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Polygonal Association Scheme and PBIB(3) Designs in Two Replicates

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SUMMARY

Partially Balanced Incomplete Block (PBIB) designs are a well-known class of incomplete block designs useful in agricultural research which are based on concept of association schemes. Here, a three-associate class polygonal association scheme has been defined. A method of constructing PBIB(3) designs based on polygonal association scheme has been described. The designs obtained by this method require only two replications and hence reduce the requirement of experimental material. Further, the efficiency of these designs has also been worked out and is found to be quite high.

Keywords: Association scheme; Efficiency; Partially balanced incomplete block design; Resolvable; Three associate class.

1. INTRODUCTION

Incomplete block designs are being used extensively in scientific experiments where a high degree of experimental control is necessary to cope with the variability present in the experimental material. Partially Balanced Incomplete Block (PBIB) designs with m-associate classes developed by Bose and Nair (1939) are a very general class of incomplete block designs which include Balanced Incomplete Block (BIB) designs and the lattice designs as special cases. Two-associate-class PBIB designs have been extensively studied in the literature and have been catalogued by Clatworthy (1973).

A BIB design or a PBIB design with two associate classes may not be available for a given set of parameters. In such situations, a PBIB design with three associate classes can be useful. Further, if there is a constraint on resources and the experimenter wants to economize on the use of experimental material, three-associate class PBIB designs can be used. Several research workers have contributed to the development of 3-class association schemes and designs based on them. As the required number of replications is less in PBIB(3) designs, this class of block designs are widely acceptable and appreciated by agricultural research workers.

Vartak (1955), Sharma and Das (1985) and Suen (1989) studied rectangular PBIB(3) designs in which number of treatments, $v = mn$; $m, n \geq 2$. Raghavarao and Chandrasekhararao (1964) proposed cubic PBIB(3) designs for $v = s^3$ ($s \geq 2$) treatments. The nested group divisible (NGD) class of PBIB(3) designs for $v = mns$ treatments ($m, n, s \geq 2$) introduced by Roy (1953) were subsequently studied by Raghavarao (1960). Bhagwandas *et al.* (1992) have given some methods of constructing these designs. Some E-optimal NGD designs have been obtained by Sinha and Kageyama (1992) and Sinha (1994).

The two associate class triangular designs were generalized to extended triangular PBIB(3) designs (John, 1966) for $v = (s + 2)(s + 3)(s + 4)/6$. Saha *et al.* (1973), Agarwal and Nair (1984) and Agarwal (1998) proposed cyclic PBIB(3) designs for a prime or prime power value of v. Rao (1956) developed circular lattices which were essentially PBIB(3) designs for $v = 2n^2$ treatments. These designs were subsequently generalized by Varghese and Sharma (2004) to accommodate $2sn^2$ treatments; $n, s \geq 2$. Sharma *et al.*

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(2010) defined tetrahedral and cubical association schemes with related PBIB(3) designs for v = 6m and v = 8m respectively. A series of PBIB(3) designs having n(n − 2)/4 treatments using the concept of triangular association scheme were given by Kipkemoi *et al.* (2014). Sharma and Garg (2018) obtained PBIB designs by treating columns of a Youden square as blocks.

Here, we propose a three-associate class polygonal association scheme and a method of constructing designs based on it. The PBIB(3) designs for v <100 obtained using the proposed method have been listed in section 7 along with the efficiencies.

2. MODEL AND EXPERIMENTAL SETUP

Consider the following linear additive fixed effects block model for v treatments, replicated r times, arranged in b blocks of size k each:

$$y_{uj} = \mu + \tau_u + \beta_j + e_{uj}.$$

Here, y_{uj} is the response from a unit in the j^{th} (j = 1, 2, …, b) block receiving u^{th} (u = 1,2,…,v) treatment, μ is the general mean, τ_u and β_j are the u^{th} treatment effect and the j^{th} block effect, respectively. e_{uj} are independent random errors normally distributed with zero mean and constant variance σ^2. The model can be rewritten as:

$$\mathbf{y} = \boldsymbol{\mu}\mathbf{1} + \boldsymbol{\Delta}_1'\boldsymbol{\tau} + \boldsymbol{\Delta}_2'\boldsymbol{\beta} + \mathbf{e}, \quad (2.1)$$

where **y** is a $n\times1$ vector of observations, μ is the general mean effect, **1** is the $n\times1$ vector of unities, $\boldsymbol{\Delta}_1'$ is the observation-treatment matrix of order $n\times v$, $\boldsymbol{\tau}$ is $v\times1$ vector of treatment effects, $\boldsymbol{\Delta}_2'$ is the observation-block matrix of order $n\times b$, $\boldsymbol{\beta}$ is $b\times1$ vector of block effects, and **e** is $n\times1$ vector of random error terms.

The reduced normal equations pertaining to treatment effects under the above model is $\mathbf{C}\hat{\boldsymbol{\tau}} = \mathbf{Q}$, which can be solved to get $\hat{\boldsymbol{\tau}} = \mathbf{C}^{-}\mathbf{Q}$, where $\mathbf{C}^{-}$ is the generalized inverse of **C** with $\mathbf{p}'\boldsymbol{\tau} = 0$.

Here,

$$\mathbf{C} = r\mathbf{I}_v - \frac{1}{k}\mathbf{N}\mathbf{N}' \text{ and} \quad (2.2)$$

$\mathbf{Q}$ [= $(Q_1, Q_2, \ldots, Q_v)'$]

are the information matrix and vector of adjusted treatment totals, respectively for equi-replicated and proper incomplete block designs. Here, $\mathbf{N} = \boldsymbol{\Delta}_1\boldsymbol{\Delta}_2'$ is the incidence matrix of treatments Vs. blocks incidence matrix of order $(v \times b)$, $\mathbf{I}_v$ is an identity matrix of order v.

It may be noted here that

$$\mathbf{N}\mathbf{N}' = ((n_{ii'})) = \begin{cases} r, & \text{if } i = i' \ (=1, 2, \ldots, v) \\ \lambda_1, \lambda_2 \ldots, \lambda_m & \text{if } i \neq i' \ (=1, 2, \ldots, v) \text{ are} \\ & 1^{st}, 2^{nd}, \ldots, m^{th} \text{ associates} \end{cases}$$

for a PBIB(m) design. (2.3)

3. POLYGONAL ASSOCIATION SCHEME

Consider p concentric polygons each having s edges and s vertices. Arrange v = psm (s > 4) treatments on the vertices of p polygons such that each vertex contains exactly m distinct treatments. Consider two adjacent sectors with pm common treatments. The association scheme is defined as follows:

Treatment β is the first associate of α, if β and α are on the same common portion of adjacent sectors; the second associate, if β and α lie on other portion of the adjacent sectors; and third associate, otherwise. The parameters of the association scheme are:

v (= psm), n_1 = pm – 1, n_2 = 2pm, n_3 = (s – 3)pm, and

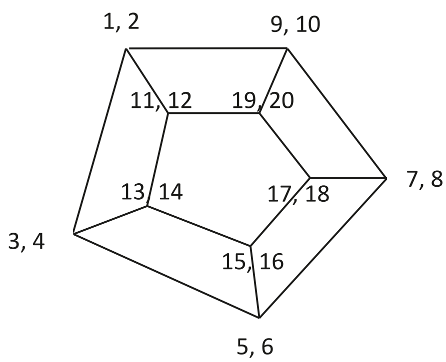
$$\mathbf{P}_1 = \begin{bmatrix} p(m-1) & 0 & 0 \\ 0 & 2pm & 0 \\ 0 & 0 & (s-3)pm \end{bmatrix},$$

$$\mathbf{P}_2 = \begin{bmatrix} 0 & pm-1 & 0 \\ pm-1 & 0 & pm \\ 0 & pm & (s-4)pm \end{bmatrix} \text{ and}$$

$$\mathbf{P}_3 = \begin{bmatrix} 0 & 0 & pm-1 \\ 0 & pm & pm \\ pm-1 & pm & (s-5)pm \end{bmatrix}.$$

Here, n_i is the number of i^{th} (i = 1, 2, 3) associates of a given treatment. $\mathbf{P}_i = ((p^i_{jk}))$, p^i_{jk} represents the number of treatments common to the j^{th} associates of the α and k^{th} associates of β, where α and β are mutually i^{th} associates (i, j, k = 1, 2, 3).

Example 3.1: Let p = 2, s = 5 and m = 2 giving v = 20. An arrangement of these treatments on the 2 polygons each of 5 edges and 5 vertices with two distinct treatments on each vertex is as given below.



Here, $n_1 = 3, n_2 = 8, n_3 = 8$ and the various associates of treatments, say 1, 2, 3 and 5 are as given below.

Different associates of treatments 1, 2, 3 and 5

Treatment	1st Associates	2nd Associates	3rd Associates
1	2, 11, 12	3, 4, 9, 10, 13, 14, 19, 20	5, 6, 7, 8, 15, 16, 17, 18
2	1, 11, 12	3, 4, 9, 10, 13, 14, 19, 20	5, 6, 7, 8, 15, 16, 17, 18
3	4, 13, 14	1, 2, 5, 6, 11, 12, 15, 16	7, 8, 9, 10, 17, 18, 19, 20
5	6, 15, 16	3, 4, 7, 8, 13, 14, 17, 18	1, 2, 9, 10, 11, 12, 19, 20

$$\mathbf{P}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 4 \end{bmatrix}, \mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 4 & 4 \\ 3 & 4 & 0 \end{bmatrix}.$$

A method of constructing PBIB(3) designs following polygonal association scheme with minimum replications is now described.

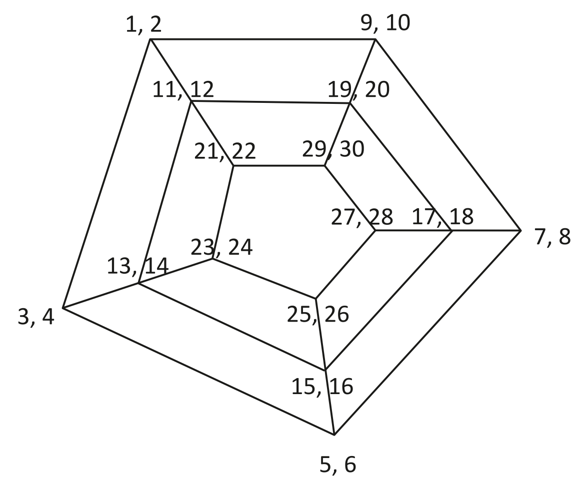
4. METHOD OF CONSTRUCTION

The blocks of the design are formed by combining all the treatments on the 2p vertices of a sector. This results into a series of PBIB(3) designs with parameters $v = psm$ $(s > 4)$, $b = s$, $r = 2$, $k = 2pm$, $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 0$. It is important to mention here that $\frac{k}{v} = \frac{2pm}{psm} = \frac{2}{s}$ $(s > 4)$ indicating $k < \frac{v}{2}$. Thus, the designs obtained have a small block size and also have minimum number of replications ($r = 2$). Hence, the proposed designs require less experimental units as compared to most of those available in literature.

Example 4.1: Let $v = psm = 2 \times 5 \times 2 = 20$. The PBIB(3) design with parameters $v = 20$, $b = 5$, $r = 2$, $k = 8$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$ is obtained as given below:

Block	Treatments							
1	1	2	3	4	11	12	13	14
2	3	4	5	6	13	14	15	16
3	5	6	7	8	15	16	17	18
4	7	8	9	10	17	18	19	20
5	9	10	1	2	11	12	19	20

Example 4.2: Let $p = 3$, $s = 5$ and $m = 2$. An arrangement of 30 treatments on the 3 polygons each of 5 edges and 5 vertices with two distinct treatments on each vertex is as given below.



The PBIB(3) design with parameters $v = 30$, $b = 5$, $r = 2$, $k = 12$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$ is obtained as given below:

Block	Treatments											
1	1	2	3	4	11	12	13	14	21	22	23	24
2	3	4	5	6	13	14	15	16	23	24	25	26
3	5	6	7	8	15	16	17	18	25	26	27	28
4	7	8	9	10	17	18	19	20	27	28	29	30
5	9	10	1	2	11	12	19	20	29	30	21	22

It can be seen that there is a considerable amount of saving in resources. The information matrix (**C**) for estimating the contrasts pertaining to treatment effects is obtained by developing a SAS code in PROC IML. SAS code along with output generated for Example 4.1 is given in Appendix A.

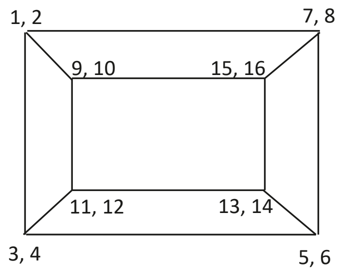
In general, $\mathbf{C} = ((c_{gh}))$; $g, h = 1,2,\ldots,v$ will have following entries:

$$c_{gg} = \frac{2pm - 1}{pm}, \; c_{gh(i)} = \frac{-\lambda_i}{2pm}, \; i = 1, 2, 3; \lambda_i = 2, 1, 0$$

g and h being mutually i^{th} associates, appear together λ_i times at n_i positions in each row of **C**.

5. PARTICULAR CASES

Case 5.1 (For s = 4): Let p = 2 and m = 2. Arrangement of 16 treatments on the vertices of p = 2 polygons of s = 4 edges with each vertex containing m = 2 distinct treatments is as follows:



Here, $n_1 = 3$, $n_2 = 8$, $n_3 = 4$ and the three associates of treatments, say 1, 2, 3 and 5 are as given below.

Different associates of treatments 1, 2, 3 and 5

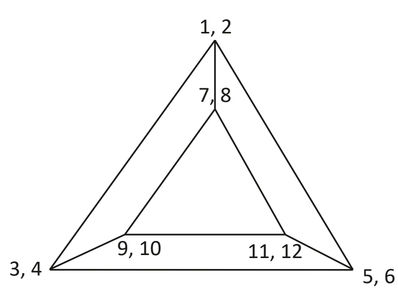
Treatment	1st Associates	2nd Associates	3rd Associates
1	2, 9, 10	3, 4, 7, 8, 11, 12, 15, 16	5, 6, 13, 14
2	1, 9, 10	3, 4, 7, 8, 11, 12, 15, 16	5, 6, 13, 14
3	4, 11, 12	1, 2, 5, 6, 9, 10, 13, 14	7, 8, 15, 16
5	6, 13, 14	3, 4, 7, 8, 11, 12, 15, 16	1, 2, 9, 10

And $\mathbf{P}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\mathbf{P}_2 = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$, $\mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 8 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

The PBIB(3) design with parameters v = 16, b = 4, r = 2, k = 8, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$ is obtained as given below:

Block	Treatments							
1	1	2	3	4	9	10	11	12
2	3	4	5	6	11	12	13	14
3	5	6	7	8	13	14	15	16
4	1	2	7	8	9	10	15	16

Case 5.2 (For s = 3): Let p = 2 and m = 2. Arrangement of 12 treatments on the vertices of p = 2 polygons of s = 3 edges with each vertex containing m = 2 distinct treatments is as follows:



Here, $n_1 = 3$ and $n_2 = 8$. This reduces to a two associate class association scheme and the associates of treatments, say 1, 2 and 5 are as given below.

Different associates of treatments 1, 2 and 5

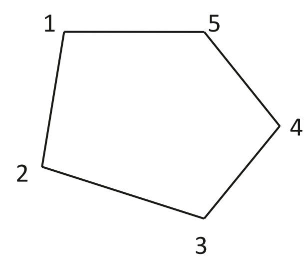
Treatment	1st Associates	2nd Associates
1	2, 7, 8	3, 4, 5, 6, 9, 10, 11, 12
2	1, 7, 8	3, 4, 5, 6, 9, 10, 11, 12
5	6, 11, 12	1, 2, 3, 4, 7, 8, 9, 10

And $\mathbf{P}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$, $\mathbf{P}_2 = \begin{bmatrix} 0 & 3 \\ 3 & 4 \end{bmatrix}$

The PBIB(2) design with parameters v = 12, b = 3, r = 2, k = 8, $\lambda_1 = 2$, $\lambda_2 = 1$ is obtained as:

Block	Treatments							
1	1	2	3	4	7	8	9	10
2	3	4	5	6	9	10	11	12
3	1	2	5	6	7	8	11	12

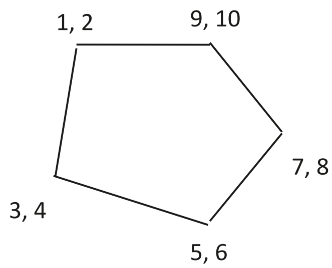
Case 5.3 (For p = 1, m = 1, s ≥ 4): Let s = 5. Following is the arrangement of 5 treatments on the polygon with 5 edges and 5 vertices:



This reduces to a two associate class association scheme and the symmetric PBIB(2) design with parameters v = 5, b = 5, r = 2, k = 2, $\lambda_1 = 1$, $\lambda_2 = 0$ is as follows:

Block	Treatments	
1	1	2
2	2	3
3	3	4
4	4	5
5	5	1

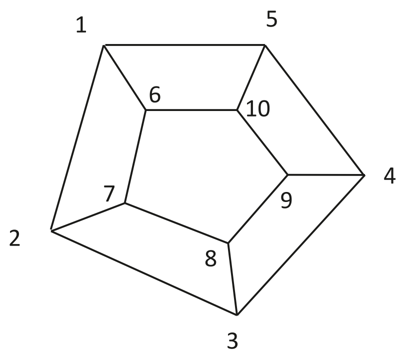
Case 5.4 (For p = 1, m = 2, s ≥ 4): Let s = 5. Following is the arrangement of 10 treatments on the polygon with 5 edges and 5 vertices:



This is a three associate class association scheme and the PBIB(3) design with parameters v = 10, b = 5, r = 2, k = 4, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$ is as follows:

Block	Treatments			
1	1	2	3	4
2	3	4	5	6
3	5	6	7	8
4	7	8	9	10
5	9	10	1	2

Case 5.5 (For p = 2, m = 1, s ≥ 4): Let s = 5. Following is the arrangement of 10 treatments on the 2 polygons with 5 edges each and 5 vertices with one treatment on each vertex:



PBIB(3) design with parameters v = 10, b = 5, r = 2, k = 4, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$ is as follows:

Block	Treatments			
1	1	2	6	7
2	2	3	7	8
3	3	4	8	9
4	4	5	9	10
5	1	5	6	10

6. AVERAGE VARIANCE FACTOR AND CANONICAL EFFICIENCY FACTOR

With underlying Model (2.1) and Equation (2.3), PBIB(3) designs obtained from the proposed method will have the concurrence matrix, $\mathbf{NN'}$ as:

$$\mathbf{NN'} = r\mathbf{I}_v - \sum_{i=1}^{3} \lambda_i \mathbf{A}_i \,, \tag{6.1}$$

with $\mathbf{A_i}$, the association matrix defined as: $\mathbf{A_i} = (a_{\alpha\beta})$ is a symmetric matrix of order v with elements 0's and 1's with $a_{\alpha\beta} = 1$ if the treatments α and β are i^{th} associates to each other and $a_{\alpha\beta} = 0$, otherwise.

Now, the concurrence matrix for proposed class of polygonal designs is obtained as:

$\mathbf{NN'} = 2\mathbf{I}_v - 2\mathbf{A}_1 - \mathbf{A}_2$, as r = 2, $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 0$.

Finally, considering the Model (2.1) and using (2.2), the general form of the information matrix (**C**) pertaining to the v treatments for the polygonal designs, is obtained as:

$$\mathbf{C} = r\mathbf{I}_v - \frac{1}{k}\mathbf{NN'} = \left(2 - \frac{2}{k}\right)\mathbf{I}_v + \frac{1}{k}\left(2\mathbf{A}_1 + \mathbf{A}_2\right). \tag{6.2}$$

The best linear unbiased estimator of $\boldsymbol{\tau}$ is $\mathbf{p'}\hat{\tau} = \mathbf{p'C^-Q}$ with $V(\mathbf{p'}\hat{\boldsymbol{\tau}}) = \sigma^2 \mathbf{p'C^-p}$.

$V(\mathbf{p'}\hat{\boldsymbol{\tau}})$ will yield three types of variances for the proposed class of designs. Now, the Average Variance Factor (AVF) of estimated elementary contrasts between treatment effects can be obtained by computing the average of variances across all possible elementary contrasts. The Canonical Efficiency Factor (CEF) of the proposed polygonal designs compared to an equi-replicate, proper and orthogonal design (randomized complete block design) with the same number of treatments and replications is the inverse value of AVF. Assuming equal estimated variance in both cases, the CEF of the proposed design can also be computed in terms of Eigen Values (EV) of **C** matrices as:

$$\text{CEF} = \frac{\text{Harmonic mean of non-zero EV of } \mathbf{C} \text{ of proposed design}}{\text{Harmonic mean of non-zero EV of } \mathbf{C} \text{ of the orthogonal design}} \tag{6.3}$$

$$= \frac{1}{2} \times \text{Harmonic mean of non-zero EV of } \mathbf{C} \text{ of proposed design}.$$

7. LIST OF DESIGNS

A list of PBIB(3) designs obtained using the method described has been prepared for v < 100 and given in Table 7.1. The list contains the parameters, AVF of the contrasts of treatment effects and the CEF compared to an orthogonal design with same replications. It is seen that the efficiency is quite high.

Table 7.1. List of PBIB(3) Designs in 2 Replicates Based on Polygonal Association Scheme

S. No.	p	s	m	v	b	k	n_1	n_2	n_3	AVF	CEF
1	2	5	2	20	5	8	3	8	8	1.211	0.826
2	2	5	3	30	5	12	5	12	12	1.138	0.879
3	2	5	4	40	5	16	7	16	16	1.103	0.907
4	2	5	5	50	5	20	9	20	20	1.082	0.925
5	2	5	6	60	5	24	11	24	24	1.068	0.937
6	2	5	7	70	5	28	13	28	28	1.058	0.945
7	2	5	8	80	5	32	15	32	32	1.051	0.952
8	2	5	9	90	5	36	17	36	36	1.045	0.957
9	3	5	2	30	5	12	5	12	12	1.170	0.855
10	3	5	3	45	5	18	8	18	18	1.091	0.917
11	3	5	4	60	5	24	11	24	24	1.068	0.937
12	3	5	5	75	5	30	14	30	30	1.054	0.949
13	3	5	6	90	5	36	17	36	36	1.045	0.957
14	2	6	2	24	6	8	3	8	12	1.290	0.775
15	2	6	3	36	6	12	5	12	18	1.190	0.840
16	2	6	4	48	6	16	7	16	24	1.160	0.862
17	2	6	5	60	6	20	9	20	30	1.113	0.898
18	2	6	6	72	6	24	11	24	36	1.094	0.914
19	2	6	7	84	6	28	13	28	42	1.080	0.926
20	2	7	2	28	7	8	3	8	16	1.370	0.730
21	2	7	3	42	7	12	5	12	24	1.244	0.804
22	2	7	4	56	7	16	7	16	32	1.182	0.846
23	2	7	5	70	7	20	9	20	40	1.145	0.873
24	2	7	6	84	7	24	11	24	48	1.120	0.892
25	3	7	2	42	7	12	5	12	24	1.244	0.804
26	3	7	3	63	7	18	8	18	36	1.161	0.861
27	3	7	4	84	7	24	11	24	48	1.120	0.892

8. DISCUSSION AND CONCLUSION

It can be seen that for even values of s, the PBIB(3) designs obtained would be resolvable in two replicates. Further, the complementary of the PBIB design obtained with parameters v = psm (s > 4), b = s, r = 2, k = 2pm, $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 0$ is also a PBIB design with parameters $v^* = v = psm$, $b^* = b = s$, $r^* = s - 2$, $k^* = (s - 2)pm$, $\lambda_1^* = s - 2$, $\lambda_2^* = s - 3$ and $\lambda_3^* = s - 4$ following the same association scheme.

There exists some series of PBIB(3) designs in two replicates, viz. circular lattice designs (Rao, 1956), circular designs (Das, 1960; Kulshreshtha *et al.*, 1971; Saha *et al.*, 1974), generalized circular lattice designs (Varghese and Sharma, 2004) and tetrahedral designs by Sharma *et al.* (2010). Circular lattice and generalized circular lattice designs are available for $v = 2s^2$ and $v = 2ms^2$ treatments respectively i.e. only for even number of treatments. 3-associate class circular designs can be obtained for v = nm (n arcs of size m each) treatments in two replications provided n = 4 or 5 (with m >1) and n = 5 or 6 (with m = 1) and only two consecutive arcs are combined to form blocks. Again, tetrahedral designs exist only for v = 6m treatments. PBIB(3) designs obtained here are more general for treatments structure covering wider range of parametric combinations. These are quite efficient and offer a useful solution for situations wherein experimental material is scarce. Hence, these designs can be advantageously used by researchers for efficient conduct of their experiments.

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APPENDIX A

SAS code for computing information matrix (C) for estimating the contrasts pertaining to treatment effects, its Average Variance Factor and Canonical Efficiency Factor of PBIB design.

```
proc iml;
a={1 2 3 4 11 12 13 14,
3 4 5 6 13 14 15 16,
5 6 7 8 15 16 17 18,
7 8 9 10 17 18 19 20,
9 10 1 2 11 12 19 20};
Blocksize={8};
a1=countn(loc(a));
*print a1;
m=j(a1,1,1);/*mean vector*/
*print m;
trt=j(a1,max(a),0);/*design matrix -obs VS direct treatment*/
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
if a[i,j]>0 then
do;
trt[k,a[i,j]]=1;
k=k+1;
end;
end;
end;
*print trt;
Block=j(a1,nrow(a),0);/*design matrix–obs VS Blocks*/
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
if a[i,j]>0 then do;
Block[k,i]=1;
k=k+1;
end;
end;
end;
*print block;
x=m||trt||block;/*design matrix*/
*print x[format=3.0];
x1=trt;
x2=m||Block;
c_mat=(x1`*x1)-(x1`*x2*(ginv(x2`*x2))*x2`*x1)/*C matrix*/;
print c_mat;
t1=comb(max(a),2);
cot=j(t1,Max(a),0);
k=1;
do i=1 to max(a)-1;
do j=i+1 to max(a);
cot[k,i]=1;
cot[k,j]=-1;
```

```
k=k+1;
end;
end;
*print cot;
covt=cot*(ginv(c_mat))*cot`;
vart1=diag(covt);
onet=j(t1,1,1);
variance=vart1*onet;
*print covt;
*print variance;
av_var=variance[+, ]/nrow(variance);
rep1=trt`*trt;
print av_var;
eig=eigval(c_mat);
eig1=eig[loc(eig>0.0000001),];/*positive eigen values*/
*print rep1;
eig2=eig1/(rep1[1,1]);
eig3=1/eig2;
CanEffFactor_trt=nrow(eig3)/sum(eig3);
print CanEffFactor_trt;
*print eig;
quit;
```

SAS Output Displaying the C Matrix of PBIB(3) Design Obtained as in Example 4.1 with Parameters v = 20, b = 5, r = 2, k = 8, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$, the Average Variance Factor and the Canonical Efficiency Factor

c_mat_round

	COL1	COL2	COL3	COL4	COL5	COL6	COL7	COL8	COL9	COL10	COL11	COL12	COL13	COL14	COL15	COL16	COL17	COL18	COL19	COL20
ROW1	1.75	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125
ROW2	-0.25	1.75	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125
ROW3	-0.125	-0.125	1.75	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0
ROW4	-0.125	-0.125	-0.25	1.75	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0
ROW5	0	0	-0.125	-0.125	1.75	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0
ROW6	0	0	-0.125	-0.125	-0.25	1.75	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0
ROW7	0	0	0	0	-0.125	-0.125	1.75	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125
ROW8	0	0	0	0	-0.125	-0.125	-0.25	1.75	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125
ROW9	-0.125	-0.125	0	0	0	0	-0.125	-0.125	1.75	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25
ROW10	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	1.75	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25
ROW11	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	1.75	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125
ROW12	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	1.75	-0.125	-0.125	0	0	0	0	-0.125	-0.125
ROW13	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	1.75	-0.25	-0.125	-0.125	0	0	0	0
ROW14	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	1.75	-0.125	-0.125	0	0	0	0
ROW15	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	1.75	-0.25	-0.125	-0.125	0	0
ROW16	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	1.75	-0.125	-0.125	0	0
ROW17	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	1.75	-0.25	-0.125	-0.125
ROW18	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	1.75	-0.125	-0.125
ROW19	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	1.75	-0.25
ROW20	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	-0.25	-0.125	-0.125	0	0	0	0	-0.125	-0.125	-0.25	1.75

av_var
1.2105263

CanEffFactor_trt
0.826087