Resolvable Dichotomized Split-Set Partially Balanced Incomplete Block Designs

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SUMMARY

A four-associate class association scheme named as Dichotomized Split-Set (DiSS) association scheme is defined for $v = 2(p-1)p$ number of treatments and a method for constructing Partially Balanced Incomplete Block (PBIB) designs based on this association scheme is developed. The proposed designs are cost effective in terms of resources as they require lesser replications. They are resolvable; hence they possess high application potential in areas like multi-site varietal trials where experimenters generally prefer incomplete block designs. The efficiency factors for these designs are computed in comparison to an orthogonal block design and are found to be quite high. For easy generation of these designs for any $v = 2p(p - 1)$; $p \geq 3$, an R package called "ResPBIBD" is developed.

Keywords: Association scheme; Resolvable; Canonical efficiency factor; Partially balanced incomplete block designs.

1. INTRODUCTION

Agricultural experimentation often involves varietal trials with a large number of treatments to be tested across multiple locations. However, experimental resources may not permit a large number of replications of the newly developed varieties. Further, when location or batch effects need to be studied, a resolvable block design is an option. Because replication is also taken into account when analysing the data, experimenters can obtain additional information on spatial or temporal variation present in the experimental units by using such a design. Again, Incomplete Block Designs (IBDs) with minimum number of replications are highly relevant from a research perspective.

The majority of IBDs used in practise are either Balanced Incomplete Block (BIB) designs or two-associate class Partially Balanced Incomplete Block (PBIB) designs (Bose and Nair, 1939), both of which have been extensively investigated in the literature. PBIB(3) and PBIB(4) designs are useful alternatives

to these designs because they can be used in situations where BIB and PBIB(2) designs are either unavailable or require a significant amount of resources.

Some well-known higher associate class PBIB designs, that are extensively studied in literature, are nested group divisible designs (Roy, 1953 and Raghavarao, 1960), cubic designs (Raghavarao and Chandrasekhararao, 1964), right angular designs [Tharthare (1963, 1965)], extended triangular designs (John, 1966) and circular designs (Das, 1960 and Saha *et al.* 1974) with varying association schemes.

The simple lattice designs developed by Yates are among the earliest resolvable incomplete block designs (1936). With two replications, Bose and Nair (1962) discussed the construction of a more general class of these designs. Rao (1956), Patterson and Williams (1976) and Williams *et al.,* (1976) described construction procedures for obtaining some important classes of resolvable designs. A general method for constructing a series of resolvable PBIB(3) designs

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with two replications was discussed by Varghese and Sharma (2004). A special kind of resolvable designs called α-designs was given by Patterson and Williams (1976) to meet up the requirements of the experimenter. Parsad *et al.* (2007) brought forward a monograph on α-designs in which construction, randomization, A-and D-efficiency computations and analysis of α-designs are detailed.

In the present study, a new series of resolvable PBIB(4) designs are constructed, and their explicit characterization properties are discussed along with their association scheme. An R-package for generating these designs has also been developed that increases the adaptability of these designs.

2. DEFINITIONS

2.1 Partially balanced incomplete block [PBIB(4)] designs

Following Bose and Nair (1939), an incomplete block design for v treatments is said to be four-associate class PBIB design, if the experimental material can be divided into b blocks of sizes k $(< v)$ such that:

- each of the treatments occurs in r blocks,

- there exists an abstract relation between treatments satisfying the following:

  a) two treatments are either 1st or 2nd or 3rd or 4th associates, the relation of association being symmetrical *i.e.,* if treatment α is the ith associate of β, then β is also the $i^{th} (i = 1, 2, 3, 4)$ associate of α,
  b) each treatment has exactly n_i, i^{th} associates,
  c) given any two treatments that are mutually i^{th} associates, the number of treatments common to the jth associates of the first and k^{th} associates of the second is $p_{jk}^i (i, j, k = 1, 2, 3, 4)$ and

- two treatments that are mutually i^{th} associates occur together in exactly λ_i blocks.

The parametric relationships to be satisfied by the four-class association schemes as well as designs are as given:

(i) $\sum\limits_{i=1}^{4} n_i = v - 1$

(ii) $\sum\limits_{k=1}^{4} p_{jk}^i = n_j - \delta_{ij}; \; \delta_{ij} = \begin{cases} 0, & i \neq j = 1, 2, 3, 4 \\ 1, & i = j = 1, 2, 3, 4 \end{cases}$

(iii) $n_i p_{jk}^i = n_j p_{ik}^j = n_k p_{ij}^k; \; i, j, k = 1, 2, 3, 4$

(iv) $vr = bk$

(v) $\sum\limits_{i=1}^{4} n_i \lambda_i = r(k-1)$

2.2 Resolvable designs

If the b blocks of a design having block size k can be divided into r sets such that each set contains a complete replication of treatments, then the design is said to be resolvable.

3. MODEL AND EXPERIMENTAL SET-UP

The proposed resolvable DiSS designs can be analysed by considering the following nested model:

$$y_{il(j)} = \mu + \tau_i + \gamma_j + \beta_{l(j)} + e_{il(j)} \qquad (1)$$

where, $i = 1, 2, ..., v, \; j = 1, 2, ..., r$ and $l = 1, 2, ..., b$. Here, $y_{il(j)}$ is the response of ith treatment belonging to block l, nested within j^{th} replication, μ is the general mean, τ_i is the effect of the ith treatment, γ_j is the effect of jth replicate, $\beta_{l(j)}$ is the effect of lth block belonging to the jth replicate and $e_{il(j)}$ is the random error component which follows identically and independently distributed following N(0, σ^2).

However, to study the properties of PBIB designs, a simple block design model may be considered, which is written as follows:

$$\mathbf{y} = \mu\mathbf{1} + \Delta_1\boldsymbol{\tau} + \Delta_2\boldsymbol{\beta} + \boldsymbol{\epsilon} \qquad (2)$$

where $\mathbf{y}$ is the $(n \times 1)$ vector of observations, μ is the general mean, $\boldsymbol{\tau}$ is the $(v \times 1)$ vector of treatment effects, $\boldsymbol{\beta}$ is the $(b \times 1)$ vector of block effects, $\mathbf{1}$ is the $(n \times 1)$ vector of unities, Δ_1 and Δ_2 are the $(n \times v)$ observations vs treatments and $(n \times b)$ observations vs blocks design matrices, respectively and $\boldsymbol{\epsilon}$ is the $(n \times 1)$ vector of errors follows *iid* $N(0, \sigma^2 \mathbf{I}_i)$, where, $\mathbf{I}$ is the identity matrix of order n. Evidently,

$$\Delta_1 = ((d_{zi})) = \begin{cases} 1, & \text{if } z^{th} \text{observation comes from } i^{th} \text{treatment} \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta_2 = ((u_{zl})) = \begin{cases} 1, & \text{if } z^{th} \text{observation comes from } l^{th} \text{block} \\ 0, & \text{otherwise} \end{cases}$$

where, $z = 1, 2, ..., n$, $i = 1, 2, ..., v$ and $l = 1, 2, ..., b$.

The reduced normal equations pertaining to treatment effects under the above model is $\mathbf{C}\hat{\boldsymbol{\tau}} = \mathbf{Q}$

with $\mathbf{1'\hat{\tau}} = 0$, where $\mathbf{C}(= r\mathbf{I} - \mathbf{N}\mathbf{K}^{-1}\mathbf{N'})$, and $\mathbf{Q}$ [$= (Q_1, Q_2, ..., Q_v)'$] are the information matrix and vector of adjusted treatment totals, respectively. Here, $\mathbf{N}$ is the incidence matrix of the design of order ($v \times b$) and $\mathbf{K}$ is the diagonal matrix of block size. It has been noted that

$$\mathbf{NN'} = ((\, n_{ii'}\,))$$

$$= \begin{cases} r, & \text{if } i = i' (= 1, 2, ..., v) \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 & \text{if } i \neq i' \text{ are 1}^{st} \text{ or 2}^{nd} \text{ or 3}^{rd} \text{ or} \\ & \quad 4^{th} \text{ associates, respectively} \end{cases}$$

The normal equations can be solved to get $\hat{\tau} = \mathbf{C^- Q}$, where $\mathbf{C^-}$ is the generalized inverse of $\mathbf{C}$ and $\mathbf{p'\hat{\tau}}$ be an estimable contrast of treatment effects such that $\mathbf{p'}\mathbf{1} = 0$. The best linear unbiased estimator of $\mathbf{p'}\tau$ is $\mathbf{p'\hat{\tau}} = \mathbf{p'C^-Q}$ with $V(\mathbf{p'\hat{\tau}}) = \sigma^2 \mathbf{p'C^-p}$. The treatment sum of squares (adjusted for block effects) is $\mathbf{Q'C^-Q}$ with $(v - 1)$ degrees of freedom (d.f.), the block sum of squares (unadjusted) with $(b - 1)$ d.f. and the error sum of squares with $(n - vb + v + b - 1)$ d.f. can be obtained in the usual manner.

4. DiSS ASSOCIATION SCHEME

Let $v = 2p(p - 1)$; $p \geq 3$, where $V = \{1, 2, 3, ..., v\}$ is the set of treatments. Consider two subsets of V, *viz.,* R and S, where,

$R = \{\, r_h | h = 1, 2, ..., (p - 1)\, \}$, $r_h = (h - 1)p + 1, (h - 1)p + 2, ..., hp$ and

$S = \{\, s_h | h = 1, 2, ..., (p - 1)\, \}$, $s_h = p^2 + p(h - 2) + 1, p^2 + p(h - 2) + 2, ..., p^2 + p(h - 1)$.

Clearly, R, S, r_h and s_h satisfies the following:

$R \cap S = \varnothing$ (a null set), $R \cup S = V$, and

$n(R) = n(S) = p(p - 1)$, $n(r_h) = p = n(s_h)$, where $n(.)$ denotes cardinality of set.

Association scheme is defined as:

For only $\theta \in V$, $\theta \in r_h$ or s_h.

Let $\theta \in r_h$, the association scheme is defined as:

- first associates of θ are remaining elements of r_h
- second associates of θ are elements corresponding to subsets s_h
- third associates of θ are $\bigcup_{i \neq h = 1}^{p-1} r_i$
- fourth associates of θ are $\bigcup_{i \neq h = 1}^{p-1} s_i$.

The parameters of this association scheme are obtained as: $v = 2(p - 1)p$, $n_1 = p - 1$, $n_2 = p$, $n_3 = p(p - 2)$, $n_4 = p(p - 2)$,

$$\mathbf{P}_1 = \begin{bmatrix} p\text{-}2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p(p\text{-}2) & 0 \\ 0 & 0 & 0 & p(p\text{-}2) \end{bmatrix},$$

$$\mathbf{P}_2 = \begin{bmatrix} 0 & p\text{-}1 & 0 & 0 \\ p\text{-}1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p(p\text{-}2) \\ 0 & 0 & p(p\text{-}2) & 0 \end{bmatrix},$$

$$\mathbf{P}_3 = \begin{bmatrix} 0 & 0 & p\text{-}1 & 0 \\ 0 & 0 & 0 & p \\ p\text{-}1 & 0 & p(p\text{-}3) & 0 \\ 0 & p & 0 & p(p\text{-}3) \end{bmatrix},$$

$$\mathbf{P}_4 = \begin{bmatrix} 0 & 0 & 0 & p\text{-}1 \\ 0 & 0 & p & p \\ 0 & p & 0 & p(p\text{-}3) \\ p\text{-}1 & 0 & p(p\text{-}3) & 0 \end{bmatrix}.$$

5. METHODS OF CONSTRUCTION OF Diss DESIGNS

Consider again the arrangement of $v = 2(p - 1)p$, where $(p \geq 3)$, treatments as arranged in Section 4. Then, the method of construction of the proposed designs is explained in the following four steps:

Step 1: Each block of first group of $\binom{p-1}{2}$ blocks are obtained by considering $r_h \cup r_{h'} \forall h > h' = 1, 2, ..., (p - 1)$.

Step 2: Each block of second group of $\binom{p-1}{2}$ blocks are obtained by considering $s_h \cup s_{h'} \forall h > h' = 1, 2, ..., (p - 1)$.

Step 3: Another group of $(p - 1)$ blocks are obtained where each block is formed by considering $r_h \cup s_h \forall h = 1, 2, ..., (p - 1)$.

Step 4: Finally, $(p - 1)^2$ blocks obtained in Steps 1-3 are grouped into $(p - 1)$ replications such that every treatment appears exactly once in each replication.

The designs obtained are resolvable PBIB(4) designs with parameters $v = 2(p-1)p$, $b = (p-1)^2$, $r = p-1$, $k = 2p$, $\lambda_1 = p-1$, $\lambda_2 = \lambda_3 = 1$, $\lambda_4 = 0$.

Information Matrix and variance estimates: The general expression of information matrix (C) pertaining to treatment effects for developed resolvable PBIB(4) design can be written as:

$$C = rI - a_1A_1 - a_2A_2 - a_3A_3 - a_4A_4$$

Here, I is the identity matrix of order v. $A_i = \{x_{\alpha\beta}\}$, where $x_{\alpha\beta} = 1$, if α^{th} and β^{th} combinations are i^{th} associates and $a_i = \dfrac{\lambda_i}{k}$ ($i = 1, 2, 3, 4$).

The generalized inverse of C is obtained as:

$$C^- = z_0I + \sum_i z_iA_i,$$

where, $z_0 = \dfrac{2p^3 + 3p^2 - 10p + 1}{2p(p-1)(p^2-1)}$, $z_1 = \dfrac{3p^2 - 8p + 1}{2p(p-1)(p^2-1)}$,

$z_2 = \dfrac{-p^2 + 4p - 7}{2p(p-1)(p^2-1)}$, $z_3 = \dfrac{p^2 - 4p - 1}{2p(p-1)(p^2-1)}$ and

$z_4 = \dfrac{-p^2 - 3}{2p(p-1)(p^2-1)}$.

Further, the variance of estimated elementary contrasts between effects of two lines are obtained as:

$$V(\widehat{\tau_\alpha - \tau_\beta}) = \begin{cases} \left(\dfrac{2}{p-1}\right)\sigma^2, \text{ if } \alpha \text{ and } \beta \text{ are first associates,} \\ \left(\dfrac{2(p+4)}{p(p+1)}\right)\sigma^2, \text{ if } \alpha \text{ and } \beta \text{ are second associates and} \\ \left(\dfrac{2\{(p+1)^2 - 2\}}{p(p^2-1)}\right)\sigma^2, \text{ if } \alpha \text{ and } \beta \text{ are third associates} \\ \left(\dfrac{2\{(p+1)^2 + p - 3\}}{p(p^2-1)}\right)\sigma^2, \text{ if } \alpha \text{ and } \beta \text{ are fourth associates} \end{cases}$$

The simplified expression for the estimated average variance of elementary contrasts of line effects is obtained as:

$$\bar{V} = \frac{2(2p^3 + 3p^2 - 10p + 1)}{r(2p^2 - 2p - 1)(p+1)}\sigma^2$$

Canonical efficiency factor: We know, the information matrix (C) of an equi-replicate, proper block design with v treatments and b blocks, each of size k is of the form:

$C = rI_v - \dfrac{1}{k}NN'$, where N is the incidence matrix of order (v × b).

It is also known that for an orthogonal design, $N = \dfrac{1}{n}rk' = \dfrac{rk}{n}J$, where J is the matrix of unities. Hence, for an orthogonal equi-replicate, proper block design,

$$C = r\left(I_v - \frac{1}{v}J_v\right).$$

Here C is a product of a scalar coefficient r and an idempotent matrix $\left[I_v - \dfrac{1}{v}J_v\right]$. Consequently, for C, the eigen values of C are r and 0 with multiplicities (v - 1) and 0, and the harmonic mean of non-zero eigen values (EV) of C is r.

Then the Canonical Efficiency Factor (CEF) of the proposed design compared to an equi-replicate, proper and orthogonal design is:

$$CEF = \frac{\text{Harmonic mean of non-zero EV of } C \text{ of proposed design}}{\text{Harmonic mean of non-zero EV of } C \text{ of the orthogonal design}}$$

$$= \frac{1}{r} \times \text{Harmonic mean of non-zero EV of C of proposed design}$$

For the proposed series of designs, the (v - 1) non-zero eigen values of the resulting information matrix are $(p-1)$, $\left(\dfrac{p+1}{2}\right)$, $\left(\dfrac{p-1}{2}\right)$ and 1 with multiplicities are $2(p-1)^2$, $(p-2)$, $(p-2)$ and 1, respectively. Further, the CEF of resolvable PBIB(4) designs is obtained as follows:

$$CEF = \frac{(2p^2 - 2p - 1)(p+1)}{2p^3 + 3p^2 - 10p + 1}$$

Example: Considering $p = 4$, giving rise to $v = 24$ treatments. These treatments are arranged in $R = \{r_1, r_2, r_3\}$, $S = \{s_1, s_2, s_3\}$, where $r_1 = \{1, 2, 3, 4\}$, $r_2 = \{5, 6, 7, 8\}$, $r_3 = \{9, 10, 11, 12\}$, $s_1 = \{13, 14, 15, 16\}$, $s_2 = \{17, 18, 19, 20\}$ and $s_3 = \{21, 22, 23, 24\}$. Then, using the association scheme described in Section 4, different associates of treatment 1 are listed below:

First associates	Second associates	Third associates	Fourth associates
2, 3, 4	13, 14, 15, 16	5, 6, 7, 8, 9, 10, 11, 12	17, 18, 19, 20, 21, 22, 23, 24

Further, following the method of construction detailed in the Section 5, a resolvable PBIB(4) design

in two groups of $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ blocks can be obtained by considering all possible pairs within set R and within set S, and another group of 3 distinct blocks can be obtained by pairing r_h and s_h, where $h = 1, 2, 3$ as:

Replications	Blocks	Treatments							
	B1	1	2	3	4	5	6	7	8
I	B2	13	14	15	16	17	18	19	20
	B3	9	10	11	12	21	22	23	24
	B4	1	2	3	4	9	10	11	12
II	B5	13	14	15	16	21	22	23	24
	B6	5	6	7	8	17	18	19	20
	B7	1	2	3	4	13	14	15	16
III	B8	5	6	7	8	9	10	11	12
	B9	17	18	19	20	21	22	23	24

The parameters of the resultant resolvable PBIB(4) design are: $v = 24$, $b = 9$, $r = 3$, $k = 8$, $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 1$ and $\lambda_4 = 0$. The average variance factor and canonical efficiency factor for this design are computed as 0.7942 and 0.8394, respectively.

6. R-PACKAGE: "ResPBIBD"

The construction procedure explained above may appear to be little complex for the end users. To enhance the usability and to make the users more accessible to these designs, an R-package named as "ResPBIBD" (Vinaykumar *et al.*, 2022) has been developed. This package contains several utility functions, including PBIBD3(), which is related to proposed Series of DiSS designs. This function is used to generate DiSS design as well as its parameters, variance factors within associates, average variance factors and canonical efficiency factors. The designs can be generated through "ResPBIBD" using the following steps:

Step 1: Install the the R package "ResPBIBD" using:

install.packages("ResPBIBD")

Step 2: Once the package gets installed, store this package in the library of R session for further use:

library("ResPBIBD")

Step 3: Proposed designs for v treatments can be obtained using the function PBIBD3(v, p) by fixing v[=2(p-1)p] and p (≥ 3), where, p is a positive integer. For example,
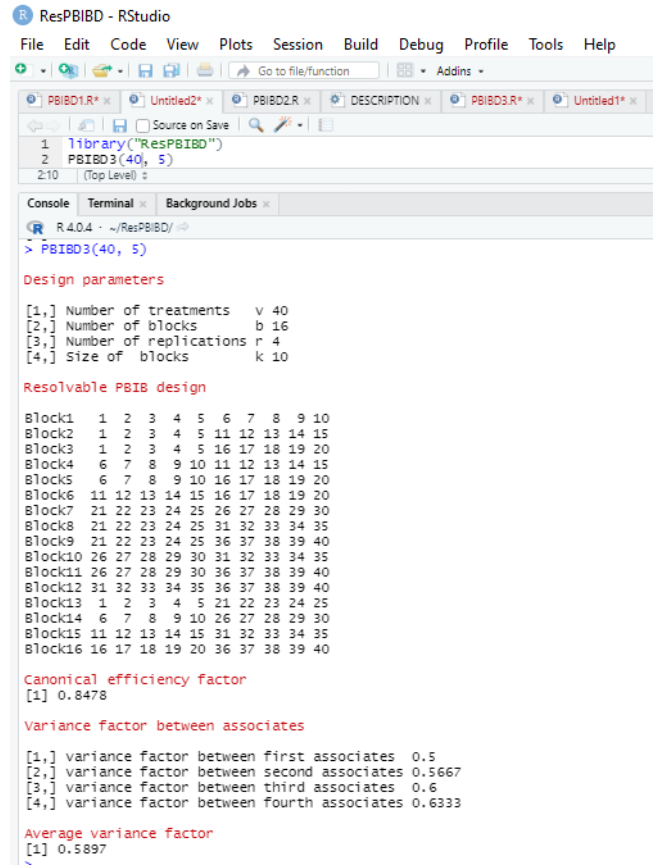
PBIBD3(40, 5)

Fig. 1. R output for DiSS PBIBD3(40, 5)

A snapshot of output generated using "ResPBIBD" for a DiSS design pertaining to v = 40 treatments is given in Fig. 1.

7. LIST OF DESIGNS

For quick reference of the end users, a list of parameters of designs (for number of treatments ≤ 500) belonging to all the classes has been prepared and given below:

Here, V_1, V_2, V_3 and V_4 represents variance factors between first, second, third and fourth associates, respectively. AVF represents Average Variance Factors and CEF represents Canonical Efficiency Factors.

The proposed DiSS designs are four associate class PBIB designs. The most popular class of existing four associate class PBIB designs are right angular designs by Tharthare (1963, 1965) and special case of circular designs by Das (1960) and Saha *et al.* (1974). Both these classes of designs are non-resolvable and are available for different treatment structure and parametric combinations. Right angular designs are obtained using Group divisible (GD) designs, whereas the proposed

Sl. No.	p	v	b	r	k	V_1	V_2	V_3	V_4	AVF	CEF
1	3	12	4	2	6	1.0000	1.1667	1.1667	1.3333	1.1818	0.8462
2	4	24	9	3	8	0.6667	0.8000	0.7667	0.8667	0.7942	0.8394
3	5	40	16	4	10	0.5000	0.6000	0.5667	0.6333	0.5897	0.8478
4	6	60	25	5	12	0.4000	0.4762	0.4476	0.4952	0.4659	0.8586
5	7	84	36	6	14	0.3333	0.3929	0.3690	0.4048	0.3835	0.8691
6	8	112	49	7	16	0.2857	0.3333	0.3135	0.3413	0.3252	0.8786
7	9	144	64	8	18	0.2500	0.2889	0.2722	0.2944	0.2818	0.8871
8	10	180	81	9	20	0.2222	0.2545	0.2404	0.2586	0.2484	0.8946
9	11	220	100	10	22	0.2000	0.2273	0.2152	0.2303	0.2219	0.9012
10	12	264	121	11	24	0.1818	0.2051	0.1946	0.2075	0.2004	0.9071
11	13	312	144	12	26	0.1667	0.1868	0.1777	0.1886	0.1827	0.9124
12	14	364	169	13	28	0.1538	0.1714	0.1634	0.1729	0.1677	0.9171
13	15	420	196	14	30	0.1429	0.1583	0.1512	0.1595	0.1550	0.9214
14	16	480	225	15	32	0.1333	0.1471	0.1407	0.1480	0.1441	0.9252

DiSS designs do not require any prior knowledge of existing designs. Circular designs are easy to construct, however, the efficiency is quite low. Further, for same number of treatments, the DiSS designs are available for smaller number of replications compared to right angular designs. For example:

- For v = 12, a right angular design (v = 12, b = 26, r = 13, k = 6, $\lambda_1 = 7 = \lambda_4$, $\lambda_2 = 5 = \lambda_3$) is available in 13 replications, which is quite high but the proposed DiSS designs (v = 12, b = 4, r = 2, k = 6, $\lambda_1 = 2$, $\lambda_2 = 1 = \lambda_3$ and $\lambda_4 = 0$) are available for only two replications.

- Similarly for v = 24, a right angular design (v = 24, b = 72, r = 12, k = 4, $\lambda_1 = 4$, $\lambda_2 = 2 = \lambda_3$, $\lambda_4 = 1$) is available in 12 replications, which is quite high but the proposed DiSS designs are available for only three replications.

The proposed DiSS designs have an additional advantage of resolvability.

8. CONCLUSION

A new four-class association scheme called DiSS association scheme has been proposed along with method of constructing PBIB(4) designs based on this scheme. Designs based on this association scheme are proper and resolvable. The canonical efficiency factors of the proposed series of designs are quite high in comparison to the corresponding orthogonal designs. The designs are available in fewer replications, making

them a fair choice when there is a serious resource crunch, and hence can be employed as an economical alternative to the existing popular classes of IBDs. Apart from serving the purpose of regular IBDs, they find promising application in multi-site varietal trials in breeding experiments, as these designs possess an additional property of resolvability.

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