1&-1&1\\1&-1&1&-1&1&-1&1&-1\end{pmatrix},\ W_{32}^{**}=\begin{pmatrix}1&1&-1&-1&1&1&-1&-1\\-1&-1&1&1&-1&-1&1&1\\0&0&0&0&0&0&0&0\\-1&-1&1&1&-1&-1&1&1\\1&1&-1&-1&1&1&-1&-1\end{pmatrix},$$

$$W_{33}^{**}=\begin{pmatrix}1&-1&-1&1&1&-1&-1&1\\-1&1&1&-1&-1&1&1&-1\\0&0&0&0&0&0&0&0\\-1&1&1&-1&-1&1&1&-1\\1&-1&-1&1&1&-1&-1&1\end{pmatrix},\ W_{34}^{**}=\begin{pmatrix}1&1&1&1&-1&-1&-1&-1\\-1&-1&-1&-1&1&1&1&1\\0&0&0&0&0&0&0&0\\-1&-1&-1&-1&1&1&1&1\\1&1&1&1&-1&-1&-1&-1\end{pmatrix}$$

$$W_{35}^{**}=\begin{pmatrix}1&-1&1&-1&-1&1&-1&1\\-1&1&-1&1&1&-1&1&-1\\0&0&0&0&0&0&0&0\\-1&1&-1&1&1&-1&1&-1\\1&-1&1&-1&-1&1&-1&1\end{pmatrix},\ W_{36}^{**}=\begin{pmatrix}1&1&-1&-1&-1&-1&1&1\\-1&-1&1&1&1&1&-1&-1\\0&0&0&0&0&0&0&0\\-1&-1&1&1&1&1&-1&-1\\1&1&-1&-1&-1&-1&1&1\end{pmatrix},$$

$$W_{37}^{**}=\begin{pmatrix}1&-1&-1&1&-1&1&1&-1\\-1&1&1&-1&1&-1&-1&1\\0&0&0&0&0&0&0&0\\-1&1&1&-1&1&-1&-1&1\\1&-1&-1&1&-1&1&1&-1\end{pmatrix}$$

For first set, after replacing the row having all elements zero of each **W**** matrix with second to eighth row of $\mathbf{H}_8$ subsequently and we get,

$$W_{11}^{*}=\begin{pmatrix}1&-1&1&-1&1&-1&1&-1\\-1&1&-1&1&-1&1&-1&1\\1&-1&-1&1&1&-1&-1&1\\1&-1&1&-1&1&-1&1&-1\\-1&1&-1&1&-1&1&-1&1\end{pmatrix},\ W_{12}^{*}=\begin{pmatrix}1&1&-1&-1&1&1&-1&-1\\-1&-1&1&1&-1&-1&1&1\\1&1&-1&-1&1&1&-1&-1\\1&1&-1&-1&1&1&-1&-1\\-1&-1&1&1&-1&-1&1&1\end{pmatrix},$$

$$W_{13}^{*}=\begin{pmatrix}1&-1&-1&1&1&-1&-1&1\\-1&1&1&-1&-1&1&1&-1\\1&-1&1&-1&1&-1&1&-1\\1&-1&-1&1&1&-1&-1&1\\-1&1&1&-1&-1&1&1&-1\end{pmatrix},\ W_{14}^{*}=\begin{pmatrix}1&1&1&1&-1&-1&-1&-1\\-1&-1&-1&-1&1&1&1&1\\1&1&1&1&-1&-1&-1&-1\\1&1&1&1&-1&-1&-1&-1\\-1&-1&-1&-1&1&1&1&1\end{pmatrix}$$

$$W_{15}^{*}=\begin{pmatrix}1&-1&1&-1&-1&1&-1&1\\-1&1&-1&1&1&-1&1&-1\\1&-1&-1&1&-1&1&1&-1\\1&-1&1&-1&-1&1&-1&1\\-1&1&-1&1&1&-1&1&-1\end{pmatrix},\ W_{16}^{*}=\begin{pmatrix}1&1&-1&-1&-1&-1&1&1\\-1&-1&1&1&1&1&-1&-1\\1&1&-1&-1&-1&-1&1&1\\1&1&-1&-1&-1&-1&1&1\\-1&-1&1&1&1&1&-1&-1\end{pmatrix},$$

$$W_{17}^{*}=\begin{pmatrix}1&-1&-1&1&-1&1&1&-1\\-1&1&1&-1&1&-1&-1&1\\1&-1&1&-1&-1&1&-1&1\\1&-1&-1&1&-1&1&1&-1\\-1&1&1&-1&1&-1&-1&1\end{pmatrix}$$

For second set, after replacing the row having all elements zero of each **W** matrix with second to eighth row of $\mathbf{H}_8$ subsequently and we get,

$$W_{21}^{*}=\begin{pmatrix}1&-1&1&-1&1&-1&1&-1\\1&-1&1&-1&1&-1&1&-1\\1&-1&-1&1&1&-1&-1&1\\-1&1&-1&1&-1&1&-1&1\\-1&1&-1&1&-1&1&-1&1\end{pmatrix},\ W_{22}^{*}=\begin{pmatrix}1&1&-1&-1&1&1&-1&-1\\1&1&-1&-1&1&1&-1&-1\\1&1&-1&-1&1&1&-1&-1\\-1&-1&1&1&-1&-1&1&1\\-1&-1&1&1&-1&-1&1&1\end{pmatrix},$$

$$W_{23}^{*}=\begin{pmatrix}1&-1&-1&1&1&-1&-1&1\\1&-1&-1&1&1&-1&-1&1\\1&-1&1&-1&1&-1&1&-1\\-1&1&1&-1&-1&1&1&-1\\-1&1&1&-1&-1&1&1&-1\end{pmatrix},\ W_{24}^{*}=\begin{pmatrix}1&1&1&1&-1&-1&-1&-1\\1&1&1&1&-1&-1&-1&-1\\1&1&1&1&-1&-1&-1&-1\\-1&-1&-1&-1&1&1&1&1\\-1&-1&-1&-1&1&1&1&1\end{pmatrix}$$

$$W_{25}^{*}=\begin{pmatrix}1&-1&1&-1&-1&1&-1&1\\1&-1&1&-1&-1&1&-1&1\\1&-1&-1&1&-1&1&1&-1\\-1&1&-1&1&1&-1&1&-1\\-1&1&-1&1&1&-1&1&-1\end{pmatrix},\ W_{26}^{*}=\begin{pmatrix}1&1&-1&-1&-1&-1&1&1\\1&1&-1&-1&-1&-1&1&1\\1&1&-1&-1&-1&-1&1&1\\-1&-1&1&1&1&1&-1&-1\\-1&-1&1&1&1&1&-1&-1\end{pmatrix},$$

$$W_{27}^* = \begin{pmatrix} 1\text{-}1\text{-}1\ 1\text{-}1\ 1\ 1\text{-}1 \\ 1\text{-}1\text{-}1\ 1\text{-}1\ 1\ 1\text{-}1 \\ 1\text{-}1\ 1\text{-}1\text{-}1\ 1\text{-}1\ 1 \\ \text{-}1\ 1\ 1\text{-}1\ 1\text{-}1\text{-}1\ 1 \\ \text{-}1\ 1\ 1\text{-}1\ 1\text{-}1\text{-}1\ 1 \end{pmatrix}$$

For third set, after replacing the row having all elements zero of each **W** matrix with second to eighth row of $\mathbf{H}_8$ subsequently and we get,

$$W_{31}^* = \begin{pmatrix} 1\text{-}1\ 1\text{-}1\ 1\text{-}1\ 1\text{-}1 \\ \text{-}1\ 1\text{-}1\ 1\text{-}1\ 1\text{-}1\ 1 \\ 1\text{-}1\text{-}1\ 1\ 1\text{-}1\text{-}1\ 1 \\ \text{-}1\ 1\text{-}1\ 1\text{-}1\ 1\text{-}1\ 1 \\ 1\text{-}1\ 1\text{-}1\ 1\text{-}1\ 1\text{-}1 \end{pmatrix}, W_{32}^* = \begin{pmatrix} 1\ 1\text{-}1\text{-}1\ 1\ 1\text{-}1\text{-}1 \\ \text{-}1\text{-}1\ 1\ 1\text{-}1\text{-}1\ 1\ 1 \\ 1\ 1\text{-}1\text{-}1\ 1\ 1\text{-}1\text{-}1 \\ \text{-}1\text{-}1\ 1\ 1\text{-}1\text{-}1\ 1\ 1 \\ 1\ 1\text{-}1\text{-}1\ 1\ 1\text{-}1\text{-}1 \end{pmatrix},$$

$$W_{33}^* = \begin{pmatrix} 1\text{-}1\text{-}1\ 1\ 1\text{-}1\text{-}1\ 1 \\ \text{-}1\ 1\ 1\text{-}1\text{-}1\ 1\ 1\text{-}1 \\ 1\text{-}1\ 1\text{-}1\ 1\text{-}1\ 1\text{-}1 \\ \text{-}1\ 1\ 1\text{-}1\text{-}1\ 1\ 1\text{-}1 \\ 1\text{-}1\text{-}1\ 1\ 1\text{-}1\text{-}1\ 1 \end{pmatrix}, W_{34}^* = \begin{pmatrix} 1\ 1\ 1\ 1\text{-}1\text{-}1\text{-}1\text{-}1 \\ \text{-}1\text{-}1\text{-}1\text{-}1\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\text{-}1\text{-}1\text{-}1\text{-}1 \\ \text{-}1\text{-}1\text{-}1\text{-}1\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\text{-}1\text{-}1\text{-}1\text{-}1 \end{pmatrix}$$

$$W_{35}^* = \begin{pmatrix} 1\text{-}1\ 1\text{-}1\text{-}1\ 1\text{-}1\ 1 \\ \text{-}1\ 1\text{-}1\ 1\ 1\text{-}1\ 1\text{-}1 \\ 1\text{-}1\text{-}1\ 1\text{-}1\ 1\ 1\text{-}1 \\ \text{-}1\ 1\text{-}1\ 1\ 1\text{-}1\ 1\text{-}1 \\ 1\text{-}1\ 1\text{-}1\text{-}1\ 1\text{-}1\ 1 \end{pmatrix}, W_{36}^* = \begin{pmatrix} 1\ 1\text{-}1\text{-}1\text{-}1\text{-}1\ 1\ 1 \\ \text{-}1\text{-}1\ 1\ 1\ 1\ 1\text{-}1\text{-}1 \\ 1\ 1\text{-}1\text{-}1\text{-}1\text{-}1\ 1\ 1 \\ \text{-}1\text{-}1\ 1\ 1\ 1\ 1\text{-}1\text{-}1 \\ 1\ 1\text{-}1\text{-}1\text{-}1\text{-}1\ 1\ 1 \end{pmatrix},$$

$$W_{37}^* = \begin{pmatrix} 1\text{-}1\text{-}1\ 1\text{-}1\ 1\ 1\text{-}1 \\ \text{-}1\ 1\ 1\text{-}1\ 1\text{-}1\text{-}1\ 1 \\ 1\text{-}1\ 1\text{-}1\text{-}1\ 1\text{-}1\ 1 \\ \text{-}1\ 1\ 1\text{-}1\ 1\text{-}1\text{-}1\ 1 \\ 1\text{-}1\text{-}1\ 1\text{-}1\ 1\ 1\text{-}1 \end{pmatrix}$$

In each set, the 7 **W** matrices are mutually orthogonal to each other i.e. the grand total of all the entries in the Hadamard product of any two distinct **W**-matrices reduces to zero.

From (3.3), we get

$$C = \begin{pmatrix} 38.40 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 38.40 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 38.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 38.40 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 38.40 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 38.40 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 38.40 \end{pmatrix} \text{ and}$$

$\det(\mathbf{C}) = (38.40)^7 = 1.23\times10^{11}$

$\det(\mathbf{I}(\theta)) = 5^8\times1.23\times10^{11} = 4.81\times10^{16}$ (from 3.2)

Any one of the three sets of **W** matrices results the same value of $\det(\mathbf{I}(\theta))$ (= 4.81×10^{16}) which is the unique upper bound as mentioned in 3.4 for the set of seven D-optimal covariates in a CRD having 8 treatments with 5 replications.

4. COVARIATE DESIGNS UNDER NON-REGULAR CASES IN RCBD SET-UP

For two-way layout, the set-up can be written as

$$(\mathbf{Y},\ \mu\mathbf{1}+\mathbf{X}_1\tau+\mathbf{X}_2\beta+\mathbf{Z}\boldsymbol{\gamma},\ \sigma^2\mathbf{I}) \quad (4.1)$$

where μ, as usual, stands for the general effect, τ^{vx1}, β^{bx1} represent vectors of treatment and block effects, respectively, $\mathbf{X}_1^{nxv}$ and $\mathbf{X}_2^{nxb}$ are the corresponding incidence matrices, respectively. **Y** and **Z** as usual, represents an observation vector of order nx1 and the design matrix of order nxc corresponding to vector of covariate effects $\boldsymbol{\gamma}^{cx1}$, respectively.

The information matrix for the whole set of parameters $\eta = (\mu, \tau', \beta', \gamma')'$ underlying a design d with $\mathbf{X}_{1d}$, $\mathbf{X}_{2d}$ and $\mathbf{Z}_d$ as the versions of $\mathbf{X}_1$, $\mathbf{X}_2$ and **Z** in (4.1):

$$I_d(\eta) = \begin{pmatrix} n & 1'X_{1d} & 1'X_{2d} & 1'Z_d \\ & X'_{1d}X_{1d} & X'_{1d}X_{2d} & X'_{1d}Z_d \\ & & X'_{2d}X_{2d} & X'_{2d}Z_d \\ & & & Z'_dZ_d \end{pmatrix} \quad (4.2)$$

Dutta *et al.* (2010b) considered D-optimal design when n=2 (mod 4) under non-regular cases in block design set-up.

The situations may arise, where the conditions n=0 (mod 4) with $\mathbf{Z}'\mathbf{X}_1 \neq \mathbf{0}$ and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$ will exist. In that case, simultaneous estimation of ANOVA parameters and γ-parameters are not possible orthogonally and/or most efficiently. Here, the D-optimality criterion may be considered to give an efficient allocation of treatments and covariates in RCBD set-up. A block design for given b (odd number) and v (such that $\mathbf{H}_v$ exist), the reduced normal equation for estimation of γ is given by following the section 4.4 of Das *et al.* (2015):

$$(\mathbf{Z}'\mathbf{Q}\mathbf{Z})\gamma = \mathbf{Z}'\mathbf{Q}\mathbf{y}$$

$$\hat{\gamma} = (\mathbf{Z}'\mathbf{Q}\mathbf{Z})^{-1}\,\mathbf{Z}'\mathbf{Q}\mathbf{y}$$

Where, $\mathbf{Q} = (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}')$, $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$

Hence, the information matrix for γ is given by

$\mathbf{I}(\gamma) = \mathbf{Z}'\mathbf{Q}\mathbf{Z}$

or, $\mathbf{I}(\gamma) = \mathbf{Z}'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}')\mathbf{Z}$

or, $\mathbf{I}(\gamma) = \mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z}$

or, $\det(\mathbf{I}(\gamma)) = \det(\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z})$

Since **Q** is non-negative definite, it follows that

$\mathbf{Z}'\mathbf{Q}\mathbf{Z} \le \mathbf{Z}'\mathbf{Z}$ (in Lowener order sense; Pukelsheim 1993) and equality comes when $\mathbf{Z}'\mathbf{X}_1 = \mathbf{0}$ and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$.

But in the present situation, i.e., n = 0 (mod 4) with $\mathbf{Z}'\mathbf{X}_1 \neq \mathbf{0}$ and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$, it follows that

$\mathbf{Z}'\mathbf{Q}\mathbf{Z} < \mathbf{Z}'\mathbf{Z}$.

Now, the problem is that of selecting **Z**-matrix with $|z_{ij}^{(t)}| \le 1$ satisfying $\mathbf{Z}'\mathbf{X}_1 \neq \mathbf{0}$ and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$ such that the covariate design will be D-optimal, i.e., $\det(\mathbf{I}(\gamma))$ or $\det(\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z})$ should be maximum when $\mathbf{Z} \in \mathcal{Z}$, $\mathcal{Z} = \{\mathbf{Z}: z_{ij}^{(t)} \in [-1, 1]\ \forall\, i, j\}$. So, the contribution from $\mathbf{Z}'\mathbf{Z}$ should be maximum and contribution from the part of $\mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z}$ should be minimum.

4.1 Conditions for D-optimality

We have already observed that when n=0 (mod 4) with $\mathbf{Z}'\mathbf{X}_1 \neq \mathbf{0}$ and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$, it is impossible to estimate γ-components most efficiently in the sense of attaining the lower bound σ^2/n to the variance of the estimated covariate parameters. Thus, in the above case, the first problem is that of choosing a matrix $\mathbf{Z}^{n\times c} = (z_{ij}^{(t)})$ with $z_{ij}^{(t)} \in [-1, 1]\ \forall\, i, j$ such that the $\det(\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z})$ is maximum subject to $\mathbf{Z}'\mathbf{X}_1 \neq \mathbf{0}$ and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$. A necessary condition for maximization of $\det(\mathbf{Z}'\mathbf{Z})$ where $\mathbf{Z} \in \mathcal{Z}$, is that $z_{ij}^{(t)} = \pm 1\ \forall\, i, j, t$ (Lemma 4.4.1 of Das *et al.*, 2015). Based on the necessary condition, we can restrict to the class $\mathcal{Z}^* = \{\mathbf{Z}: z_{ij}^{(t)} = \pm 1\ \forall\, i, j, t\}$ for finding D-optimum design.

Theorem 4.1: A covariate design $\mathbf{Z}^* \in \mathcal{Z}^*$ is D-optimal in the sense of maximizing $\det(\mathbf{Z}'\mathbf{Z})$ subject to the condition $\mathbf{Z}'\mathbf{X}_1 \neq \mathbf{0}$ and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$, if it satisfies $\mathbf{Z}^{*\prime}\mathbf{Z}^* = n\mathbf{I}_c$ and

$$a_{lm} = \pm 1, \text{ where } a_{lm} \text{ be the elements of } Z'X_1,$$
$$l = 1, 2, ..., c \text{ and } m = 1, 2, ..., v.$$

Proof: Based on the necessary condition, we can restrict to the class $\mathcal{Z}^*$ for maximization of $\det(\mathbf{Z}'\mathbf{Z})$. For any $\mathbf{Z} \in \mathcal{Z}^*$, we can write

$$\det(\mathbf{Z}'\mathbf{Z}) = \det\begin{pmatrix} n & s_{12} & \dots & s_{1c} \\ s_{12} & n & \dots & s_{2c} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s_{1c} & s_{2c} & \dots & n \end{pmatrix}$$

where, $s_{tt'} = \sum_i \sum_j z_{ij}^{(t)} z_{ij}^{(t')}$, $t \neq t' = 1, 2, \dots, c$. The $\det(\mathbf{Z}'\mathbf{Z})$ will be maximum whenever it is possible to construct $\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_c$ i.e. the covariates are mutually orthogonal to each other. So, all off-diagonal elements of $\mathbf{Z}'\mathbf{Z}$ can be zero. As the elements of $\mathbf{Z}'\mathbf{X}_1$ will be either +1 or -1 and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$, the contribution from the part of $\mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z}$ will be minimum. Thus, $\det(\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Z})$ will be maximum and the theorem is proved.

Now, we can represent any column of **Z*** in the form of a matrix **U*** of order vxb corresponding to the incidence matrix of the block design.

With the conditions $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$ and $\mathbf{Z}^{*\prime}\mathbf{Z}^* = n\mathbf{I}_c$, in addition $\mathbf{Z}'\mathbf{X}_1$ is such a matrix with all elements either +1 or -1, thus, in terms of **U*** matrix, the conditions reduce to:

$\mathbf{C}_1$: Each **U*** matrix has all column-sums equal to zero;

$\mathbf{C}_2$: Each **U*** matrix has all row-sums equal to either +1 or -1;

$\mathbf{C}_3$: The grand total of all the entries in the Hadamard product of any two distinct **U*** matrices reduces to zero.

Theorem 4.2: Under the realization of the conditions n = 0 (mod 4) with $\mathbf{Z}'\mathbf{X}_1 \neq \mathbf{0}$ and $\mathbf{Z}'\mathbf{X}_2 = \mathbf{0}$, then a set of c (= v – 1) orthogonal covariates will exist which ultimately produces a D-optimal covariate design in a RCBD set-up with v number of treatments for b odd number of blocks, provided $\mathbf{H}_v$ and $\mathbf{H}_{b-1}$ exist.

Proof (by construction)**:** For construction of the D-optimal covariate design in a Randomized Complete Block Design set-up with v number of treatments for b odd number of replications, we follow the steps given below.

Step 1. Let us consider a Hadamard matrix of order v, $\mathbf{H}_v$.

$\mathbf{H}_v = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_{v-1})$

Step 2. Let us construct a Special Array $\mathbf{H}_b$* of order b from $\mathbf{H}_{b-1}$ with one row and column with all zero elements in middle, i.e. ($\mathbf{1}^*$, $\mathbf{h}_1^*$, $\mathbf{h}_2^*$,…, $\mathbf{h}_{(b-1)/2-1}^*$, $\mathbf{0}$, $\mathbf{h}_{(b-1)/2}^*$,…, $\mathbf{h}_{b-2}^*$).

$$H_b^* = \begin{pmatrix} 1\ldots\ 1\ 0\ 1\ldots1 \\ \ldots\ldots\ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ 1\ldots\ -1\ 0\ -1\ldots1 \\ 0\ldots\ 0\ 0\ 0\ldots0 \\ 1\ldots\ 1\ 0\ -1\ldots-1 \\ \ldots\ldots\ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ \ldots\ldots\ 0\ \ldots\ldots \\ 1\ldots\ -1\ 0\ 1\ldots-1 \end{pmatrix}$$

Step 3. Using $\mathbf{H}_v$ and $\mathbf{H}_b$*, by Kronecker product of these two matrices, we get (b-2) set of (v-1) $\mathbf{U}_{ij}$* matrices of order vxb (without consider the first row and the row with all zeros), where i=1, 2,…, (v-1) and j=1, 2,…, (b-2). In each of the $\mathbf{U}_{ij}$* matrix there is one column with all elements zero in the middle.

$$U_{ij}^* = h_i \otimes h_j^{*\prime}\ ,\ \otimes \text{ denotes the Kronecker product}$$

Step 4. According to Das *et al.* (2015), maximization of the determinant of $\mathbf{Z}'\mathbf{Z}$, can be achieved by replacing the zero elements in $\mathbf{U}_{ij}$* matrix either by -1 or +1. In each of (b-2) sets, let us replace the zero elements of $\mathbf{U}_{ij}$* matrix by (i+1)th column of $\mathbf{H}_v$. In that way, we get (b-2) sets of (v-1) mutually orthogonal $\mathbf{U}_{ij}$* matrices. The matrix of orthogonal optimal covariates, $\mathbf{Z}$ of order nxc can be constructed from the (v-1) $\mathbf{U}_{ij}$* matrices in any one of j sets. It has been verified that any set of c (= v – 1) orthogonal covariates ultimately produces a D-optimal covariate design in a RCBD set-up satisfying the conditions $\mathbf{C}_1$ to $\mathbf{C}_3$ simultaneously.

Example 4.1: Let us consider a RCBD with v = 8 and b = 3. The Special Array of order 3 is

$$H_3^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

The Hadamard matrix of order 8 is

$$H_8 = \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 1&-1&-1&1&1&-1&-1&1 \\ 1&1&-1&-1&1&1&-1&-1 \\ 1&-1&1&-1&1&-1&1&-1 \\ 1&1&1&1&-1&-1&-1&-1 \\ 1&-1&-1&1&-1&1&1&-1 \\ 1&1&-1&-1&-1&-1&1&1 \\ 1&-1&1&-1&-1&1&-1&1 \end{pmatrix}$$

The one set of 7$\mathbf{U}$* matrices have been constructed using $\mathbf{H}_8$ and $\mathbf{H}_3$*, by Kronecker product of these two matrices.

The set of 7$\mathbf{U}$* matrices are

$$U_{11}^* = \begin{pmatrix} 1&0&-1 \\ -1&0&1 \\ 1&0&-1 \\ -1&0&1 \\ 1&0&-1 \\ -1&0&1 \\ 1&0&-1 \\ -1&0&1 \end{pmatrix}, U_{21}^* = \begin{pmatrix} 1&0&-1 \\ -1&0&1 \\ -1&0&1 \\ 1&0&-1 \\ 1&0&-1 \\ -1&0&1 \\ -1&0&1 \\ 1&0&-1 \end{pmatrix}, U_{31}^* = \begin{pmatrix} 1&0&-1 \\ 1&0&-1 \\ -1&0&1 \\ -1&0&1 \\ 1&0&-1 \\ 1&0&-1 \\ -1&0&1 \\ -1&0&1 \end{pmatrix},$$

$$U_{41}^* = \begin{pmatrix} 1&0&-1 \\ 1&0&-1 \\ 1&0&-1 \\ 1&0&-1 \\ -1&0&1 \\ -1&0&1 \\ -1&0&1 \\ -1&0&1 \end{pmatrix}, U_{51}^* = \begin{pmatrix} 1&0&-1 \\ -1&0&1 \\ 1&0&-1 \\ -1&0&1 \\ -1&0&1 \\ 1&0&-1 \\ -1&0&1 \\ 1&0&-1 \end{pmatrix}, U_{61}^* = \begin{pmatrix} 1&0&-1 \\ -1&0&1 \\ -1&0&1 \\ 1&0&-1 \\ -1&0&1 \\ 1&0&-1 \\ 1&0&-1 \\ -1&0&1 \end{pmatrix},$$

$$U_{71}^* = \begin{pmatrix} 1&0&-1 \\ 1&0&-1 \\ -1&0&1 \\ -1&0&1 \\ -1&0&1 \\ -1&0&1 \\ 1&0&-1 \\ 1&0&-1 \end{pmatrix}$$

In this set, after replacing the column having all elements zero of each $\mathbf{U}$* matrix with second to eighth column of $\mathbf{H}_8$ subsequently, we get,

$$U_{11}^{*}=\begin{pmatrix} 1 & 1 & -1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ -1 & -1 & 1 \end{pmatrix},\ U_{21}^{*}=\begin{pmatrix} 1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1 \end{pmatrix},\ U_{31}^{*}=\begin{pmatrix} 1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & 1 \end{pmatrix},$$

$$U_{41}^{*}=\begin{pmatrix} 1 & 1 & -1\\ 1 & 1 & -1\\ 1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ -1 & -1 & 1 \end{pmatrix},\ U_{51}^{*}=\begin{pmatrix} 1 & 1 & -1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ 1 & 1 & -1 \end{pmatrix},\ U_{61}^{*}=\begin{pmatrix} 1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1 \end{pmatrix},$$

$$U_{71}^{*}=\begin{pmatrix} 1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & -1 \end{pmatrix}$$

In this set, the 7 **U*** matrices are mutually orthogonal to each other i.e., the grand total of all the entries in the Hadamard product of any two distinct **U*** matrices reduces to zero and all column sums equal to zero and all row sums equal to either +1 or -1 in each **U*** matrix. These orthogonal covariates ultimately produce a D-optimal covariate design in a RCBD set-up satisfying the conditions $\mathbf{C}_1$ to $\mathbf{C}_3$ simultaneously.

In this example, we can also verify that $\mathbf{Z'X}_1 \neq \mathbf{0}$, $\mathbf{Z'X}_2 = \mathbf{0}$, $\mathbf{Z'Z} = n\mathbf{I}_c$, $\mathbf{Z'QZ} < \mathbf{Z'Z}$; where $\mathbf{Q} = (\mathbf{I} - \mathbf{X}(\mathbf{X'X})^{-}\mathbf{X'})$, $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ and $\det(\mathbf{I}(\gamma))$, which is unique maximum for the set of seven D-optimal covariates. In this case, $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{Z}$ are the following:

$$X_1=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix},\ X_2=\begin{pmatrix} 1&0&0\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 0&1&0\\ 0&1&0\\ 0&1&0\\ 0&1&0\\ 0&1&0\\ 0&1&0\\ 0&1&0\\ 0&1&0\\ 0&0&1\\ 0&0&1\\ 0&0&1\\ 0&0&1\\ 0&0&1\\ 0&0&1\\ 0&0&1\\ 0&0&1 \end{pmatrix},$$

$$Z=\begin{pmatrix} 1&1&1&1&1&1&1\\ -1&-1&1&1&-1&-1&1\\ 1&-1&-1&1&1&-1&-1\\ -1&1&-1&1&-1&1&-1\\ 1&1&1&-1&-1&-1&-1\\ -1&-1&1&-1&1&1&-1\\ 1&-1&-1&-1&-1&1&1\\ -1&1&-1&-1&1&-1&1\\ 1&1&1&1&1&1&1\\ -1&-1&1&1&-1&-1&1\\ 1&-1&-1&1&1&-1&-1\\ -1&1&-1&1&-1&1&-1\\ 1&1&1&-1&-1&-1&-1\\ -1&-1&1&-1&1&1&-1\\ 1&-1&-1&-1&-1&1&1\\ -1&1&-1&-1&1&-1&1\\ -1&-1&-1&-1&-1&-1&-1\\ 1&1&-1&-1&1&1&-1\\ -1&1&1&-1&-1&1&1\\ 1&-1&1&-1&1&-1&1\\ -1&-1&-1&1&1&1&1\\ 1&1&-1&1&-1&-1&1\\ -1&1&1&1&1&-1&-1\\ 1&-1&1&1&-1&1&-1 \end{pmatrix}$$

Now, we can easily find out $\mathbf{Z'X}_1$, $\mathbf{Z'X}_2$ and $\mathbf{Z'Z}$. These are given below:

$$Z'X_1 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix},$$

$$Z'X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Z'Z = \begin{pmatrix} 24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 24 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{pmatrix}$$

So, the information matrix for γ is $\mathbf{I}(\gamma) = \mathbf{Z'QZ} = \mathbf{Z'Z} - \mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z} =$

$$\begin{pmatrix} 21.33334 & 0 & 0 & 0 & -1.1E-16 & 0 & 0 \\ 0 & 21.33334 & 0 & 0 & 0 & -1.1E-16 & 0 \\ -1.1E-16 & 1.11E-16 & 21.3333 & -5.6E-17 & 0 & 0 & -2E-16 \\ -1.1E-16 & 1.11E-16 & -5.6E-17 & 21.3333 & 0 & 0 & 5.6E-17 \\ -1.1E-16 & 0 & 0 & 0 & 21.3333 & 0 & 0 \\ 0 & -1.1E-16 & 0 & 0 & 0 & 21.3333 & 0 \\ 0 & 0 & -1.7E-16 & -1.7E-16 & -1.1E-16 & 1.1E-16 & 21.3333 \end{pmatrix} \cong$$

$$\begin{pmatrix} 21.33 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 21.33 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 21.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 21.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 21.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 21.33 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 21.33 \end{pmatrix}$$

The determinant value of the information matrix for γ i.e., $\det(\mathbf{I}(\gamma)) = (21.33)^7 = 2008796709$, which is unique maximum for the set of seven D-optimal covariates in a RCBD with 8 treatments in 3 blocks.

Theorem 4.3: Under the realization of the conditions n = 0 (mod 4) with $\mathbf{Z'X}_1 \neq \mathbf{0}$ and $\mathbf{Z'X}_2 = \mathbf{0}$, then a set of c (= v – 1) orthogonal covariates will exist which ultimately produces a D-optimal covariate design in a RCBD set-up with v number of treatments for b odd number of blocks, provided $\mathbf{H}_v$ and $\mathbf{H}_{b-r}$ exists such that r(> 1) is an odd number and $\mathbf{H}_{r-1}$ exists where r is the number of rows and columns in the SA with all elements zero.

Proof (by construction)**:** For construction of the D-optimal covariate design in a Randomized Complete Block Design set-up with v number of treatments for b odd number of replications, we follow the steps given below.

Step 1. Let us consider a Hadamard matrix of order v, $\mathbf{H}_v$.

$$\mathbf{H}_v = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{v-1})$$

Step 2. Let us construct a Special Array $\mathbf{H}_b^*$ of order b from $\mathbf{H}_{b-r}$ with r rows and columns with all zero elements in middle, i.e. $(\mathbf{1}^*, \mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{(b-r)/2-1}^*, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{h}_{(b-r)/2}^*, \ldots, \mathbf{h}_{b-r-1}^*)$.

$$H_b^* = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & -1 & -1 \\ 1 & -1 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$$

Step 3. Using $\mathbf{H}_v$ and $\mathbf{H}_b^*$, by Kronecker product of these two matrices, we get (b-r-1) set of (v-1) U_{ij}^{**} matrices of order vxb (without consider the first row and the r rows with all zeros), where i=1,2,…,(v-1) and j=1,2,…,(b-r-1). In each of the $\mathbf{U}_{ij}^{**}$ matrix there are r columns with all elements zero in the middle.

$$U_{ij}^{**} = h_i \otimes h_j^{*\prime}, \; \otimes \text{ denotes the Kronecker product}$$

Step 4. As $\mathbf{H}_v$ and $\mathbf{H}_{r-1}$ both are exists, based on the Theorem 4.2, a set of c (= v – 1) orthogonal covariates will exist which ultimately produces a D-optimal covariate design in a RCBD set-up with v number of treatments for r odd number of replications. The constructed matrix of order vxr is denoted as $\mathbf{U}_{ij}^{***}$, where, i=1, 2,…, (v-1) and j=1, 2,…, (r-2).

Step 5. In each set, insert the first $\mathbf{U}^{***}$ matrix of order vxr in the U_{11}^{**} matrix, such that all the r columns with all elements zero in the middle has been replaced by +1 or -1. Let the resulting matrix be $\mathbf{U}_{11}^*$. Repeat the procedure with other $\mathbf{U}^{***}$ matrices in the remaining $\mathbf{U}_{ij}^{**}$ matrices in the same set. Thus we get (v-1)$\mathbf{U}_{ij}^*$ matrices of order vxb which are orthogonal to each other. Finally, the desired (v-1) $\mathbf{U}^*$ matrices of order vxb ultimately produces a D-optimal covariate design in a Randomized Complete Block Design set-up with v number of treatments for b odd number of replications satisfying the conditions $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$ simultaneously.

For easy understanding of the above steps, the following example will be useful.

Example 4.2: Let us consider a RCBD with v = 8 and b = 7. When r=3, the 7**U***matrices are given below:

Step 1. Let us consider a Hadamard matrix of order 8, $\mathbf{H}_8$.

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

Step 2. Let us construct a Special Array of order 7 from $\mathbf{H}_4$ with 3 rows and columns with all zero elements in the middle, $\mathbf{H}_7^*$ i.e. $(\mathbf{1^*}, \mathbf{h}_1^*, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{h}_2^*, \mathbf{h}_3^*)$.

$$H_7^* = \left(1^*, h_1{}^*, 0, 0, 0, h_2{}^*, h_3{}^*\right) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Step 3. By using $\mathbf{H}_8$ and $\mathbf{H}_7^*$, by Kronecker product of these two matrices, we get 3 set of 7 $\mathbf{U}_{ij}^*$ matrices of order 8x7 where i=1,2,…,7 and j=1,2,3. In each of the $\mathbf{U}_{ij}^{**}$ matrix there are 3 columns with all elements zero in the middle. First matrix of first set $\mathbf{U}_{11}^{**}$ is the following.

$$U_{11}^{**} = h_1 \otimes h{*}'_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Similarly, we can easily construct the others.

Step 4. As $\mathbf{H}_8$ and $\mathbf{H}_2$ exists, based on the Theorem 4.2, a set of c (= 7) orthogonal covariates will exist which ultimately produces a D-optimal covariate design in a Randomized Complete Block Design set-up with 8 number of treatments for 3 odd number of replications. The constructed matrix of order 8x3 is denoted as $\mathbf{U}_{ij}^{***}$, where, i=1, 2,…, 7 and j=1.

$$U_{11}^{***} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, U_{21}^{***} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, U_{31}^{***} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix},$$

$$U_{41}^{***} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, U_{51}^{***} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, U_{61}^{***} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

$$U_{71}^{***} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

Step 5. In each set, insert the first $\mathbf{U}_{11}^{***}$ matrix of order 8x3 in the $\mathbf{U}_{11}^{**}$ matrix, such that all the 3 columns with all elements zero in the middle has been replaced by either +1 or -1. Let the resulting matrix be $\mathbf{U}_{11}^{**}$. Repeat the procedure with other $\mathbf{U}^{***}$ matrices in the remaining $\mathbf{U}_{ij}^{**}$ matrices in the same set. Thus we get 7 $\mathbf{U}_{ij}^*$ matrices of order 8x7 which are orthogonal to each other. Here, $\mathbf{U}_{11}^{***}$ matrix is inserted in $\mathbf{U}_{11}^{**}$ matrix and we get the following matrix $\mathbf{U}_{11}^*$.

$$U^*_{11} = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Similarly, we can find out the others. Here, from each set, finally, the desired 7**U*** matrices of order 8x7 ultimately produces a D-optimal covariate design in a Randomized Complete Block Design set-up with 8 number of treatments for 7 odd number of replications satisfying the conditions $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$ simultaneously.

For this RCBD with v=8 and b=7, the other possible alternative is for 7 D-optimum covariates, use SA of order 7 with r=5.

Corollary 4.1: The D-optimal covariate design in RCBD developed by theorem 4.3 is true for CRD with similar v and b (number of replications).

Proof: Straight forward from the definition of CRD.

5. CONCLUSION

New D-optimal covariate designs in CRD and RCBD set ups have been presented in section 3 and 4. In Theorem 3.1 and Theorem 4.2, if $\mathbf{H}_v$ and $\mathbf{H}_{b-1}$ exist, then c=(v – 1) D-optimal covariates will exist in a CRD and RCBD set ups with v number of treatments for b odd number of replications (or blocks). In Theorem 4.3, if $\mathbf{H}_v$ and $\mathbf{H}_{b-r}$ exists, then c=(v – 1) D-optimal covariates will exist in a RCBD set up with v number of treatments for b odd number of blocks provided r (> 1) is a odd number and $\mathbf{H}_{r-1}$ exist. The developed D-optimal covariate designs under the above three Theorems are not available in the present day literature. The designs with optimal covariates can be applied in practical situation under ANCOVA model. Interested readers may follow Sinha (2009) and Das *et al.* (2015) for the purpose of practical application.

REFERENCES

Das, K., Mandal, N.K. and Sinha, B.K. (2003). Optimal experimental designs with covariates. *J. Stat. Plan. Inference*, **115**, 273-285.

Das, P., Dutta, G., Mandal, N.K. and Sinha, B.K. (2015). *Optimal covariate design-Theory and applications*. Springer, India.

Das, H., Majumder, A., Kumar, M. and Nishad, D. (2020). New Series of Optimal Covariate Designs in CRD and RCBD set-ups. *J. Ind. Soc.Agric. Stat.*, **74(1)**, 41-50.

Dey, A. and Mukerjee, R. (2006). D-optimal designs for covariate models. *Statistics,* **40**, 297-305.

Dutta, G. (2004). Optimum choice of covariates in BIBD set-up. *Cal. Stat. Assoc. Bull*., **55**, 39-55.

Dutta, G. (2009). *Optimum designs for covariates models*. (Unpublished Ph.D. Thesis submitted to the University of Kalyani, Kalyani, West Bengal)

Dutta, G., Das, P. and Mandal, N.K. (2007). Optimum choice of covariates for a series of SBIBDs obtained through projective geometry. *J. Modern Appld. Stat. Methods*, **6**, 649-656.

Dutta, G., Das, P. and Mandal, N.K. (2009a). Optimum covariate designs in split-plot and strip-plot design set-ups. *J. Appld. Stat*., **36**, 893-906.

Dutta, G., Das, P. and Mandal, N.K. (2009b). Optimum covariate designs in partially balanced incomplete block (PBIB) design set-ups. *J. Stat. Plan. Inference,* **139**, 2823-2835.

Dutta, G., Das, P. and Mandal, N.K. (2010a). Optimum covariate designs in binary proper equireplicate block design set-up. *Discrete Mathematics,* **310**, 1037-1049.

Dutta, G., Das, P. and Mandal, N.K. (2010b). D-optimal designs for covariate parameters in block design set-up. *Commun. Stat. Theory Methods*, **39**, 3434-3443.

Dutta, G., Das, P. and Mandal, N.K. (2014). D-Optimal designs for covariate models. *Commun. Stat. Theory Methods,* **43**, 165-174.

Pukelsheim F. (1993). *Optimal design of experiments*. Wiley, New York.

Rao, P.S.S.N.V.P.., Rao, S.B., Saha, G.M. and Sinha, B.K. (2003). Optimal designs for covariates' models and mixed orthogonal arrays. Electron Notes. *Discret. Math*., **15**, 157-160.

Sinha, B.K. (2009). A reflection on the choice of covariates in the planning of experimental designs. *J. Indian Soc. Agric. Stat*., **63(3)**, 219-225.

Troya, Lopes, J. (1982a). Optimal designs for covariate models. *J. Stat. Plan. Inference*, **6**, 373-419.

Troya, Lopes, J. (1982b). Cyclic designs for a covariate model. *J. Stat. Plan. Inference,* **7**, 49-75.