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Robustness of General Efficiency Balanced Block Designs against Missing Observations

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SUMMARY

In the present communication, some results on robustness of generalized efficiency balanced (GEB) block designs with equal block sizes (binary and non-binary) are presented. The necessary & sufficient conditions are obtained for robustness as per criteria 1 (Ghosh, 1982) when the missing observations appear in the following patterns: (i) t (≥1) observations of a design pertaining to the same treatment are missing and (ii) all observations in a block are missing but the residual design still becomes connected. Sufficient conditions for robustness according to Criteria 1 are examined in each type of GEB designs. Another criterion (Criteria 2) for robustness of a block design is the small amount loss of efficiency values of the residual design after deletion of t (≥1) observations. The declining trend of efficiency values after deletion of treatments one by one are presented graphically for measuring the robustness of designs under study.

Keywords: GEB designs; Conditions for robustness; Connected designs and Efficiency of a block design.

1. INTRODUCTION

A design with one or more missing observations can be analysed with higher efficiency values and having connectedness property will be termed as a robust design. Several authors had investigated on the robustness properties of block designs against missing observations such as, Ghosh (1982), Ghosh *et al.* (1983), Kageyama and Mukherjee (1986), Srivastava *et al.* (1991), Gupta and Srivastava (1996), Bhar and Dey (2003).

A criterion of robustness of designs (in particular, incomplete block designs) was introduced by Ghosh (1982). According to the above criterion (to be called Criteria 1), an incomplete block design is robust against the loss of **t** (≥1) observations of any treatment if the residual design (obtained after deleting **t** observations of any one of v treatments) remains connected. It was shown by Ghosh (1982) that Balanced Incomplete Block (BIB) designs are robust according to criteria 1 against the loss of any (r-1) observations of a particular treatment, where r is the common replication of the original BIB design. Similar results on certain Partially Balanced Incomplete Block (PBIB) designs were also obtained by Ghosh *et al.* (1983). Godolphin and Warren (2011) derived improved conditions for robustness of binary block designs against the loss of whole blocks. Bhar (2014) identified E-efficiency criteria as an alternate of A-efficiency criteria to judge the efficiency of the residual design. Dutta *et al.* (2020) reported some results on robust designs against the presence of outliners in ANCOVA model. Ekpo *et al.* (2021) compared the robustness of two PBIB designs using optimality criteria. Another criterion of robustness (to be called Criteria 2) is in terms of the efficiency of the residual design. As per this criterion, a connected design is said to be robust if the efficiency of the residual design relative to the original one is not too small. The papers by Kageyama and Mukerjee (1986), Srivastava *et al.* (1991), Duan and Kageyama (1996), Srivastava *et al.* (1996) are available in this spirit.

Godolphin and Warren (2021) developed a set of measures that enable non-isomorphic BIBDs with same parameters to be ranked. Their investigation suggested

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that there are some correspondences between robustness against becoming disconnected and rankings associated with A-efficiency. Hemavathi *et al.* (2022) examined the robustness of sequential third-order rotatable design and investigated the loss of information when one or two observations pertaining to experimental run (s) is (are) missing, which are at different radii from the design centre and found that the maximum loss of information occurs when the observation at the design points which are at higher radii from the design centre is lost and the design has the minimum efficiency.

In the field of agriculture, we often see that some of the observations in field experiment are missing or lost. Therefore, it will be a big headache for researchers to deal with that kind of dataset. Here lies the importance of robustness in block designs, e.g., BIB designs, VB designs and some efficiency balanced designs. Utilizing the above two criteria of robustness, researchers can easily handle the datasets with one or more than one missing observations.

The existing literature reveals that the most studies on robustness of block designs, using either Criteria 1 or 2 have been restricted to specific classes of designs with a specific pattern of missing observations. The purpose of this communication is to present some results on robustness of generalized efficiency balanced (GEB) block designs having non-equi-replicated treatments (binary and non-binary). The necessary & sufficient conditions for GEB designs are verified for robustness as per criteria 1, when the missing observations appear in the following patterns: (i) t (≥1) observations pertaining to the same treatment are missing once or multiple times (ii) all the observations are missing in a block. The efficiency of residual designs after deletion of t (≥1) treatments from different types of GEB designs are also examined.

Keeping in mind the importance of the robustness property in block designs, this article mainly aims to discuss the robustness properties and characteristics of generalized efficiency balanced (GEB) block designs. The declining trend of efficiency values after deletion of treatments one by one from the GEB designs developed from different sources are presented by line diagrams.

2. MATERIALS AND METHODS

In what follows, some basic results are presented.

2.1 Model and C- matrix structure of GEB designs:

Let us consider a block design D with treatments labelled as 1, ..., v, v+1. The set of treatments are arranged in b blocks of equal sizes having (k+1) elements from each block. $\boldsymbol{r} = (r_1, r_2, ..., r_v, r_{v+1})'$ is the replication vector where r_{v+1} denoting the replication number of control (or added) treatment and r_i denoting the replication number of test (or original) treatments, (i=1, 2, ..., v). A Fixed effects additive model is considered for analysis of a block design will be:

$$\mathbf{Y} = \boldsymbol{\mu}\mathbf{1} + \mathbf{X}\boldsymbol{\tau} + \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \qquad \dots (2.1)$$

where, **Y** is a n × 1 vector of observations, n is the no. of components, **μ** is a general mean, **1** is a n × 1 vector of ones, **X** is a n × (v+1) incidence matrix of observations versus treatments, **τ** is a (v+1) × 1 vector of treatment effects, **Z** is a n × b incidence matrix of observations versus blocks, **β** is a b × 1 vector of block effects and **ε** is a n × 1 vector of errors, N $(0, \sigma^2 I)$.

General Efficiency Balanced (GEB) Block Design (Kageyama and Mukerjee;1986**).** The information matrix C of GEB design following model (2.1) is of the form:

$$C = d\,(S - g^{-1}ss'),$$

where, $s = (s_1, s_2, \dots, s_{v+1})'$, S= diag $(s_1, s_2, \dots, s_{v+1})$, $g = \sum_{i=1}^{v+1} s_i$ and d be a constant.

Majumder *et al.*(2013) extended the concept of GEB designs for correlated observations.

Generalised inverse matrix of the C matrix of GEB designs with one extra treatment added to each block in a BIB design (*v, b, r, k* and *λ*)

When one extra treatment is added to each block of a BIB design, then the information matrix (C) of order v+1 will be:

$$C = \begin{bmatrix} \boldsymbol{M_d} & c1_v \\ c1_v' & d \end{bmatrix}, \text{ where, } \mathbf{M_d} = (a-b)\left[I + \frac{J}{\left(\frac{a-b}{b}\right)} \right]$$

I is an identity matrix of order v and J ($=1_v1_v'$) is a matrix of all unit elements of order v. Here a, b, c, d are the elements of the C matrix of the GEB design.

Since M_d is positive definite and non-singular matrix of order v, a generalized inverse of C (as given by Srivastava *et.al*., 1996) will be C^g. Then,

$$C^g = \begin{bmatrix} \mathbf{M_d^{-1}} & 0 \\ 0' & 0 \end{bmatrix}$$

The concept or idea of a GEB design is presented in the following example for easy understanding.

Example 2.1: Let D be a symmetric BIB design with parameters $v=b=7$, $r=k=4$, $\lambda=2$. One extra treatment is added to each block of the previously defined design. Let D* be the resultant GEB design with parameters $v^*=8$, $b^*=7$, $\boldsymbol{r}^*=(4.\mathbf{1'}_7, 7)$, $k^*=5$, $\lambda^*=(2, 4)$. The design D^* is given in the following:

Block1	1	3	2	6	8
Block2	2	4	3	7	8
Block3	3	5	4	1	8
Block4	4	6	5	2	8
Block5	5	7	6	3	8
Block6	6	1	7	4	8
Block7	7	2	1	5	8

2.2 Robustness conditions of block designs

To begin with, it is introduced some notation to be followed throughout. All matrices and vectors are real, vectors being written as column vectors. We denote a n component vectors of all unities by 1_n and by I_n, an identity matrix of order n. For a matrix A, A' = transpose matrix, M(A) = column span (range), A^- = generalized inverse matrix (g- inverse) and A^+ = Moore penrose inverse.

In what follows, six theorems, two corollaries and related conditions on robustness properties of block designs, developed by Dey (1993) are presented, which are mainly applied on the results of traditional binary block designs either BIB designs, variance balanced (VB) designs or efficiency balanced (EB) designs. An attempt has been made to apply the above theorems on non-traditional (*e.g.*, non-equireplicate, binary and non-binary) designs like GEB designs, which are not VB or EB.

Interested readers may go through the article, Dey (1993), for the proofs of the six theorems and two corollaries as given below.

DEFINITION 2.1 (Dey, 1986)**:** A block design with v treatments is said to be connected if the rank of the C matrix of the design is v-1.

THEOREM 2.1: Let A & B be a pair of symmetric, non-negative definite matrices of order n, let A= B + GG', where, G is a nxm matrix such that $M(G) \subset M(A)$, then Rank(A)= Rank(B), if and only if I_m -G' A^- G is positive definite.

THEOREM 2.2: Let A, B, G be as in Theorem 2.1 and suppose that I_m-G' A^- G=I_m-G' A^+ G is positive definite. Then, $B^+ = A^+ + A^+ G\left(I_m - G'A^+G\right)^{-1} G' A^+$.

2.2.1 Conditions for robustness when t(≥1) observations pertaining to the same treatment are lost

Consider a connected, binary block design d_0 with v treatments, b blocks and constant block size k. Let t (≥ 1) of the bk observations be missing and let all these t observations pertain to the same treatment. Without loss of generality, it may assume that these t observations pertain to the first treatment in the first t blocks. It further assumes that these t affected blocks are not all identical. Let the residual design, obtained by deleting these t observations from d_0 be called d_t. If N_0 (respectively, N_t) is the incidence matrix of d_0 (respectively, d_t), then

$$N_0 = \begin{bmatrix} 1_t' & e' \\ F & M \end{bmatrix}, \quad N_t = \begin{bmatrix} 0' & e' \\ F & M \end{bmatrix},$$

where e is a (0, 1) matrix of order 1×(b-t) & F & M are (0, 1) matrices of orders (v-1)×t & (v-1)×(b-t) respectively. Denote by $C_0(C_t)$, the usual C-matrix of $d_0(d_t)$. Then, it can be shown, after some routine algebra, that

$C_0 = C_t + UU'$, where U is a v×t matrix, given by:

$$U = \left\{K(K-1)\right\}^{-1/2} \begin{bmatrix} (K-1)1_t' \\ -F \end{bmatrix}, \text{ Clearly, } 1_v' U = 0'.$$

Also, since Rank (C_0) = v-1 (as d is assumed to be connected) & $1_v' C_0 = 0'$, if follows that $M(U) \subset M(C_0)$. Thus, Rank (C_0) = Rank(C_t) and Theorem 2.1 can be established. Thus, the rank of the residual design (d_t) after deletion of t (≥1) from design d_0 with v treatments is v-1 and the residual design is connected (see Definition 2.1).

THEOREM 2.3: The design d_0 is robust against the loss of any t (≥1) observations pertaining to the same treatment according to Criteria 1 if and only if I_t-U' C_0^- U is positive definite.

Corollary 2.1: The design d_0 is robust against the loss of any single observation according to Criterion 1 if and only if, $u' C_0^- u < 1$, where, $u' = \{K(K-1)\}^{-1/2}$ (K-1, -f') and f is a (0, 1) vector representing the incidence of the (v-1) unaffected treatments in the first block containing the missing observation.

This result of Corollary 2.1 has been obtained by Ghosh *et al.* (1991) in terms of the Moore-Penrose C_0^+, using a different approach.

The necessary & sufficient condition for robustness given in Theorem 2.3 is not very convenient in the sense that its verification depends on the structure (incidence) of the unaffected treatments in the t affected blocks through the matrix U. A simpler sufficient condition in terms of the smallest positive eigen value of C_0 is given as:

THEOREM 2.4: The design d_0 is robust as per criterion 1 against the loss of t(>1) observations pertaining to the same treatment, if t does not exceed the smallest positive eigen value of C_0.

Corollary 2.2: The design d_0 is robust against the loss of a single observation, according to Criterion 1, if the smallest positive eigen value of C_0 is strictly larger than unity.

2.2.2 Conditions for robustness when all observations in a block are missing

Suppose d_0 is a connected, binary block design with *v* treatments, *b* blocks and block size *k*, and suppose that for some reason, all the observations in a block are missing. Without loss of generality, let the missing observations pertain to the first *k* treatments in the first block. If $C_0(C_k)$ denotes the C-matrix of d_0 (residual design d_k), then it can be shown that

$C_0 = C_k + VV'$, where V is a $v \times k$ matrix given by $V' = [I_k - k^{-1}J_k, 0]$

It is easily seen that $1_v' V = 0'$ and hence M (V) ⊂M (C_0). Thus, using Theorem 2.1, we arrive at the result.

THEOREM 2.5: The design d_0 is robust as per Criterion 1 against the loss of all observations in a block if and only if $I_K - V' C_0^- V$ is positive definite.

Note that $VV' = \begin{bmatrix} I_k - k^{-1}J_k & 0 \\ 0 & 0 \end{bmatrix}$,

which is a symmetric, idempotent matrix with rank (k–1), and thus, λ_{max} (VV') = 1. Hence, proceeding as in the proof of Theorem 2.4, we arrive at the following sufficient condition.

THEOREM 2.6: The design d_0 is robust as per Criterion 1 against the loss of all observations in a block if the smallest positive eigen value of C_0 is strictly larger than unity.

2.3 Efficiency of the Residual Design

Criterion 1 of robustness is in terms of the connectedness of the residual design. However, even if a design is robust according to Criterion 1, the residual design may have poor efficiency relative to the original design. It is, therefore, of interest to examine the efficiency of the residual design and decide robustness on the basis of Criterion 2. If d_0 is a binary block design with constant block size and d_t is the residual design, we take as a measure of efficiency of the residual design, the quantity E has given by

$$E = \frac{\text{Sum of reciprocals of non zero eigenvalues of } C_0}{\text{Sum of reciprocals of non zero eigenvalue of } C_t}$$

$$= \frac{tr\left(C_0^+\right)}{tr\left(C_t^+\right)}$$, where $C_0(C_t)$ is the C matrix of d_0 (d_t).

As the efficiency (E) is nothing but the ratio between the trace values of C_0^+ and C_t^+ matrices, thus, the efficiency value will be decreased if the number of deleted observations of the original design (d_0) will increase.

3. APPLICATION OF ABOVE THEOREMS FOR DIFFERENT TYPES OF GEB DESIGNS

In the present section of the article, results are shown for robust properties of GEB block designs using different design parameters after deletion of a single treatment or multiple treatments. Different types of GEB block designs are considered and the properties of robustness for every design are examined.

3.1 Robustness properties of GEB designs developed from symmetric BIB design using criterion 1 & criterion 2

Let us consider a GEB design D^* with parameters v^*=8, b^*=7, $\mathbf{r}^*$=(4.$\mathbf{1}'_7$, 7), k^*=5, λ^*=(2, 4) developed from a symmetric BIB design D with parameters v=b=7, r=k=4, λ=2 as given in example 2.1.

So, the information matrix ($\mathbf{C_0}$) of the GEB design D^* is

$$\mathbf{C_0} = \begin{bmatrix} \boldsymbol{M_d} & c1_v \\ c1_v' & d \end{bmatrix}, \text{ where, } \mathbf{M_d} = (a\text{-}b)\left[I + \frac{J}{\left(\frac{a-b}{b}\right)} \right],$$

Where , a=64/20, b=-8/20, c=-16/20, d=112/20

Now, treatment 1 is deleted from 1st block. So, for the residual design, information matrix will be

$$\mathbf{C_t} = \begin{pmatrix} e & f & f & b & b & f & b & g \\ f & h & i & b & b & i & b & j \\ f & i & h & b & b & i & b & j \\ b & b & b & a & b & b & b & c \\ b & b & b & b & a & b & b & c \\ f & i & i & b & b & h & b & j \\ b & b & b & b & b & b & a & c \\ g & j & j & c & c & j & c & k \end{pmatrix}$$

where, a =64/20, b =-8/20, c =-16/20, e =48/20, f =-4/20, g =-12/20, h =63/20, I =-9/20, j =-17/20, k =111/20. From the relationship, $\mathbf{C_0} = \mathbf{C_t} + \mathbf{UU'}$, we get, $\mathbf{U'} = 1/\sqrt{20}$ (4, -1, -1, 0, 0, -1, 0, -1).

Now, according to Theorem 2.3, the design D is robust against the loss of any t ($\geq$1) observations pertaining to the same treatment according to Criteria 1 because

$$\left|\mathbf{I_t} - \mathbf{U'C_0^-U}\right| = 0.7292\ (>0).$$

According to Theorem 2.4, the residual design is also robust as per criteria 1 against loss of 1 observation pertaining to the first treatment as t=1 does not exceed the smallest positive eigen value of C_0 i.e. 3.6.

To find the efficiency of the design D we have to find C_0^+ & C_t^+. Then, $\text{tr}(C_0^+)$=2.823 and

$\text{tr}(C_t^+)$=2.9247.

So, efficiency of the residual design = $\dfrac{\text{tr}\left(C_0^+\right)}{\text{tr}\left(C_t^+\right)} = \dfrac{2.823}{2.9247} = 0.9652$

Next, treatment 1 is deleted from 1st & 3rd blocks. From the relationship, $C_0=C_t+UU'$, we get, $\left|I_t - U'C_0^-U\right|$ =0.4726(>0), which is positive definite. So, we can conclude that according to Theorem 2.3, the residual design is robust against the loss of 2 observations pertaining to the first treatment according to Criteria 1. According to Theorem 2.4, the residual design is robust as per criteria 1 against loss of 2 observations pertaining to the first treatment as t =2 does not exceed the smallest positive eigen value of C_0 i.e. 3.6.Here, $\text{tr}(C_t^+)$=3.1201.

So, efficiency of the residual design = $\dfrac{\text{tr}\left(C_0^+\right)}{\text{tr}\left(C_t^+\right)} = \dfrac{2.823}{3.1201} = 0.9048.$

Next, treatment 1 is deleted from 1st, 3rd & 6th blocks. Here, $\left|I_t - U'C_0^-U\right|$ =0.2297(>0), which is positive definite. So, we can conclude that according to Theorem 2.3, the residual design is robust against the loss of 3 observations pertaining to the first treatment according to Criteria 1 as $I_t\text{-}U'C_0^-U$ is positive definite. According to Theorem 2.4 also, the residual design is robust as per criteria 1 against loss of 3 observations pertaining to the first treatment as t =3 does not exceed the smallest positive eigen value of C_0 i.e. 3.6.

Here, $\text{tr}(C_t^+)$=3.6865 and the efficiency of the residual design = $\dfrac{\text{tr}\left(C_0^+\right)}{\text{tr}\left(C_t^+\right)} = \dfrac{2.823}{3.6865} = 0.7658.$

Table 1. Efficiency values of the residual GEB designs with respect to missing observations of 1st treatment (previously existing treatment of the BIB design)

Serial No.	Missing treatment no.	No. of missing observations	Blocks in which missing observation appears	Efficiency of the residual design over full GEB
1.	1	1	1	0.9652
2.	1	2	1, 3	0.9048
3.	1	3	1, 3, 6	0.7658

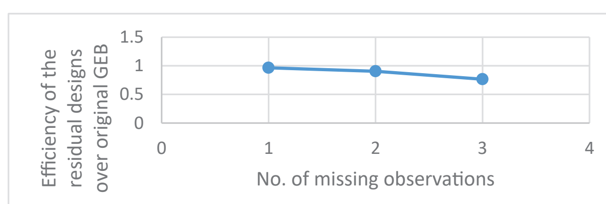

Fig. 1. Line diagram of Efficiency of the residual GEB designs over original GEB developed from BIB design

Now, the added treatment 8 is deleted from 1st, 2nd, 3rd, 4th, 5th and 6th blocks taking one at time, two at a time and so on up to 5 at a time. The results are presented in Table 2 and the declining trend of efficiency values are presented in Fig. 2.

Table 2. Efficiency values of the residual GEB designs with respect to missing observations for 8^{th} treatment (added treatment to the BIB design)

Serial No.	Missing treatment number	No. of missing observations	Blocks in which missing observation appears	Determinant of $I_t - U'C_0^- U$	Smallest possible nom-zero Eigen value of C_0	Efficiency of the residual designs over full GEB
1.	8	1	1	0.8333	3.6	0.9878
2.	8	2	1, 2	0.675	3.6	0.9724
3.	8	3	1, 2, 3	0.5251	3.6	0.9514
4.	8	4	1, 2, 3, 4	0.3829	3.6	0.9198
5.	8	5	1, 2, 3, 4, 5	0.248	3.6	0.8467
6.	8	6	1, 2, 3, 4, 5, 6	0.12	3.6	0.7361

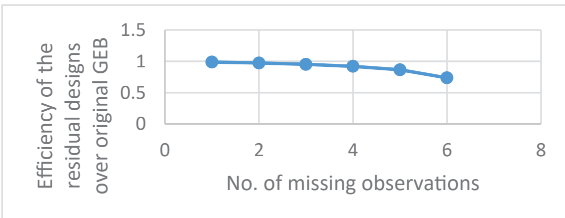

Fig. 2. Line diagram of Efficiency of the residual designs over original GEB for deletion of treatment no. 8 (newly added treatment to the BIB design)

Now, 1^{st} block is deleted from GEB D* with parameters $v^*=8$, $b^*=7$, $r^*=(4, 7)$, $k^*=5$, $\lambda^*=(2, 4)$.

From the relationship, $C_0=C_k+VV'$, we get,

$$V'=1/5\begin{pmatrix} 4 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 & 0 & 0 \end{pmatrix}$$

Here, $\left|I_k - V'C_0^- V\right|=0.272(>0)$, which is positive definite.

So, we can conclude that according to theorem 2.5, the residual design is robust against the loss all observations in a block as $I_k - V'C_0^- V$ is positive definite as per criterion 1.

According to theorem 2.6, the residual design is also robust as per criteria 1 against loss of all observations in a block as the smallest positive eigen value of C i.e. 3.6 is strictly larger than unity. So, tr(C_t^+)=3.1782 and the efficiency of the residual design

$$= \frac{tr\left(C_0^+\right)}{tr\left(C_t^+\right)} = \frac{2.823}{3.1782} = 0.8882.$$

3.2 The robust GEB designs developed from asymmetric binary Variance Balanced (VB) design

Let D be a asymmetric binary VB design with parameters $v=8$, $b=14$, $r=7$, $k=4$, $\lambda=3$. One extra treatment is added to each block of the previously defined design. Let D* be the resultant GEB design with parameters $v^*=9$, $b^*=14$, $\mathbf{r}^*=(7.\ \mathbf{1}'_7,\ 14)$, $k^*=5$, $\lambda^*=(3, 7)$. The design D* is the following:

Bl.1	Bl.2	Bl.3	Bl.4	Bl.5	Bl.6	Bl.7	Bl.8	Bl.9	Bl.10	Bl.11	Bl.12	Bl.13	Bl.14
1	2	3	4	5	6	7	1	2	3	4	5	6	7
2	3	4	5	6	7	1	2	3	4	5	6	7	1
4	5	6	7	1	2	3	4	5	6	7	1	2	3
7	1	2	3	4	5	6	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9	9	9	9	9	9

Table 3. Efficiency values of the residual GEB designs with respect to missing observations of 1^{st} treatment (previously existing treatment of the BIB design)

Serial No.	Missing treatment no.	No. of missing observations	Blocks in which missing observation appears	Smallest possible nom- zero Eigen value of C_0	Efficiency of the residual design over full GEB
1.	1	1	1	6.2	0.9866
2.	1	2	1, 2	6.2	0.9684
3.	1	3	1, 2, 5	6.2	0.9443
4.	1	4	1, 2, 5, 7	6.2	0.9052
5.	1	5	1, 2, 5, 7, 8	6.2	0.8404
6.	1	6	1, 2, 5, 7, 8, 12	6.2	0.6909

Table 4. Efficiency values of the residual GEB designs with respect to missing observations of 9th treatment (added treatment to the BIB design)

Serial No.	Missing treatment number	No. of missing observations	Blocks in which missing observation appears	Smallest possible nom-zero Eigen value of C_0	Efficiency of the residual designs over full GEB
1.	9	1	1	6.2	0.9959
2.	9	2	1, 2	6.2	0.9901
3.	9	3	1, 2, 3	6.2	0.9857
4.	9	4	1, 2, 3, 4	6.2	0.9796
5.	9	5	1, 2, 3, 4, 5	6.2	0.9727
6.	9	6	1, 2, 3, 4, 5, 6	6.2	0.9642
7.	9	7	1, 2, 3, 4, 5, 6, 7	6.2	0.9525
8.	9	8	1, 2, 3, 4, 5, 6, 7, 8	6.2	0.9415
9.	9	9	1, 2, 3, 4, 5, 6, 7, 8, 9	6.2	0.9261
10.	9	10	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	6.2	0.9034
11.	9	11	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	6.2	0.8685
12.	9	12	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	6.2	0.8089
13.	9	13	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13	6.2	0.6695

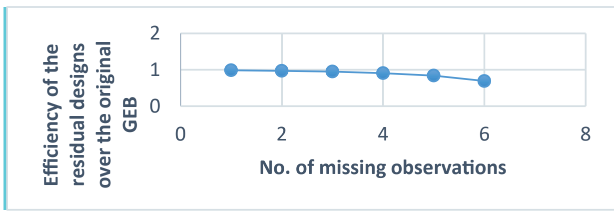

Fig. 3. Line diagram of Efficiency of the residual designs over original GEB design for treatment no. 1 (existing treatment of the VB design)

Fig. 4. Line diagram of Efficiency of the residual designs over original GEB design for treatment no. 9 (added treatment)

3.3 The efficiency values of the robust GEB designs developed from non-binary Variance Balanced (VB) design

Let us consider the cyclic incomplete block design (with imposed association relation among the treatments) D_1 with the following parameters (v =8, b_1 = 8, r_1 = 4, k_1 = 4, λ_{11} = 2, λ_{12} = 1 , n_1 = 5, n_2 = 2) & another design D_2 with parameters (v =8, b_2 =8, r_2 =2, k_2 =4, λ_{21} =0 λ_{22} =1) the treatments having same association relation & construct the design D^* (D^* is a GEB design with parameters: v^* =9, b^* =16, $r^{*'} = \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 16 \end{pmatrix}$, k^* =4, s =1/2, z = 1, g =5, a =10) with t =1, p =0, q =2 as following

Bl.1	Bl.2	Bl.3	Bl.4	Bl.5	Bl.6	Bl.7	Bl.8	Bl.9	Bl.10	Bl.11	Bl.12	Bl.13	Bl.14	Bl.15	Bl.16
1	2	3	4	5	6	7	8	1	1	2	2	3	3	4	5
2	3	4	5	6	7	8	1	4	6	5	7	6	8	7	8
3	4	5	6	7	8	1	2	9	9	9	9	9	9	9	9
5	6	7	8	1	2	3	4	9	9	9	9	9	9	9	9

Table 5. Efficiency values of the residual non- binary GEB designs with respect to missing observations of 1st treatment (previously existing treatment of the VB design)

Serial No.	Missing treatment no.	No. of missing observations	Blocks in which missing observation appears	Smallest possible non- zero Eigen value of C_0	Efficiency of the residual design over full GEB
1.	1	1	1	5.0	0.9809
2.	1	2	1, 5	5.0	0.9544
3.	1	3	1, 5, 7	5.0	0.9129
4.	1	4	1, 5, 7, 8	5.0	0.8428
5.	1	5	1, 5, 7, 8, 9	5.0	0.7005

Table 6. Efficiency values of the residual non-binary GEB designs with respect to missing observations of 9th treatment (added treatment to the VB design)

Sl. No.	Missing treatment number	No. of missing observations	Blocks in which missing observation appears	Smallest possible non-zero Eigen value of C_0	Efficiency of the residual designs over full GEB
1.	9	1	9(1T)	5.0	0.9964
2.	9	2	9(2T)	5.0	0.9863
3.	9	3	9(2T), 10(1T)	5.0	0.9809
4.	9	4	9(2T), 10(2T)	5.0	0.9684
5.	9	5	9(2T), 10(2T), 11(1T)	5.0	0.9632
6.	9	6	9(2T), 10(2T), 11(2T)	5.0	0.9512
7.	9	7	9(2T), 10(2T), 11(2T), 12(1T)	5.0	0.9444
8.	9	8	9(2T), 10(2T), 11(2T), 12(2T)	5.0	0.9256
9.	9	9	9(2T), 10(2T), 11(2T), 12(2T), 13(1T)	5.0	0.9175
10.	9	10	9(2T), 10(2T), 11(2T), 12(2T), 13(2T)	5.0	0.8935
11.	9	11	9(2T), 10(2T), 11(2T), 12(2T), 13(2T), 14(1T)	5.0	0.8813
12.	9	12	9(2T), 10(2T), 11(2T), 12(2T), 13(2T), 14(2T)	5.0	0.8432
13.	9	13	9(2T), 10(2T), 11(2T), 12(2T), 13(2T), 14(2T), 15(1T)	5.0	0.8171
14.	9	14	9(2T), 10(2T), 11(2T), 12(2T), 13(2T), 14(2T), 15(2T)	5.0	0.7176
15.	9	15	9(2T), 10(2T), 11(2T), 12(2T), 13(2T), 14(2T), 15(2T), 16(1T)	5.0	0.6368

*1T means one time or once.

*2T means two time or twice.

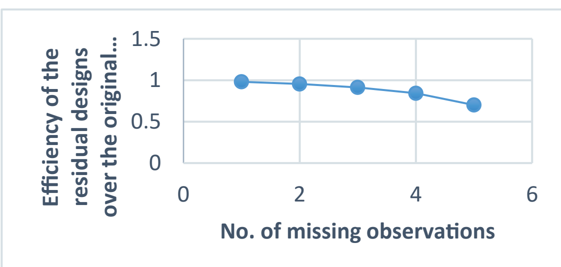

Fig. 5. Line diagram of Efficiency of the residual designs over original non- binary GEB design for treatment no. 1 (existing treatment of the VB design)

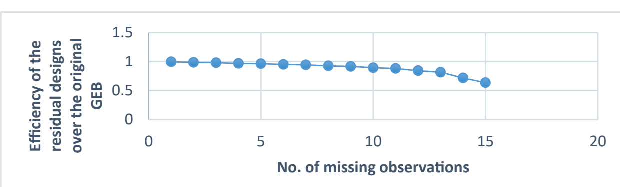

Fig. 6. Line diagram of Efficiency of the residual designs over original non- binary GEB design for treatment no. 9 (added treatment to the VB design)

4. SUMMARY & CONCLUSION

It has been observed that all the residual designs after deleting the observations from the GEB designs developed from either BIB designs (Symmetric or asymmetric) or VB designs, are robust. The designs will be robust after deleting either any treatment from the existing BIB designs (e.g. 1st) or the added treatment (e.g. 8th) under criteria 1 according to theorem 2.3 and 2.4. But the efficiency values will decrease when the number of missing observations is increased for both of the treatments, e.g., 1st and 8th treatments, respectively. Comparing the tables 3.1 and 3.2 and figures 3.1 and 3.2, it has been observed that when the number of replication is large for any particular treatment then one or two missing observations don't make drastic changes in efficiency values over the original design. But, when more than half of replication number of a treatment is missing, the efficiency of the design will be extremely low and one can't opt for such designs. It can be concluded that if the number of replications less with respect to a particular treatment then a single missing observation plays a significant role in making a quite low efficiency of the residual design. For example, in the design of example 2.1, when the 1st treatment is deleted from the 1st block, the efficiency of the residual design over the original GEB is 0.9652, whereas, missing of the 8th treatment from the 1st block gave the efficiency of the residual design as 0.9878 over the original GEB design which is greater than previous case (as discussed in section 3.1). Besides, when the 1st treatment is deleted for two times then the efficiency of the residual design is 0.9048 over the original GEB design, whereas, deleting the 8th treatment for two times, the efficiency of the residual design is 0.9724 over the original GEB, which is greater than previous case. So, it can be concluded that in case of treatment with higher replication value, loss of a single observation is not that much harmful and robustness criteria have maintained. But it is also true that missing of a huge number of observations of a particular treatment makes the design inefficient though it fulfils the robustness criteria 1.It has also been observed that when one block is lost then the residual design is a robust design with high enough efficiency value. Thus, it can be concluded from the study that GEB designs developed from either BIB designs (symmetric or asymmetric) or from non-binary variance balanced designs will always be robust after deletion of t(≥1) observations of any particular treatment from a GEB designs either from the existing treatments in BIB design (or VB design) or from the added treatment. Even, they will be robust after deletion of all treatments in a block of the GEB design.

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REFERENCES

Bhar, L. and Dey, A. (2003). Robustness of nested balanced incomplete block designs against missing data. *Journal of the Indian Society of Agriculture Statistics,* **56(1),** 25-38.

Bhar, L. (2014). Robustness of variance balanced block designs. *Sankhya B*, **76**, 305-316.

Dey, A. (1986). *Theory of block design*. A Halsted Press book. John Wiley, New York.

Dey, A. (1993). Robustness of block designs against missing data. *Statistica Sinica*, **03**, 219-231.

Duan, X. and Kageyama, S. (1996). Robustness of variance balanced designs against unavailability of some observations. *Hiroshima Mathematical Journal*, **26**, 351-362.

Dutta, G., Mandal, N.K. and Das, P. (2020). Designs robust against presence of an outlier in an analysis of covariance model. *Journal of the Indian Society for Probability and Statistics*. **21**, 315-327.

Ekpo, A., Ikughur, J.A., Nja, M.E. and Nwaosu, S.C. (2021). Comparison of robustness of two partially balanced incomplete block designs [PBIBD (2)] using optimality criteria. *Asian Journal of Mathematics & Statistics*, **14(1),** 1-12.

Ghosh, S. (1982). Robustness of BIBD against the unavailability of data. *Journal of Statistical Planning and Inference,* **6**, 29-32.

Ghosh, S., Rao, S.B. and Singhi, N.(1983). On a robustness property of PBIBD. *Journal of Statistical Planning and Inference*, **8(3),** 355-363.

Godolphin, J.D. and Warren, H.R. (2011). Improved conditions for robustness of binary block designs against the loss of whole blocks. *Journal of Statistical Planning and Inference*, **141**, 3498-3505.

Godolphin, J. and Warren, H.R. (2021). Investigation into the robustness of balanced incomplete block designs. *Statistics and Applications*, **19(1),** 467-481.

Gupta, V.K. and Srivastava, R. (1992). Investigations on robustness of block designs against missing observations. *Sankhya: The Indian Journal of Statistics,* **54(1)**, 100-105.

Hemavathi, M., Varghese, E., Shekhar, S., Athulya, C.K., Ebeneezar, S., Gills, R. and Jaggi, S. (2022). Robustness of sequential third-order response surface design to missing observations. *Journal of Taibah University for Science***, 16(1)**, 270-279.

Kageyama, S. and Mukerjee, R. (1986). General balanced designs through reinforcement. *Sankhya B*, **48**, 380-387.

Majumder, A., Patil, S.G. and Manjunatha, G.R. (2013). General efficiency balanced (GEB) block designs with correlated observations for even number of treatments. *Calcutta Statistical Association Bulletin*, **65**, 257-260.

Srivastava, R., Gupta, V.K. and Dey, A. (1991). Robustness of some designs against missing data. *Journal of Applied Statistics,* **18**, *313–318.*

Srivastava, R., Rajender, P. Gupta, V.K. (1996). Robustness of block designs making test treatments control comparison against a missing observation. *Sankhya* B, **58**, 407-413.