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Logarithmic type Direct and Synthetic Estimators using Bivariate Auxiliary Information with an Application to Real Data

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SUMMARY

This research work addresses bivariate auxiliary information-based logarithmic type direct and synthetic estimators for domain means in simple random sampling (SRS). The mean square error (MSE) of the suggested estimators is obtained, approximately to the first order. The efficiency standards by which the superiority of the suggested estimators is asserted are established. To demonstrate the superiority of the suggested estimators, a simulation investigation employing an artificially constructed normal population through the R programming language is also conducted. The analysis of real data from Swedish municipalities and the paddy crop acreage in the Mohanlal Ganj tehsil, Uttar Pradesh, India, also provides some applicability for the suggested estimators.

Keywords: Domain means; Direct and synthetic estimators; Auxiliary information; Simple random sampling.

MSC2020: 62D05.

1. INTRODUCTION

Small area estimation (SAE) is a branch of statistical research that combines survey sampling, statistical models, and conclusions regarding a finite population. The requirement for small area estimation methodologies has also arisen as a result of agricultural development planning for quick production increase, to gather information regarding different sectors of economy, section of people, geographical regions, cultivable land, water, minerals, oils, etc. Large-scale surveys can give information at higher levels of aggregation at the national and state levels due to sample designs. Recently, the government has changed its planning priorities from the macro to the micro levels, recognizing the need of accurate data at the lower levels of aggregation, such as tehsil, block, and village panchayat, for the effective and optimal use of economic resources.

The concept of a small area emerges from large-scale surveys where it is crucial to estimate not only the variables of the total or mean population but also the parameters of the so-called domain of subpopulations. These domain estimators are referred to as direct estimators if they just rely on domain-specific sample data. A direct estimator may also employ the available auxiliary data relating to the parameter of interest. In his book, Rao (2003) provides a detailed explanation of the estimator based on the direct technique of estimation.

Estimates of the parameters for these domains may not be accurate when the domain sizes are very small since the traditional direct sampling technique may not adequately reflect such domains in the sample. In such conditions, the estimate of the population parameter is done using an indirect (synthetic) estimator. With the use of data from units belonging to other domains that are similar to the small domain of interest, the main aim

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of these approaches is to elaborate the effective sample size for each domain. Tikkiwal and Ghiya (2000) suggested generalized synthetic estimator for domain mean for the estimation of crop acreage. Tikkiwal *et al.* (2013) examined the performance of the generalized regression estimator for small domains. The reader is recommended to refer Sisodia and Singh (2001), Sisodia and Chandra (2012), Sharma and Sisodia (2016) for more detailed study about SAE.

The survey researchers improve the efficiency of their proposed estimators by efficiently utilizing the auxiliary information. These information's are associated with the auxiliary variable which are strongly correlated with the variable under study. The estimation of parameters that are relevant in small areas is therefore typically done using auxiliary data in sample surveys. A miniscule work has been done for domain mean estimation using bivariate auxiliary information in SRS. Khare and Ashutosh (2018) extended the generalized synthetic estimator of Tikkiwal and Ghiya (2000) under SRS using bivariate auxiliary information. These authors studied bivariate auxiliary information based direct and synthetic estimators separately. The objective of this manuscript is different from above mentioned studies. Here, we propose a logarithmic type direct and synthetic estimators simultaneously for the estimation of domain mean under SRS using bivariate auxiliary information.

1.1 Notations

Let's assume that an estimate-required finite population $\Omega=(\Omega_1,\Omega_2,\dots,\Omega_N)$ consists of A non-overlapping small regions, or domains Ω_a of size N_a. Depending on the circumstance, the domains may take many different forms and represent small areas of a sampled population, such as a district, tehsil, or other state-level unit. Let y represent the variable being studied. Assume, therefore, that the auxiliary data, represented by x and z, is likewise accessible. A simple random sample of size n is picked without replacement with the condition that n_a, a=1,2,…,A units in the sample 's' originate from the small area 'A'. Hence, $\sum N_a = N$ and $\sum n_a = n$. The study variable y, together with the auxiliary variables x and z, have the following notations for its population and sample means:

Let $\bar{Y}$ and $\bar{Y}_a$ be the population means based on N and N_a observations, respectively, on the variable y, while $\bar{y}$ and $\bar{y}_a$ be the sample means based on N and N_a observations, respectively, on the variable y. Let $\bar{X}$ and $\bar{Z}$ be the population means of auxiliary variables x and z, respectively, based on N observations, while $\bar{X}_a$ and $\bar{Z}_a$ be the population means of auxiliary variables x and z for a small domain based on N_a observations. Let $\bar{x}$ and $\bar{z}$ be the sample means based on n observations on variables x and z, respectively, while $\bar{x}_a$ and $\bar{z}_a$ be the sample mean based on n_a observations on variables x and z, respectively. We consider the following notations in order to determine the variables of the direct estimators:

$$\bar{y}_a = \bar{Y}_a(1+e_0)\bar{x}_a = \bar{X}_a(1+e_1)\bar{z}_a = \bar{Z}_a(1+e_2)$$

such that $E(e_i)=0, |e_i|<1; i=0,1,2,$
$E(e_0^2) = f_a C_{y_a}^2, E(e_1^2) = f_a C_{x_a}^2, E(e_2^2) = f_a C_{z_a}^2$ and
$E(e_0 e_1) = f_a \rho_{y_a x_a} C_{y_a} C_{x_a},\ E(e_0 e_2) = f_a \rho_{y_a z_a} C_{y_a} C_{z_a}$
and $E(e_1 e_2) = f_a \rho_{x_a z_a} C_{x_a} C_{z_a}$
where $f_a = (N_a - n_a)/N_a n_a$,
$S_{x_a}^2 = (N_a-1)^{-1}\sum_{i=1}^{N_a}(X_{ai}-\bar{X}_a)^2 S_{y_a}^2$
$= (N_a-1)^{-1}\sum_{i=1}^{N_a}(Y_{ai}-\bar{Y}_a)^2 S_{z_a}^2$
$= (N_a-1)^{-1}\sum_{i=1}^{N_a}(Z_{ai}-\bar{Z}_a)^2$,
$C_{x_a} = S_{x_a}/\bar{X}_a, C_{y_a} = S_{y_a}/\bar{Y}_a, C_{z_a} = S_{z_a}/\bar{Z}_a, S_{x_a y_a}$
$= (N_a-1)^{-1}\sum_{i=1}^{N_a}(X_{ai}-\bar{X}_a)(Y_{ai}-\bar{Y}_a)$
and $C_{x_a y_a} = \rho_{x_a y_a} C_{x_a} C_{y_a}$, respectively such that $X_{ai}, a=1,2,\dots,A$ and $i=1,2,\dots,N_a$ denote the i^{th} observation of small domain a of the population for the variable x and Y_{ai}, denote the i^{th} observation of small domain a for the variable y.

Similarly, we consider the following notations in order to determine the variables of the synthetic estimators:

$$\bar{y} = \bar{Y}(1+\varepsilon_0), \bar{x} = \bar{X}(1+\varepsilon_1), \bar{z} = \bar{Z}(1+\varepsilon_2),$$

such that
$E(\varepsilon_i)=0, |\varepsilon_i|<1; i=0,1,2, E(\varepsilon_0^2) = fC_y^2, E(\varepsilon_1^2)$
$= fC_x^2, E(\varepsilon_2^2) = fC_z^2, E(\varepsilon_0\varepsilon_1) = f\rho_{yx}C_yC_x, E(\varepsilon_0\varepsilon_2)$
$= f\rho_{yz}C_yC_z, E(\varepsilon_1\varepsilon_2) = f\rho_{xz}C_xC_z$,

where

$f = (N-n)/Nn, S_y^2 = (N-n)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$, $S_x^2 = (N-n)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})^2, S_z^2 = (N-n)^{-1}\sum_{i=1}^{N}(z_i - \bar{Z})^2$, $C_y = S_y/\bar{Y}, C_x = S_x/\bar{X}, C_z = S_z/\bar{Z}$, $S_{yx} = (N-n)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X}), S_{yz} = (N-n)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})(z_i - \bar{Z}), S_{xz} = (N-n)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})(z_i - \bar{Z}), C_{yx} = \rho_{yx}C_yC_x$, $C_{yz} = \rho_{yz}C_yC_z$, and $C_{xz} = \rho_{xz}C_xC_z$.

Section 2 contains a summary for both direct and synthetic methods of estimation. With the use of additional data, we suggest a bivariate auxiliary information based logarithmic type direct and synthetic estimators for calculating domain mean in Section 3. In Section 5, the effectiveness of the suggested estimators has been evaluated through the use of a simulation experiment. In Section 4, the application of the proposed direct and synthetic estimators is further illustrated using Sweden municipalities data and paddy crop acreage data of Mohanlal Ganj tehsil, Uttar Pradesh, India. Section 6 of this paper contains a brief conclusion.

2. EXISTING DIRECT AND SEPARATE ESTIMATORS FOR DOMAIN MEAN

The present section provides all prominent existing direct and synthetic estimators of domain mean based on bivariate auxiliary information.

2.1 Direct estimators

The variance of the mean per unit estimator $\bar{y}_{m,a}^d = \bar{y}_a$ is reported below as

$$V\left(\bar{y}_{m,a}^d\right) = f_a\bar{Y}_a^2C_{y_a}^2$$

The direct ratio estimator based on bivariate auxiliary information is reported below as

$$\bar{y}_{r,a}^d = \bar{y}_a\left(\frac{\bar{X}_a}{\bar{x}_a}\right)\left(\frac{\bar{Z}_a}{\bar{z}_a}\right)$$

The MSE of the estimator $\bar{y}_{r,a}^d$ is reported below as

$$MSE\left(\bar{y}_{r,a}^d\right) = f_a\bar{Y}_a^2\left(C_{y_a}^2 + C_{x_a}^2 + C_{z_a}^2 - 2\rho_{y_ax_a}C_{y_a}C_{x_a} - 2\rho_{y_az_a}C_{y_a}C_{z_a} - 2\rho_{y_az_a} + 2\rho_{x_az_a}C_{x_a}C_{z_a}\right)$$

Table 1. Few members of the generalized class of direct estimators $\bar{y}_{(j),a}^d$.

Members of the estimators $\bar{y}_{(j),a}^d, j = 1, 2, \ldots, 8$	a	b	c	d
$\bar{y}_{(1),a}^d = \bar{y}_a\left(\frac{\bar{X}_a + \beta_2(x_a)}{\bar{x}_a + \beta_2(x_a)}\right)\left(\frac{\bar{Z}_a + \beta_2(z_a)}{\bar{z}_a + \beta_2(z_a)}\right)$	1	$\beta_2(x_a)$	1	$\beta_2(z_a)$
$\bar{y}_{(2),a}^d = \bar{y}_a\left(\frac{\bar{X}_a + C_{x_a}}{\bar{x}_a + C_{x_a}}\right)\left(\frac{\bar{Z}_a + C_{z_a}}{\bar{z}_a + C_{z_a}}\right)$	1	C_{x_a}	1	C_{z_a}
$\bar{y}_{(3),a}^d = \bar{y}_a\left(\frac{C_{x_a}\bar{X}_a + \beta_2(x_a)}{C_{x_a}\bar{x}_a + \beta_2(x_a)}\right)\left(\frac{C_{z_a}\bar{Z}_a + \beta_2(z_a)}{C_{z_a}\bar{z}_a + \beta_2(z_a)}\right)$	C_{x_a}	$\beta_2(x_a)$	C_{z_a}	$\beta_2(z_a)$
$\bar{y}_{(4),a}^d = \bar{y}_a\left(\frac{\bar{X}_a + \rho_{y_ax_a}}{\bar{x}_a + \rho_{y_ax_a}}\right)\left(\frac{\bar{Z}_a + \rho_{y_az_a}}{\bar{z}_a + \rho_{y_az_a}}\right)$	1	$\rho_{y_ax_a}$	1	$\rho_{y_az_a}$
$\bar{y}_{(5),a}^d = \bar{y}_a\left(\frac{C_{x_a}\bar{X}_a + \rho_{y_ax_a}}{C_{x_a}\bar{x}_a + \rho_{y_ax_a}}\right)\left(\frac{C_{z_a}\bar{Z}_a + \rho_{y_az_a}}{C_{z_a}\bar{z}_a + \rho_{y_az_a}}\right)$	C_{x_a}	$\rho_{y_ax_a}$	C_{z_a}	$\rho_{y_az_a}$
$\bar{y}_{(6),a}^d = \bar{y}_a\left(\frac{\rho_{y_ax_a}\bar{X}_a + C_{x_a}}{\rho_{y_ax_a}\bar{x}_a + C_{x_a}}\right)\left(\frac{\rho_{y_az_a}\bar{Z}_a + C_{z_a}}{\rho_{y_az_a}\bar{z}_a + C_{z_a}}\right)$	$\rho_{y_ax_a}$	C_{x_a}	$\rho_{y_az_a}$	C_{z_a}
$\bar{y}_{(7),a}^d = \bar{y}_a\left(\frac{\rho_{y_ax_a}\bar{X}_a + \beta_2(x_a)}{\rho_{y_ax_a}\bar{x}_a + \beta_2(x_a)}\right)\left(\frac{\rho_{y_az_a}\bar{Z}_a + \beta_2(z_a)}{\rho_{y_az_a}\bar{z}_a + \beta_2(z_a)}\right)$	$\rho_{y_ax_a}$	$\beta_2(x_a)$	$\rho_{y_az_a}$	$\beta_2(z_a)$
$\bar{y}_{(8),a}^d = \bar{y}_a\left(\frac{S_{x_a}\bar{X}_a + \beta_2(x_a)}{S_{x_a}\bar{x}_a + \beta_2(x_a)}\right)\left(\frac{S_{z_a}\bar{Z}_a + \beta_2(z_a)}{S_{z_a}\bar{z}_a + \beta_2(z_a)}\right)$	S_{x_a}	$\beta_2(x_a)$	S_{z_a}	$\beta_2(z_a)$

The direct generalized ratio estimator based on bivariate auxiliary information is reported below as

$$\bar{y}^{d}_{(j),a} = \bar{y}_a\left(\frac{a\bar{X}_a + b}{a\bar{x}_a + b}\right)\left(\frac{c\bar{Z}_a + d}{c\bar{z}_a + d}\right)$$

where $a, b, c,$ and d are either real values or function of known population parameters such as coefficient of kurtosis, standard deviation, coefficient of variation of auxiliary variables x and z, and coefficient of correlation of study and auxiliary variables. Following Sisodia and Dwivedi (1981), we provide some sub class of the estimator $\bar{y}^{d}_{(j),a}$ in Table 1 for different values of $a, b, c,$ and d. The MSE of the estimator $\bar{y}^{d}_{(j),a}$ is reported below as

$$MSE\left(\bar{y}^{d}_{(j),a}\right) = f_a\bar{Y}_a^2\left(C_{y_a}^2 + \upsilon_1^2 C_{x_a}^2 + \upsilon_2^2 C_{z_a}^2 - 2\upsilon_1\rho_{y_a}\right.$$

where

$\upsilon_1 = a\bar{X}_a/(a\bar{X}_a + b)$ and $\upsilon_2 = c\bar{Z}_a/(c\bar{Z}_a + d)$.

2.2 Synthetic estimators

The variance of the estimator $\bar{y}^{s}_{m,a} = \bar{y}$ is reported below as

$$V\left(\bar{y}^{s}_{m,a}\right) = (\bar{Y} - \bar{Y}_a)^2 + f\bar{Y}^2 C_y^2$$

The synthetic ratio estimator based on bivariate auxiliary information is reported below as

$$\bar{y}^{s}_{r,a} = \bar{y}\left(\frac{\bar{X}_a}{\bar{x}}\right)\left(\frac{\bar{Z}_a}{\bar{z}}\right)$$

Under the synthetic assumption $\bar{Y}_a = \bar{Y}(\bar{X}_a/\bar{X})(\bar{Z}_a/\bar{Z})$, the MSE of the estimator $\bar{y}^{s}_{r,a}$ is reported below as

$$MSE\left(\bar{y}^{s}_{r,a}\right) = \bar{Y}_a^2 f\left(C_y^2 + C_x^2 + C_z^2 - 2\rho_{yx}C_yC_x - 2\rho_{yz}C_yC_z + 2\rho_{xz}C_xC_z\right)$$

The synthetic generalized ratio estimator based on bivariate auxiliary information is given as

$$\bar{y}^{s}_{(j),a} = \bar{y}\left(\frac{e\bar{X}_a + f}{e\bar{x} + f}\right)\left(\frac{g\bar{Z}_a + h}{g\bar{z} + h}\right)$$

where $e, f, g,$ and h are either real values or function of known population parameters such as coefficient of kurtosis, standard deviation, coefficient of variation of auxiliary variables and coefficient of correlation of study variable and auxiliary variables. Following Sisodia and Dwivedi (1981), we provide some sub class of the estimator $\bar{y}^{s}_{(j),a}$ in Table 2 for different values of $e, f, g,$ and h.

Table 2. Few members of the generalized class of synthetic estimators $\bar{y}^{s}_{(j),a}$

Members of the estimators $\bar{y}^{s}_{(j),a}, j = 1, 2, \ldots, 8$	e	f	g	h
$\bar{y}^{s}_{(1),a} = \bar{y}\left(\frac{\bar{X}_a + \beta_2(x)}{\bar{x} + \beta_2(x)}\right)\left(\frac{\bar{Z}_a + \beta_2(z)}{\bar{z} + \beta_2(z)}\right)$	1	$\beta_2(x)$	1	$\beta_2(z)$
$\bar{y}^{s}_{(2),a} = \bar{y}\left(\frac{\bar{X}_a + C_x}{\bar{x} + C_x}\right)\left(\frac{\bar{Z}_a + C_z}{\bar{z} + C_z}\right)$	1	C_x	1	C_z
$\bar{y}^{s}_{(3),a} = \bar{y}\left(\frac{C_x\bar{X}_a + \beta_2(x)}{C_x\bar{x} + \beta_2(x)}\right)\left(\frac{C_z\bar{Z}_a + \beta_2(z)}{C_z\bar{z} + \beta_2(z)}\right)$	C_x	$\beta_2(x)$	C_z	$\beta_2(z)$
$\bar{y}^{s}_{(4),a} = \bar{y}\left(\frac{\bar{X}_a + \rho_{yx}}{\bar{x} + \rho_{yx}}\right)\left(\frac{\bar{Z}_a + \rho_{yz}}{\bar{z} + \rho_{yz}}\right)$	1	ρ_{yx}	1	ρ_{yz}
$\bar{y}^{s}_{(5),a} = \bar{y}\left(\frac{C_x\bar{X}_a + \rho_{yx}}{C_x\bar{x} + \rho_{yx}}\right)\left(\frac{C_z\bar{Z}_a + \rho_{yz}}{C_z\bar{z} + \rho_{yz}}\right)$	C_x	ρ_{yx}	C_z	ρ_{yz}
$\bar{y}^{s}_{(6),a} = \bar{y}\left(\frac{\rho_{yx}\bar{X}_a + C_x}{\rho_{yx}\bar{x} + C_x}\right)\left(\frac{\rho_{yz}\bar{Z}_a + C_z}{\rho_{yz}\bar{z} + C_z}\right)$	ρ_{yx}	C_x	ρ_{yz}	C_z
$\bar{y}^{s}_{(7),a} = \bar{y}\left(\frac{\rho_{yx}\bar{X}_a + \beta_2(x)}{\rho_{yx}\bar{x} + \beta_2(x)}\right)\left(\frac{\rho_{yz}\bar{Z}_a + \beta_2(z)}{\rho_{yz}\bar{z} + \beta_2(z)}\right)$	ρ_{yx}	$\beta_2(x)$	ρ_{yz}	$\beta_2(z)$
$\bar{y}^{s}_{(8),a} = \bar{y}\left(\frac{S_x\bar{X}_a + \beta_2(x)}{S_x\bar{x} + \beta_2(x)}\right)\left(\frac{S_z\bar{Z}_a + \beta_2(z)}{S_z\bar{z}_a + \beta_2(z)}\right)$	S_x	$\beta_2(x)$	S_z	$\beta_2(z)$

Under the synthetic assumption the $\bar{Y}_a = \bar{Y}\{(e\bar{X}_a + f/e\bar{X} + f)\}\{(g\bar{Z}_a + h/g\bar{Z} + h)\}$, the MSE of the estimator $\bar{y}^s_{(j),a}$ is reported below as

$$MSE(\bar{y}^s_{(j),a}) = \bar{Y}_a^2 f\left(C_y^2 + \upsilon_3^2 C_x^2 + \upsilon_4^2 C_z^2 - 2\upsilon_3\rho_{yx}C_yC_x - 2\upsilon_4\rho_{yz}C_yC_z + 2\upsilon_3\upsilon_4\rho_{xz}C_xC_z\right)$$

where $\upsilon_3 = e\bar{X}/e\bar{X} + f$ and $\upsilon_4 = g\bar{Z}/g\bar{Z} + h$.

3. PROPOSED DIRECT AND SYNTHETIC ESTIMATORS

Motivated by the study of Bhushan and Kumar (2022), we propose a logarithmic type direct and synthetic estimators for domain mean using bivariate auxiliary information under SRS.

3.1 Direct estimator

Assuming that the auxiliary variables $x_a > 0$ and $z_a > 0$ we propose the following logarithmic type direct estimator based on bivariate auxiliary information as

$$\bar{y}^d_{bk,a} = \bar{y}_a\left[1 + \lambda_a log\left(\frac{\bar{x}_a}{\bar{X}_a}\right)\right]\left[1 + \delta_a log\left(\frac{\bar{z}_a}{\bar{Z}_a}\right)\right]$$

where λ_a and δ_a are suitably chosen constants.

To find the MSE and minimum MSE of the proposed direct estimator $\bar{y}^d_{bk,a}$, we utilize thenotations provided in the earlier section and rewrite the proposed direct estimator $\bar{y}^d_{bk,a}$ as

$$\bar{y}^d_{bk,a} - \bar{Y}_a = \bar{Y}_a\left\{e_0 + \lambda_a e_1 + \delta_a e_2 - \frac{\lambda_a}{2}e_1^2 - \frac{\delta_a}{2}e_2^2 + \lambda_a e_0 e_1 + \delta_a e_0 e_2 + \lambda_a\delta_a e_1 e_2\right\}$$

Squaring and taking expectation on both sides to the above expression provides

$$MSE(\bar{y}^d_{bk,a}) = f_a\bar{Y}_a^2\left(C_{y_a}^2 + \lambda_a^2 C_{x_a}^2 + \delta_a^2 C_{z_a}^2 + 2\lambda_a\rho_{y_ax_a}C_{y_a}C_{x_a} + 2\delta_a\rho_{y_az_a}C_{y_a}C_{z_a} + 2\lambda_a\delta_a\rho_{x_az_a}C_{x_a}C_{z_a}\right) \quad (1)$$

Minimization of (1) regarding λ_a and δ_a provides the optimum values of λ_a and δ_a as

$$\lambda_{a(opt)} = \left(\frac{C_{y_a}}{C_{x_a}}\right)\left(\frac{\rho_{y_az_a}\rho_{x_az_a} - \rho_{y_ax_a}}{1-\rho^2_{x_az_a}}\right) \text{ and}$$

$$\delta_{a(opt)} = \left(\frac{C_{y_a}}{C_{z_a}}\right)\left(\frac{\rho_{y_az_a}\rho_{x_az_a} - \rho_{y_az_a}}{1-\rho^2_{x_az_a}}\right)$$

It is to be noted that the optimum values of λ_a and δ_a need the prior knowledge of $C_{y_a}, C_{x_a}, \rho_{y_az_a}, \rho_{x_az_a}$, and $\rho_{y_az_a}$ that can be obtained from past data or experience gathered in due course of time. In case, if the practitioner fails to guess the values of la and da by utilizing his all resources, it is worth advisable to replace λ_a and δ_a by its consistent estimate given below:

$$\hat{\lambda}_{a(opt)} = \left(\frac{\hat{C}_{y_a}}{\hat{C}_{x_a}}\right)\left(\frac{\hat{\rho}_{y_az_a}\hat{\rho}_{x_az_a} - \hat{\rho}_{y_ax_a}}{1-\hat{\rho}^2_{x_az_a}}\right) \text{ and}$$

$$\hat{\delta}_{a(opt)} = \left(\frac{\hat{C}_{y_a}}{\hat{C}_{z_a}}\right)\left(\frac{\hat{\rho}_{y_az_a}\hat{\rho}_{x_az_a} - \hat{\rho}_{y_az_a}}{1-\hat{\rho}^2_{x_az_a}}\right)$$

The minimum MSE of the proposed direct estimator $\bar{y}^d_{bk,a}$ is obtained by putting the values $\lambda_{a(opt)}$ and $\delta_{a(opt)}$ in (1) as

$$MSE(\bar{y}^d_{bk,a})_{min} = f_a\bar{Y}_a^2 C_{y_a}^2\left(1 - R^2_{y_a\cdot x_az_a}\right) \quad (2)$$

where $R^2_{y_a\cdot x_az_a}$ is the multiple correlation coefficient of y_a on x_a and z_a in domain a.

3.2 Synthetic estimator

Assuming that the auxiliary variables $x > 0$ and $z > 0$, we propose the following logarithmic type synthetic estimator based on bivariate auxiliary information as

$$\bar{y}^s_{bk,a} = \bar{y}\left[1 + \lambda log\left(\frac{\bar{x}}{\bar{X}_a}\right)\right]\left[1 + \delta log\left(\frac{\bar{z}}{\bar{Z}_a}\right)\right]$$

where λ and δ are suitably chosen scalars.

To find the MSE and minimum MSE of the proposed synthetic estimator $\bar{y}^s_{bk,a}$, we utilize the notations provided in the earlier section and rewrite the proposed synthetic estimator $\bar{y}^s_{bk,a}$ as

$$\bar{y}^s_{bk,a} = \bar{Y}(1+\varepsilon_0)\left[1 + \lambda A + \lambda\left(\varepsilon_1 - \frac{\varepsilon_1^2}{2} + \cdots\right)\right]\left[1 + \delta B + \delta\left(\varepsilon_2 - \frac{\varepsilon_2^2}{2} + \cdots\right)\right]$$

where $A = log\left(\frac{\bar{X}}{\bar{X}_a}\right)$ and $B = log\left(\frac{\bar{Z}}{\bar{Z}_a}\right)$. Further, neglecting the terms having power greater than 2 and subtracting $\bar{Y}_a$ both sides to the above expression, we get

$$\bar{y}^s_{bk,a} - \bar{Y}_a = \bar{Y}\left\{(1+\lambda A)(1+\delta B) + (1+\lambda A)\delta\left(\varepsilon_2 - \frac{\varepsilon_2^2}{2}\right) + (1+\delta B)\lambda\left(\varepsilon_1 - \frac{\varepsilon_1^2}{2}\right) + \lambda\delta\varepsilon_1\varepsilon_2 + (1+\lambda A)(1+\delta B)\varepsilon_0 + (1+\lambda A)\delta\varepsilon_0\varepsilon_2 + (1+\delta B)\lambda\varepsilon_0\varepsilon_1\right\} - \bar{Y}_a$$

Squaring and taking expectation on both sides to the above expression provides

$$MSE\left(\bar{y}^s_{bk,a}\right)=\left[\begin{array}{c}\bar{Y}_a^2 f\left\{\begin{array}{l}(1+\delta B)^2\lambda^2C_x^2+(1+\lambda A)^2\delta^2C_z^2+\\(1+\lambda A)^2(1+\delta B)^2C_y^2+2\lambda\delta(1+\lambda A)(1+\\\delta B)\rho_{xz}C_xC_z+2\lambda(1+\lambda A)(1+\delta B)^2\rho_{yx}C_yC_x\\2\delta(1+\lambda A)^2(1+\delta B)\rho_{yz}C_yC_z\end{array}\right\}\\+\{\bar{Y}(1+\lambda A)(1+\delta B)-\bar{Y}_a\}\bar{Y}f\\\left\{\begin{array}{c}-(1+\delta B)\lambda\frac{C_x^2}{2}-(1+\lambda A)\frac{C_z^2}{2}\\+\lambda\delta\rho_{xz}C_xC_z+(1+\lambda A)\delta\rho_{yz}C_yC_z\\+(1+\delta B)\lambda\rho_{yx}C_yC_x\end{array}\right\}\\+\{\bar{Y}(1+\lambda A)(1+\delta B)-\bar{Y}_a\}^2\end{array}\right]$$

Under the synthetic assumptions $\bar{Y}_a=\bar{Y}(1+\lambda A),\bar{Y}_a=\bar{Y}(1+\delta B)$ and $\bar{Y}_a=\bar{Y}(1+\lambda A)(1+\delta B)$,the above MSE expression becomes

$$MSE\left(\bar{y}^s_{bk,a}\right)=f\bar{Y}_a^2\left(C_y^2+\lambda^2C_x^2+\delta^2C_z^2+2\lambda\rho_{yx}C_yC_x+2\delta\rho_{yz}C_yC_z+2\lambda\delta\rho_{xz}C_xC_z\right)\tag{3}$$

Minimization of (12) regarding λ and δ provides the optimum values of λ and δ as

$$\lambda_{(opt)}=\left(\frac{C_y}{C_x}\right)\left(\frac{\rho_{yz}\rho_{xz}-\rho_{yx}}{1-\rho_{xz}^2}\right)\text{ and}$$

$$\delta_{(opt)}=\left(\frac{C_y}{C_z}\right)\left(\frac{\rho_{yx}\rho_{xz}-\rho_{yx}}{1-\rho_{xz}^2}\right)$$

It is to be noted that the optimum values of λ and δ need the prior knowledge of $C_y, C_x, C_z, \rho_{yz}, \rho_{xz}$, and ρ_{yx} that can be obtained from past data or experience gathered in due course of time. In case, if the practitioner fails to guess the values of λ and δ by utilizing his all resources, it is worth advisable to replace λ and δ by its consistent estimate given below:

$$\hat{\lambda}_{(opt)}=\left(\frac{\hat{C}_y}{\hat{C}_x}\right)\left(\frac{\hat{\rho}_{yz}\hat{\rho}_{xz}-\hat{\rho}_{yx}}{1-\hat{\rho}_{xz}^2}\right)\text{ and}$$

$$\hat{\delta}_{(opt)}=\left(\frac{\hat{C}_y}{\hat{C}_z}\right)\left(\frac{\hat{\rho}_{yx}\hat{\rho}_{xz}-\hat{\rho}_{yx}}{1-\hat{\rho}_{xz}^2}\right)$$

The minimum MSE of the proposed direct estimator $\bar{y}^s_{bk,a}$ is obtained by putting the values of $\lambda_{(opt)}$ and $\delta_{(opt)}$. in (3) as

$$MSE\left(\bar{y}^s_{bk,a}\right)_{min}=f\bar{Y}_a^2C_y^2\left(1-R^2_{y.xz}\right).\tag{4}$$

where $R^2_{y.xz}$ is the multiple correlation coefficient of y on x and z.

Corollary 3.1. *The proposed synthetic estimator $\bar{y}^s_{bk,a}$ outperforms the proposed direct estimator $\bar{y}^d_{bk,a}$ iff*

$$R_{y.xz}>\sqrt{1-\frac{f_aC^2_{y_a}}{fC_y^2}\left(1-R^2_{y_a.x_az_a}\right)}\tag{5}$$

and contrariwise. Otherwise, both are equally effective when the equality in (5) holds.

Proof. *We arrive at (5) by comparing the minimal MSEs of the suggested direct and synthetic estimators from (2) and (4), respectively.*

Further, we compare the MSEs of the proposed direct and synthetic estimators with the MSEs of the existing direct and synthetic estimators and the conditions are derived. Under these conditions, the proposed estimators outperform the reviewed estimators.

$$MSE\left(\bar{y}^d_{bk,a}\right)<MSE\left(\bar{y}^d_{m,a}\right)\Rightarrow R^2_{y_a.x_az_a}>1$$

$$MSE\left(\bar{y}^d_{bk,a}\right)<MSE\left(\bar{y}^d_{r,a}\right)\Rightarrow$$

$$R^2_{y_a.x_az_a}>-\frac{1}{C^2_{y_a}}\left(\begin{array}{c}C^2_{x_a}+C^2_{z_a}-2\rho_{y_ax_a}C_{y_a}C_{x_a}\\-2\rho_{y_az_a}C_{y_a}C_{z_a}+2\rho_{x_az_a}C_{x_a}C_{z_a}\end{array}\right)$$

$$MSE\left(\bar{y}^d_{bk,a}\right)<MSE\left(\bar{y}^d_{(j),a}\right)\Rightarrow$$

$$R^2_{y_a.x_az_a}>-\frac{1}{C^2_{y_a}}\left(\begin{array}{c}C^2_{y_a}+\upsilon_1^2C^2_{x_a}+\upsilon_2^2C^2_{z_a}-2\upsilon_1\rho_{y_ax_a}C_{y_a}C_{x_a}\\-2\upsilon_2\rho_{y_az_a}C_{y_a}C_{z_a}+2\upsilon_1\upsilon_2\rho_{x_az_a}C_{x_a}C_{z_a}\end{array}\right)$$

$$MSE\left(\bar{y}^s_{bk,a}\right)<MSE\left(\bar{y}^s_{m,a}\right)\Rightarrow R^2_{y.xz}>1-\frac{\left\{\left(\bar{Y}-\bar{Y}_a\right)^2+f\bar{Y}^2C_y^2\right\}}{f\bar{Y}_a^2C_y^2}$$

$$MSE\left(\bar{y}^s_{bk,a}\right)<MSE\left(\bar{y}^s_{r,a}\right)\Rightarrow$$

$$R^2_{y.xz}>-\frac{1}{C_y^2}\left(\begin{array}{c}C_x^2+C_z^2-2\rho_{yx}C_yC_x\\-2\rho_{yz}C_yC_z+2\rho_{xz}C_xC_z\end{array}\right)$$

$$MSE\left(\bar{y}^s_{bk,a}\right)<MSE\left(\bar{y}^s_{(j),a}\right)\Rightarrow$$

$$R^2_{y.xz}>-\frac{1}{C_y^2}\left(\begin{array}{c}C_y^2+\upsilon_3^2C_x^2+\upsilon_4^2C_z^2-2\nu_3\rho_{yx}C_yC_x\\-2\nu_4\rho_{yz}C_yC_z+2\nu_3\nu_4\rho_{xz}C_xC_z\end{array}\right)$$

4. SIMULATION STUDY

In this section, the suggested direct and synthetic estimators are put to the test via a simulation study on an artificial normal population generated through the R software. The normal population of size $N=12000$ is produced utilizing the parameters $\bar{Y}=16, \bar{X}=17, \bar{Z}=22, \sigma_y=70, \sigma_x=69, \sigma_z=80$, and different combinations of correlation coefficients ρ_{yx}, ρ_{yz}, and ρ_{xz} described in the Tables 3-4.

Six equal domains of size 2000 each make up the population. The descriptive statistics for each domain are computed from a simple random sample of size 400 that is taken from each domain. Now, using the formulas shown below, the MSE and percent relative efficiency (PRE) for direct and synthetic estimators are determined.

$$MSE\left(\overline{y}^{d}_{*,a}\right)=\frac{1}{18,000}\sum_{i=1}^{18000}\left(\overline{y}^{d}_{*,a}-\overline{Y}_{a}\right)^{2}$$

$$MSE\left(\overline{y}^{s}_{*,a}\right)=\frac{1}{18,000}\sum_{i=1}^{18000}\left(\overline{y}^{s}_{*,a}-\overline{Y}_{a}\right)^{2}$$

$$PRE\left(\overline{y}^{d}_{m,a},\overline{y}^{d}_{*,a}\right)=\frac{MSE\left(\overline{y}^{d}_{m,a}\right)}{MSE\left(\overline{y}^{d}_{*,a}\right)}\times 100$$

$$PRE\left(\overline{y}^{s}_{m,a},\overline{y}^{s}_{*,a}\right)=\frac{MSE\left(\overline{y}^{s}_{m,a}\right)}{MSE\left(\overline{y}^{s}_{*,a}\right)}\times 100$$

where

$\overline{y}^{d}_{*,a}=\overline{y}^{d}_{m,a},\overline{y}^{d}_{r,a},\overline{y}^{d}_{(j),a},\overline{y}^{d}_{bk,a}$ *and* $\overline{y}^{s}_{*,a}=\overline{y}^{s}_{m,a},\overline{y}^{s}_{r,a},\overline{y}^{s}_{(j),a},\overline{y}^{s}_{bk,a}$.

The simulation results for direct and synthetic estimators are reported in Tables 3-4.

4.1 Discussion of simulation results

The simulation results of Table 3 show that the suggested direct estimator $\overline{y}^{d}_{bk,a}$ outperform the traditional direct estimators, namely, direct mean per unit estimator $\overline{y}^{d}_{m,a}$, direct ratio estimator $\overline{y}^{d}_{r,a}$, direct generalized ratio estimators $\overline{y}^{d}_{(j),a}$, in terms of lesser MSE and increased PRE, respectively, for varied values of correlation coefficients in each of the 6 domains. The simulation results of Table 4 show that the suggested synthetic estimator $\overline{y}^{s}_{bk,a}$ outperforms the traditional synthetic estimators, namely, synthetic mean per unit estimator $\overline{y}^{s}_{m,a}$, synthetic ratio estimator $\overline{y}^{s}_{r,a}$, synthetic generalized ratio estimators $\overline{y}^{s}_{(j),a}$, in terms of lesser MSE and increased PRE, respectively, for varied values of correlation coefficients in each of the 6 domains.

Table 3. MSE and PRE of direct estimators for hypothetically drawn population

	ρ_{yx} ρ_{yz} ρ_{xz}	**0.9 0.9 0.9**		**0.8 0.8 0.8**		**0.7 0.7 0.7**		**0.9 0.8 0.7**		**0.8 0.7 0.6**		**0.7 0.8 0.9**		**0.6 0.7 0.8**	
Domains	**Estimators**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
1	$\overline{y}^{d}_{m,a}$	9.8	100.0	9.7	100.0	9.9	100.0	9.8	100.0	9.8	100.0	9.7	100.0	10.0	100.0
	$\overline{y}^{d}_{r,a}$	7.6	129.9	8.0	121.6	10.9	91.0	6.3	155.2	8.2	120.1	12.7	76.9	14.3	70.0
	$\overline{y}^{d}_{(1),a}$	7.6	129.1	8.1	119.7	11.0	90.7	6.3	155.3	8.2	120.7	12.8	75.8	14.2	70.3
	$\overline{y}^{d}_{(2),a}$	4.4	222.2	4.6	211.9	6.8	145.3	3.7	263.8	5.5	178.0	7.4	131.7	10.4	96.6
	$\overline{y}^{d}_{(3),a}$	7.6	129.7	8.0	121.2	10.9	90.9	6.3	155.2	8.2	120.3	12.6	76.7	14.3	70.1
	$\overline{y}^{d}_{(4),a}$	6.5	150.2	7.0	137.7	10.0	99.7	5.5	178.0	7.5	131.8	11.3	85.5	13.4	74.7
	$\overline{y}^{d}_{(5),a}$	7.2	135.4	7.7	125.1	10.7	92.9	6.1	161.4	7.9	123.4	12.3	78.7	14.0	71.3
	$\overline{y}^{d}_{(6),a}$	4.2	234.6	4.2	232.2	6.0	165.2	3.4	286.9	4.9	198.2	6.5	150.1	9.0	111.1
	$\overline{y}^{d}_{(7),a}$	7.6	129.1	8.1	119.2	11	90.6	6.3	155.1	8.2	120.7	12.8	75.4	14.2	70.6
	$\overline{y}^{d}_{(8),a}$	7.6	129.9	8.0	121.6	10.9	91.0	6.3	155.2	8.2	120.1	12.6	76.9	14.3	70.0
	$\boldsymbol{\overline{y}^{d}_{bk,a}}$	**1.5**	**654.9**	**2.8**	**342.7**	**4.3**	**230.6**	**1.3**	**726.1**	**2.9**	**341.3**	**3.6**	**271.1**	**5.1**	**197.0**

2	$\overline{y}^d_{m,a}$	9.4	100.0	10.1	100.0	9.7	100.0	9.3	100.0	9.2	100.0	9.8	100.0	9.3	100.0
	$\overline{y}^d_{r,a}$	6.7	139.1	8.7	116.6	9.9	98.0	5.6	168.1	7.5	123.4	13.8	70.5	17.6	53.1
	$\overline{y}^d_{(1),a}$	6.8	137.7	8.7	115.5	10.0	97.2	5.6	166.0	7.6	121.9	13.9	69.8	17.7	52.7
	$\overline{y}^d_{(2),a}$	3.3	284.3	5.6	180.9	6.9	140.9	2.8	336.9	4.5	206.6	9.0	108.3	10.7	86.6
	$\overline{y}^d_{(3),a}$	6.7	138.8	8.7	116.3	9.9	97.8	5.6	167.6	7.5	123.1	13.9	70.3	17.6	53.0
	$\overline{y}^d_{(4),a}$	5.6	165.6	7.8	129.9	9.2	106.1	4.7	198.8	6.7	138.2	12.6	77.4	16.2	57.6
	$\overline{y}^d_{(5),a}$	6.4	145.2	8.4	120.2	9.7	100.2	5.3	175.1	7.3	126.9	13.5	72.3	17.2	54.1
	$\overline{y}^d_{(6),a}$	3.1	302.8	5.1	197.1	6.2	157.9	2.5	368.6	4.0	231.2	7.9	122.5	8.8	105.5
	$\overline{y}^d_{(7),a}$	6.8	137.5	8.8	115.2	10.0	96.8	5.6	165.5	7.6	121.4	14.0	69.5	17.8	52.5
	$\overline{y}^d_{(8),a}$	6.7	139.1	8.7	116.6	9.9	98.0	5.6	168.1	7.5	123.4	13.8	70.5	17.6	53.1
	$\boldsymbol{\overline{y}^d_{bk,a}}$	**1.4**	**683.0**	**2.9**	**347.8**	**4.0**	**241.2**	**1.2**	**750.6**	**2.6**	**351.1**	**3.4**	**288.4**	**4.8**	**195.3**
3	$\overline{y}^d_{m,a}$	9.8	100.0	9.9	100.0	9.7	100.0	9.8	100.0	9.8	100.0	10.1	100.0	9.5	100.0
	$\overline{y}^d_{r,a}$	9.2	107.2	9.3	106.8	15.1	64.4	7.6	129.4	10.3	95.5	10.4	70.5	13.0	72.9
	$\overline{y}^d_{(1),a}$	9.0	108.9	9.3	107.3	15.2	64.0	7.5	130.5	10.2	96.2	10.3	69.8	12.9	73.5
	$\overline{y}^d_{(2),a}$	4.7	209.9	5.4	184.8	9.4	103.8	3.9	249.2	6.1	160.5	7.0	108.3	9.1	104.9
	$\overline{y}^d_{(3),a}$	9.1	107.6	9.3	106.9	15.1	64.3	7.6	129.7	10.3	95.7	10.3	70.3	13.0	73.1
	$\overline{y}^d_{(4),a}$	7.8	126.2	8.2	120.9	13.8	70.5	6.5	150.8	9.2	106.6	9.5	77.4	12.2	78.0
	$\overline{y}^d_{(5),a}$	8.8	111.7	9.0	110.1	14.8	65.8	7.3	134.5	10.0	98.2	10.1	72.3	12.8	74.2
	$\overline{y}^d_{(6),a}$	4.4	224.0	4.9	203.9	8.0	121.7	3.5	276.4	5.3	184.6	6.3	122.5	7.8	121.9
	$\overline{y}^d_{(7),a}$	9.0	109.1	9.3	107.4	15.2	63.8	7.5	130.7	10.2	96.4	10.3	69.5	12.9	73.8
	$\overline{y}^d_{(8),a}$	9.1	107.2	9.3	106.8	15.1	64.3	7.6	129.4	10.3	95.5	10.4	70.5	13.0	73.0
	$\boldsymbol{\overline{y}^d_{bk,a}}$	**1.5**	**661.4**	**2.8**	**359.7**	**4.2**	**229.2**	**1.3**	**755.8**	**2.8**	**353.6**	**3.7**	**288.4**	**5.1**	**188.2**
4	$\overline{y}^d_{m,a}$	9.6	100.0	9.8	100.0	9.9	100.0	9.6	100.0	9.5	100.0	9.6	100.0	9.8	100.0
	$\overline{y}^d_{r,a}$	8.7	110.5	8.8	112.0	13.3	133	7.1	134.9	9.6	99.4	10.7	90.2	11.9	82.5
	$\overline{y}^d_{(1),a}$	8.7	109.6	8.9	111.0	13.3	74.9	7.2	133.9	9.6	98.7	10.8	89.0	11.9	83.0
	$\overline{y}^d_{(2),a}$	4.1	231.4	5.4	180.7	8.7	114.4	3.5	275.3	5.5	173.5	7.3	131.9	8.2	119.9
	$\overline{y}^d_{(3),a}$	8.7	110.3	8.8	111.7	13.3	74.9	7.1	134.7	9.6	99.2	10.7	89.9	11.9	82.7
	$\overline{y}^d_{(4),a}$	7.3	131.4	7.8	125.6	12.2	81.8	6.0	158.7	8.5	111.5	9.8	98.3	11.1	88.6

	$\overline{y}^{d}_{(5),a}$	8.3	115.2	8.5	115.5	12.9	76.7	6.8	140.3	9.3	102.2	10.4	92.4	11.7	84.0
	$\overline{y}^{d}_{(6),a}$	3.9	247.5	4.9	197.7	7.6	131.1	3.1	305.8	4.8	199.4	6.6	146.5	7.2	137.5
	$\overline{y}^{d}_{(7),a}$	8.8	109.4	8.9	110.7	13.3	74.9	7.2	133.4	9.7	98.2	10.9	88.6	11.8	83.3
	$\overline{y}^{d}_{(8),a}$	8.7	110.5	8.8	112.0	13.3	74.9	7.1	134.9	9.6	99.4	10.7	90.2	11.9	82.6
	$\overline{y}^{d}_{bk,a}$	**1.4**	**681.7**	**2.8**	**354.4**	**4.1**	**241.2**	**1.3**	**758.5**	**2.7**	**354.3**	**3.4**	**280.2**	**4.9**	**199.4**
5	$\overline{y}^{d}_{m,a}$	10.1	100.0	10.3	100.0	9.8	100.0	10.1	100.0	10.0	100.0	10.3	100.0	9.7	100.0
	$\overline{y}^{d}_{r,a}$	8.1	124.1	10.1	101.6	13.1	75.4	6.7	151.6	9.0	111.9	12.5	82.4	15.4	62.9
	$\overline{y}^{d}_{(1),a}$	8.1	124.2	10.2	101.3	13.1	74.9	6.6	151.8	8.9	112.1	12.6	81.5	15.3	63.3
	$\overline{y}^{d}_{(2),a}$	3.8	265.8	6.2	167.2	8.4	117.5	3.2	316.9	5.1	197.7	8.1	126.2	9.7	100.1
	$\overline{y}^{d}_{(3),a}$	8.1	124.1	10.1	101.5	13.1	75.3	6.7	151.6	9.0	111.9	12.5	82.2	15.4	63.0
	$\overline{y}^{d}_{(4),a}$	6.8	148.4	9.0	114.1	11.9	82.5	5.6	179.7	7.9	126.2	11.3	90.5	14.2	68.2
	$\overline{y}^{d}_{(5),a}$	7.8	129.4	9.8	104.7	12.8	77.2	6.4	157.9	8.7	115.1	12.2	84.5	15.1	64.1
	$\overline{y}^{d}_{(6),a}$	3.6	283.6	5.6	183.5	7.3	134.7	2.9	348.9	4.5	223.9	7.3	141.4	8.1	119.3
	$\overline{y}^{d}_{(7),a}$	8.1	124.2	10.2	101.3	13.2	74.7	6.6	151.7	9.0	112.0	12.7	81.1	15.2	63.6
	$\overline{y}^{d}_{(8),a}$	8.1	124.1	10.1	101.6	13.1	75.4	6.7	151.6	9.0	111.9	12.5	82.4	15.4	62.9
	$\overline{y}^{d}_{bk,a}$	**1.4**	**725.4**	**2.9**	**354.4**	**4.0**	**247.2**	**1.3**	**793.3**	**2.7**	**370.9**	**3.4**	**298.5**	**4.8**	**202.3**
6	$\overline{y}^{d}_{m,a}$	9.7	100.0	9.8	100.0	9.8	100.0	9.7	100.0	9.7	100.0	9.8	100.0	9.7	100.0
	$\overline{y}^{d}_{r,a}$	7.1	136.5	8.2	119.8	10.0	97.3	6.0	163.0	7.7	125.9	13.7	71.2	16.1	60.5
	$\overline{y}^{d}_{(1),a}$	7.2	135.4	8.3	118.3	10.1	96.4	6.0	162.6	7.7	125.9	14.0	69.8	16.0	60.7
	$\overline{y}^{d}_{(2),a}$	4.1	237.3	4.7	209.0	6.4	152.4	3.4	282.5	5.1	189.0	7.9	123.9	11.2	86.7
	$\overline{y}^{d}_{(3),a}$	7.1	136.2	8.2	119.4	10.1	97.1	6.0	162.9	7.7	125.9	13.8	70.9	16.1	60.6
	$\overline{y}^{d}_{(4),a}$	6.1	158.5	7.2	135.8	9.2	106.5	5.2	188.2	7.0	138.7	12.3	79.4	15.0	64.9
	$\overline{y}^{d}_{(5),a}$	6.8	142.4	7.9	123.4	9.8	99.4	5.7	169.8	7.5	129.4	13.4	73.0	15.8	61.7
	$\overline{y}^{d}_{(6),a}$	3.9	250.2	4.3	229.3	5.7	171.9	3.2	306.6	4.6	209.6	6.9	142.3	9.6	101.4
	$\overline{y}^{d}_{(7),a}$	7.2	135.3	8.3	117.9	10.2	96.0	6.0	162.3	7.7	125.7	14.1	69.3	16.0	60.8
	$\overline{y}^{d}_{(8),a}$	7.1	136.5	8.2	119.7	10.0	97.3	6.0	163.0	7.7	125.9	13.8	71.2	16.1	60.5
	$\overline{y}^{d}_{bk,a}$	**1.4**	**682.1**	**2.8**	**353.9**	**4.1**	**237.6**	**1.3**	**761.2**	**2.7**	**356.1**	**3.5**	**280.2**	**4.9**	**197.4**

Table 4. MSE and PRE of synthetic estimators for hypothetically drawn population

	ρ_{yx} ρ_{yz} ρ_{xz}	**0.9** **0.9** **0.9**		**0.8** **0.8** **0.8**		**0.7** **0.7** **0.7**		**0.9** **0.8** **0.7**		**0.8** **0.7** **0.6**		**0.7** **0.8** **0.9**		**0.6** **0.7** **0.8**	
Domains	**Estimators**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
1	$\bar{y}^s_{m,a}$	11.5	100.0	12.7	100.0	12.4	100.0	11.5	100.0	11.2	100.0	11.7	100.0	11.4	100.0
	$\bar{y}^s_{r,a}$	1.5	774.9	1.1	1123.2	1.5	817.6	1.2	944.3	1.6	718.4	1.7	684.9	2.8	408.5
	$\bar{y}^s_{(1),a}$	1.5	777.4	1.1	1120.1	1.5	816.7	1.2	948.4	1.6	721.3	1.7	680.7	2.8	411.3
	$\bar{y}^s_{(2),a}$	0.8	1447.4	0.7	1855.4	1.0	1237.0	0.7	1748.6	1.0	1125.6	1.1	1071.3	2.0	584.1
	$\bar{y}^s_{(3),a}$	1.5	775.5	1.1	1122.3	1.5	817.4	1.2	945.3	1.6	719.1	1.7	683.9	2.8	409.3
	$\bar{y}^s_{(4),a}$	1.3	907.9	1.0	1261.6	1.4	890.5	1.1	1098.5	1.4	795.6	1.6	753.2	2.6	437.7
	$\bar{y}^s_{(5),a}$	1.4	808.6	1.1	1157.6	1.5	836.0	1.2	983.7	1.5	738.6	1.7	702.0	2.7	416.2
	$\bar{y}^s_{(6),a}$	0.8	1540.5	0.6	2044.9	0.9	1410.9	0.6	1922.9	0.9	1265.9	1.0	1213.5	1.7	682.2
	$\bar{y}^s_{(7),a}$	1.5	777.7	1.1	1119.4	1.5	816.3	1.2	948.2	1.6	721.5	1.7	678.9	2.8	413.2
	$\bar{y}^s_{(8),a}$	1.5	774.9	1.1	1123.1	1.5	817.6	1.2	944.4	1.6	718.5	1.7	684.8	2.8	408.6
	$\boldsymbol{\bar{y}^s_{bk,a}}$	**0.3**	**4017.9**	**0.4**	**3275.6**	**0.6**	**2122.5**	**0.3**	**4436.1**	**0.5**	**2055.7**	**13.9**	**2346.4**	**1.0**	**1163.4**
2	$\bar{y}^s_{m,a}$	12.5	100.0	12.9	100.0	10.5	100.0	12.5	100.0	12.5	100.0	13.9	100.0	9.7	100.0
	$\bar{y}^s_{r,a}$	1.0	1216.4	1.8	729.5	2.1	507.0	0.8	1482.5	1.1	1147.3	2.6	527.1	2.3	425.3
	$\bar{y}^s_{(1),a}$	1.0	1220.2	1.8	727.5	2.1	506.4	0.8	1488.8	1.1	1151.9	2.7	523.9	2.3	428.2
	$\bar{y}^s_{(2),a}$	0.6	2267.5	1.1	1206.5	1.4	767.2	0.5	2740.0	0.7	1796.1	1.7	825.4	1.6	607.6
	$\bar{y}^s_{(3),a}$	1.0	1217.3	1.8	729.0	2.1	506.9	0.8	1484.0	1.1	1148.4	2.6	526.3	2.3	426.0
	$\bar{y}^s_{(4),a}$	0.9	1424.7	1.6	819.6	1.9	552.2	0.7	1723.9	1.0	1270.5	2.4	579.8	2.1	455.5
	$\bar{y}^s_{(5),a}$	1.0	1269.3	1.7	751.9	2.0	518.4	0.8	1544.2	1.1	1179.6	2.6	540.3	2.2	433.3
	$\bar{y}^s_{(6),a}$	0.5	2413.0	1.0	1329.9	1.2	874.9	0.4	3012.6	0.6	2019.9	1.5	935.1	1.4	709.5
	$\bar{y}^s_{(7),a}$	1.0	1220.7	1.8	727.1	2.1	506.1	0.8	1488.4	1.1	1152.2	2.7	522.5	2.3	430.1
	$\bar{y}^s_{(8),a}$	1.0	1216.4	1.8	729.5	2.1	507.0	0.8	1482.6	1.1	1147.4	2.6	527.1	2.3	425.3
	$\boldsymbol{\bar{y}^s_{bk,a}}$	**0.2**	**6317.5**	**0.6**	**2125.1**	**0.8**	**1312.1**	**0.2**	**6978.0**	**0.4**	**3288.9**	**0.8**	**1803.2**	**0.8**	**1209.6**
3	$\bar{y}^s_{m,a}$	11.7	100.0	12.9	100.0	12.6	100.0	11.7	100.0	11.3	100.0	11.9	100.0	11.5	100.0
	$\bar{y}^s_{r,a}$	1.5	784.6	1.1	1139.3	1.5	830.7	1.2	955.9	1.6	724.9	1.7	693.1	2.8	413.6

	$\bar{y}^s_{(1),a}$	1.5	787.1	1.1	1136.3	1.5	829.7	1.2	960.1	1.6	727.8	1.7	688.8	2.8	416.5
	$\bar{y}^s_{(2),a}$	0.8	1465.4	0.7	1882.0	1.0	1256.9	0.7	1769.8	1.0	1135.6	1.1	1084.0	2.0	591.2
	$\bar{y}^s_{(3),a}$	1.5	785.2	1.1	1138.5	1.5	830.5	1.2	956.9	1.6	725.6	1.7	692.1	2.8	414.4
	$\bar{y}^s_{(4),a}$	1.3	919.2	1.0	1279.8	1.4	904.7	1.1	1111.9	1.4	802.8	1.6	762.2	2.6	443.1
	$\bar{y}^s_{(5),a}$	1.4	818.7	1.1	1174.3	1.5	849.4	1.2	995.7	1.5	745.3	1.7	710.4	2.7	421.4
	$\bar{y}^s_{(6),a}$	0.8	1559.7	0.6	2074.1	0.9	1433.6	0.6	1946.2	0.9	1277.2	1.0	1227.9	1.7	690.5
	$\bar{y}^s_{(7),a}$	1.5	787.4	1.1	1135.5	1.5	829.3	1.2	959.8	1.6	728.0	1.7	687.0	2.8	418.4
	$\bar{y}^s_{(8),a}$	1.5	784.6	1.1	1139.3	1.5	830.6	1.2	956.0	1.6	724.9	1.7	693.0	2.8	413.7
	$\boldsymbol{\bar{y}^s_{bk,a}}$	**0.3**	**4069.2**	**0.4**	**3323.2**	**0.6**	**2155.9**	**0.3**	**4491.5**	**0.6**	**2074.7**	**0.5**	**2375.5**	**1.0**	**1176.3**
4	$\bar{y}^s_{m,a}$	10.4	100.0	11.4	100.0	14.8	100.0	10.3	100.0	10.0	100.0	9.8	100.0	15.1	100.0
	$\bar{y}^s_{r,a}$	1.2	901.5	1.7	681.4	2.4	613.9	0.9	1094.6	1.3	789.5	2.2	453.6	1.8	835.1
	$\bar{y}^s_{(1),a}$	1.2	904.4	1.7	679.5	2.4	613.2	0.9	1099.4	1.3	792.6	2.2	450.8	1.8	840.8
	$\bar{y}^s_{(2),a}$	0.6	1681.5	1.0	1126.9	1.6	929.6	0.5	2024.3	0.8	1236.3	1.4	710.0	1.3	1193.1
	$\bar{y}^s_{(3),a}$	1.2	902.2	1.7	680.9	2.4	613.8	0.9	1095.7	1.3	790.2	2.2	452.9	1.8	836.5
	$\bar{y}^s_{(4),a}$	1.0	1056.0	1.5	765.5	2.2	668.7	0.8	1273.1	1.1	874.3	2.0	498.9	1.8	894.5
	$\bar{y}^s_{(5),a}$	1.1	940.7	1.6	702.3	2.4	627.8	0.9	1140.2	1.2	811.7	2.1	464.9	1.8	850.8
	$\bar{y}^s_{(6),a}$	0.6	1789.5	0.9	1242.0	1.4	1060.2	0.5	2225.8	0.7	1390.4	1.2	804.2	1.1	1393.3
	$\bar{y}^s_{(7),a}$	1.2	904.7	1.7	679.1	2.4	612.9	0.9	1099.1	1.3	792.8	2.2	449.6	1.8	844.6
	$\bar{y}^s_{(8),a}$	1.2	901.6	1.7	681.4	2.4	613.9	0.9	1094.7	1.3	789.5	2.2	453.5	1.8	835.1
	$\boldsymbol{\bar{y}^s_{bk,a}}$	**0.2**	**4679.7**	**0.6**	**1982.7**	**0.9**	**1588.9**	**0.2**	**5149.3**	**0.4**	**2261.5**	**0.6**	**1552.1**	**0.6**	**2381.7**
5	$\bar{y}^s_{m,a}$	11.2	100.0	13.4	100.0	13.1	100.0	11.1	100.0	10.7	100.0	11.6	100.0	11.4	100.0
	$\bar{y}^s_{r,a}$	1.1	997.2	1.8	751.8	2.3	567.4	0.9	1209.3	1.2	874.1	2.4	484.5	2.1	558.8
	$\bar{y}^s_{(1),a}$	1.1	1000.3	1.8	749.8	2.3	566.7	0.9	1214.5	1.2	877.6	2.4	481.5	2.0	562.6
	$\bar{y}^s_{(2),a}$	0.6	1859.7	1.1	1243.5	1.5	858.8	0.5	2236.1	0.8	1368.8	1.5	758.4	1.4	798.4
	$\bar{y}^s_{(3),a}$	1.1	997.9	1.8	751.3	2.3	567.2	0.9	1210.6	1.2	875.0	2.4	483.7	2.0	559.8
	$\bar{y}^s_{(4),a}$	1.0	1168.0	1.6	844.7	2.1	618.0	0.8	1406.4	1.1	968.0	2.2	532.8	1.9	598.6
	$\bar{y}^s_{(5),a}$	1.1	1040.5	1.7	774.9	2.3	580.1	0.9	1259.7	1.2	898.7	2.3	496.6	2.0	569.3
	$\bar{y}^s_{(6),a}$	0.6	1979.1	1.0	1370.6	1.3	979.4	0.5	2458.7	0.7	1539.4	1.4	859.1	1.2	932.4

	$\overline{y}^{s}_{(7),a}$	1.1	1000.7	1.8	749.3	2.3	566.4	0.9	1214.2	1.2	877.9	2.4	480.2	2.0	565.2
	$\overline{y}^{s}_{(8),a}$	1.1	997.2	1.8	751.8	2.3	567.3	0.9	1209.4	1.2	874.2	2.4	484.4	2.1	558.8
	$\overline{y}^{s}_{bk,a}$	**0.2**	**5178.3**	**0.6**	**2188.9**	**0.9**	**1467.0**	**0.2**	**5690.9**	**0.4**	**2505.1**	**0.7**	**1658.0**	**0.7**	**1591.3**
6	$\overline{y}^{s}_{m,a}$	10.8	100.0	11.4	100.0	12.6	100.0	10.7	100.0	10.4	100.0	10.4	100.0	12.3	100.0
	$\overline{y}^{s}_{r,a}$	1.4	753.4	1.2	929.3	1.5	838.7	1.2	918.3	1.5	700.8	1.9	548.3	2.9	425.4
	$\overline{y}^{s}_{(1),a}$	1.4	755.8	1.2	926.8	1.5	837.8	1.2	922.3	1.5	703.6	1.9	545.0	2.9	428.3
	$\overline{y}^{s}_{(2),a}$	0.8	1406.5	0.7	1535.5	1.0	1269.1	0.6	1699.5	1.0	1097.6	1.2	857.9	2.0	608.1
	$\overline{y}^{s}_{(3),a}$	1.4	753.9	1.2	928.6	1.5	838.6	1.2	919.2	1.5	701.4	1.9	547.5	2.9	426.1
	$\overline{y}^{s}_{(4),a}$	1.2	882.6	1.1	1043.9	1.4	913.5	1.0	1068.1	1.3	776.1	1.7	603.0	2.7	455.7
	$\overline{y}^{s}_{(5),a}$	1.4	786.1	1.2	957.8	1.5	857.6	1.1	956.5	1.5	720.5	1.8	562.0	2.9	433.4
	$\overline{y}^{s}_{(6),a}$	0.7	1496.9	0.7	1692.4	0.9	1447.5	0.6	1868.9	0.8	1234.5	1.1	971.8	1.7	710.1
	$\overline{y}^{s}_{(7),a}$	1.4	756.0	1.2	926.2	1.5	837.3	1.2	922.0	1.5	703.7	1.9	543.5	2.9	430.2
	$\overline{y}^{s}_{(8),a}$	1.4	753.4	1.2	929.3	1.5	838.7	1.2	918.4	1.5	700.8	1.9	548.3	2.9	425.4
	$\overline{y}^{s}_{bk,a}$	**0.3**	**3909.7**	**0.4**	**2709.9**	**0.6**	**2175.5**	**0.3**	**4318.1**	**0.5**	**2007.0**	**0.6**	**1879.4**	**1.0**	**1209.0**

The simulation results of Table 3 and Table 4 show that the PRE of the proposed direct and synthetic estimators $\overline{y}^{d}_{bk,a}$ and $\overline{y}^{s}_{bk,a}$ decreases as the correlation coefficients decrease. Moreover, from the results of Table 3 and Table 4, it can be observed that the proposed synthetic estimator outperforms the proposed direct estimator for different combinations of correlation coefficients in each domain.

5. REAL DATA APPLICATIONS

This section provides an application of the suggested direct and synthetic estimators using two real data sets.

5.1 Data set 1

We take into account real data from Swedish municipalities that was published in the book of Sarndal *et al.* (2003). The lower-level local government units in Sweden are known as municipalities. A significant component of local services, including as schools, emergency services, and physical planning, are managed by the 284 municipalities together referred to as the MU284. It has wide variations in size and other features. Sweden is divided into eight regions (domains), namely, (1). Stockholm, (2). East Middle Sweden, (3). Smaland and the islands, (4). South Sweden, (5). West Sweden, (6). North Middle Sweden, (7). Middle Norrland, and (8). Upper Norrland having sizes 25, 48, 32, 38, 56, 41, 15, and 29, respectively. Out of these eight domains, we considered the first six domains for our study. Eight variables in the data set describe the municipalities in various ways. We choose three of these eight variables, namely, REV84, P75, and ME84. The following study and auxiliary variables are taken into account in this data set:

y : *REV84*= Real estate values according to 1984 assessment (in millions of Kronor), x : *P75*= 1975 population (in thousands), and z :ME84 = Number of municipal employees in 1984. Table 5 presents the domain parameters for data set 1 which are used to tabulate the MSE and PRE of the proposed direct and synthetic estimators with the help of the following formulae:

$$PRE\left(\overline{y}^{d}_{m,a},\overline{y}^{d}_{*,a}\right)=\frac{MSE\left(\overline{y}^{d}_{m,a}\right)}{MSE\left(\overline{y}^{d}_{*,a}\right)}\times 100 \tag{6}$$

$$PRE\left(\overline{y}_{m,a}^{s}, \overline{y}_{*,a}^{s}\right) = \frac{MSE\left(\overline{y}_{m,a}^{s}\right)}{MSE\left(\overline{y}_{*,a}^{s}\right)} \times 100 \quad (7)$$

5.2 Data set 2

Like the majority of other Indian states, Uttar Pradesh is divided into a number of districts for the purposes of collecting taxes and performing other administrative tasks. Each district is further broken down into a number of revenue inspector circles (RICs), each of which is composed of a number of villages. RICs are considered as small domains in this study. Since it has been noticed that the area farmed for a certain crop changes every year, either expanding or shrinking. Therefore, for real data application, we consider the crop acreage estimation problem for the RICs of Mohanlal Ganj tehsil of Uttar Pradesh. We consider 8 RICs of Mohanlal Ganj tehsil, namely, (1). Sisendi (2). Mohanlal Ganj, (3). Nigoha, (4). Nagram, (5). Khujauli, (6). Gosaiganj, (7). Amethi, and (8). Behrauli as small domains. Out of these eight domains, we considered the first six domains for out study. The paddy crop acreage (in hectare) for the agricultural season 2018-2019 is considered as study variable y, the paddy crop acreage (in hectare) for the agricultural seasons 2017-2018 and 2016-2017 are considered as auxiliary variables x and z, respectively. For easy reference, each domain's parameters are listed in Table 6. Using these domain parameters, we have tabulated MSE and PRE of the proposed direct and synthetic estimators with the help of the formulae given in (6) and (7), respectively.

5.3 Discussion of real data results

The MSE and PRE of the direct and synthetic estimators based on data set 1 are reported in Table 7 and Table 8, respectively. From the results of Table 7, it can be seen that the proposed direct estimator $\overline{y}_{bk,a}^{d}$ attains the least MSE and maximum PRE among the existing direct estimators such as direct mean estimator $\overline{y}_{m,a}^{d}$, direct ratio estimator $\overline{y}_{r,a}^{d}$, and direct generalized ratio estimator $\overline{y}_{(j),a}^{d}$, $j = 1,2,\ldots,8$. Thus, the proposed direct estimator outperforms the existing direct estimators. The results reported in Table 8 show that the proposed synthetic estimator $\overline{y}_{bk,a}^{s}$ attains the least MSE and maximum PRE among the existing synthetic estimators such as synthetic mean estimator $\overline{y}_{m,a}^{s}$, synthetic ratio estimator $\overline{y}_{r,a}^{s}$, and synthetic generalized ratio estimator $\overline{y}_{(j),a}^{s}$, $j = 1,2,\ldots,8$. Thus, the proposed synthetic estimator outperforms the existing synthetic estimators. Moreover, the proposed synthetic estimator $\overline{y}_{bk,a}^{s}$ outperforms the proposed direct estimator $\overline{y}_{bk,a}^{d}$ in each domain except domains 4 (South Sweden), and 5 (West Sweden).

The MSE and PRE of the direct and synthetic estimators based on data set 2 are reported in Table 9

Table 5. Population parameters of different domains for data set 1

Domains	N_a	$\overline{Y}_a$	$\overline{X}_a$	$\overline{Z}_a$	S_{y_a}	S_{x_a}	S_{z_a}	$\rho_{y_a x_a}$	$\rho_{y_a z_a}$	$\rho_{x_a z_a}$
1	25	6413.32	59.52	4076.36	11317.06	128.70	8696.66	0.99	0.99	0.99
2	48	2971.10	29.17	1658.71	3334.66	35.05	2145.20	0.96	0.97	0.99
3	32	2498.75	23.94	1317.03	2040.72	20.91	1410.55	0.95	0.93	0.95
4	38	2915.53	30.63	1937.71	3094.46	41.49	3998.27	0.98	0.95	0.97
5	56	3046.46	28.71	1950.39	5278.27	59.71	6227.87	0.98	0.97	0.99
6	41	2175.32	20.98	1099.76	1693.82	17.35	1010.17	0.98	0.98	0.99

Table 6. Population parameters of different domains for data set 2

Domains	N_a	$\overline{Y}_a$	$\overline{X}_a$	$\overline{Z}_a$	S_{y_a}	S_{x_a}	S_{z_a}	$\rho_{y_a x_a}$	$\rho_{y_a z_a}$	$\rho_{x_a z_a}$
1	18	105.61	103.83	102.78	75.41	78.37	79.27	0.99	0.99	0.99
2	26	87.69	89.38	89.54	63.78	64.33	57.69	0.99	0.89	0.92
3	30	66.83	66.23	67.23	54.88	56.24	57.09	0.99	0.99	0.99
4	25	184.80	179.32	181.08	125.67	122.00	122.31	0.97	0.96	0.99
5	32	115.87	116.88	121.19	74.35	73.18	75.34	0.98	0.97	0.99
6	24	86.59	84.91	86.53	70.76	68.10	73.27	0.97	0.99	0.96

Table 7. MSE and PRE of the direct estimators for data set 1

Domain Estimators	1		2		3		4		5		6	
	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
$\overline{y}^d_{m,a}$	20492138	100	880332	100	563945	100	944969	100	203524	100	288651	100
$\overline{y}^d_{r,a}$	42812255	47	1438946	61	1163415	48	4806880	19	8753509	23	472259	61
$\overline{y}^d_{(1),a}$	25176689	81	1076688	81	738681	76	2816776	33	3523047	57	290308	99
$\overline{y}^d_{(2),a}$	40277837	50	1345180	65	1104386	51	4576514	20	8068158	25	442722	65
$\overline{y}^d_{(3),a}$	32706985	62	1126304	78	703653	80	3156198	29	4939897	41	267619	107
$\overline{y}^d_{(4),a}$	41615898	49	1362784	64	1099889	51	4639670	20	8419962	24	437830	65
$\overline{y}^d_{(5),a}$	42251904	48	1375162	64	1091529	51	4683146	20	8590816	23	431250	66
$\overline{y}^d_{(6),a}$	40261713	50	1342143	65	1101352	51	4572195	20	8054260	25	442021	65
$\overline{y}^d_{(7),a}$	25101697	81	1067074	82	724414	77	2794093	33	3483077	58	287341	100
$\overline{y}^d_{(8),a}$	42604172	48	1426081	61	1133052	49	4721600	20	8504139	23	457223	63
$\boldsymbol{\overline{y}^d_{bk,a}}$	**270936**	**7563**	**55735**	**1579**	**51340**	**1098**	**26934**	**3508**	**85259**	**2387**	**12866**	**2243**

Table 8. MSE and PRE of the synthetic estimators for data set 1

Domains Estimators	1		2		3		4		5		6	
	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
$\overline{y}^s_{m,a}$	11450378	100	334172	100	657827	100	349090	100	323812	100	1136826	100
$\overline{y}^s_{r,a}$	4440747	257	953072	35	674117	97	917749	38	1002034	32	510899	222
$\overline{y}^s_{(1),a}$	1045749	1094	224438	148	167025	393	227390	153	248273	130	126585	898
$\overline{y}^s_{(2),a}$	4094135	279	878682	38	621501	105	846116	41	923822	35	471022	241
$\overline{y}^s_{(3),a}$	1504721	760	322943	103	228420	287	310974	112	339533	95	173115	656
$\overline{y}^s_{(4),a}$	4251045	269	912358	36	645320	101	878544	39	959228	33	489074	232
$\overline{y}^s_{(5),a}$	4335811	264	930551	35	658188	99	896062	38	978355	33	498826	227
$\overline{y}^s_{(6),a}$	4085438	280	876816	38	620181	106	844319	41	921860	35	470021	241
$\overline{y}^s_{(7),a}$	1024319	1117	219839	152	155494	423	211691	164	231132	140	117845	964
$\overline{y}^s_{(8),a}$	4119419	277	884109	37	625339	105	851342	41	929527	34	473931	239
$\boldsymbol{\overline{y}^s_{bk,a}}$	**69660**	**16437**	**14950**	**2235**	**10574**	**6220**	**14396**	**2424**	**15718**	**2060**	**8014**	**14184**

Table 9. MSE and PRE of direct estimators for data set 2

Domains Estimators	1		2		3		4		5		6	
	MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
$\bar{y}^d_{m,a}$	1105	100	657	100	401	100	2526	100	748	100	568	100
$\bar{y}^d_{r,a}$	1448	76	553	118	461	87	2847	88	718	104	589	96
$\bar{y}^d_{(1),a}$	1344	82	495	132	327	122	2588	97	636	117	441	128
$\bar{y}^d_{(2),a}$	1409	78	537	122	439	91	2808	89	704	106	567	100
$\bar{y}^d_{(3),a}$	1313	84	471	139	308	130	2474	102	592	126	415	136
$\bar{y}^d_{(4),a}$	1397	79	531	123	435	92	2792	90	696	107	563	100
$\bar{y}^d_{(5),a}$	1382	80	521	126	431	93	2767	91	683	109	558	101
$\bar{y}^d_{(6),a}$	1409	78	536	122	439	91	2807	90	703	106	567	100
$\bar{y}^d_{(7),a}$	1423	77	544	120	457	87	2835	89	712	105	584	97
$\bar{y}^d_{(8),a}$	1447	76	552	119	458	87	2845	88	717	104	586	96
$\boldsymbol{\bar{y}^d_{bk,a}}$	10	10495	12	5428	2	15626	161	1559	29	2574	7	7724

Table 10. MSE and PRE of synthetic estimators for data set 2

Domain Estimators	1		2		3		4		5		6	
	MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
$\bar{y}^s_{m,a}$	142	100	601	100	1943	100	5806	100	169	100	650	100
$\bar{y}^s_{r,a}$	155	91	107	560	62	3119	476	1218	187	90	104	621
$\bar{y}^s_{(1),a}$	123	115	84	707	54	3561	417	1391	164	103	91	710
$\bar{y}^s_{(2),a}$	151	93	104	574	60	3196	464	1248	182	92	102	637
$\bar{y}^s_{(3),a}$	116	122	80	750	46	4176	355	1631	139	121	78	832
$\bar{y}^s_{(4),a}$	151	94	104	576	60	3209	462	1254	182	92	101	639
$\bar{y}^s_{(5),a}$	150	94	103	580	60	3233	459	1263	180	93	100	644
$\bar{y}^s_{(6),a}$	151	94	104	575	60	3201	464	1251	182	92	101	638
$\bar{y}^s_{(7),a}$	121	117	83	719	48	4003	371	1564	145	116	81	798
$\bar{y}^s_{(8),a}$	155	91	106	561	62	3127	475	1222	186	90	104	623
$\boldsymbol{\bar{y}^s_{bk,a}}$	**14**	**1013**	**9**	**6197**	**5**	**34505**	**43**	**13483**	**16**	**999**	**9**	**6877**

and Table 10, respectively. From the results of Table 9, it can be seen that the proposed direct estimator $\overline{y}^{d}_{bk,a}$ attains the least MSE and maximum PRE among the existing direct estimators such as direct mean estimator $\overline{y}^{d}_{m,a}$, , direct ratio estimator $\overline{y}^{d}_{r,a}$, and directgeneralized ratio estimator $\overline{y}^{d}_{(j),a}, j = 1, 2, \ldots, 8$. Thus, the proposed direct estimator $\overline{y}^{d}_{bk,a}$ outperformsthe existing direct estimators. Furthermore, the results reported in Table 10 showthat the proposed synthetic estimator $\overline{y}^{s}_{bk,a}$ attains the least MSE and maximum PRE amongthe existing synthetic estimators such as synthetic mean estimator $\overline{y}^{s}_{m,a}$, synthetic ratio estimator $\overline{y}^{s}_{r,a}$ and synthetic generalized ratio estimator $\overline{y}^{s}_{(j),a}, j = 1, 2, \ldots, 8$. Thus, the proposedsynthetic estimator outperforms the existing synthetic estimators. Moreover, the proposedsynthetic estimator dominates the proposed direct estimator in domains 1 (Sisendi), 5 (Khujauli), and 6 (Gosaiganj), whereas proposed direct estimator dominates the synthetic estimatorin domains 2 (Mohanlal Ganj), 3 (Nigoha), and 4 (Nagram).

6. CONCLUSIONS

In this work, we have suggested a logarithmic type direct and synthetic estimators for domain mean utilizing SRS. The variables of the suggested estimators are developed and compared with those of the traditional estimators, and the efficiency criteria are determined. Further, a simulation study using a hypothetically generated normal population verifies the theoretical results. The simulation results are reported by MSE and PRE from Table 3 to Table 4. The simulation results are presented in Subsection 4.1 from which we draw the conclusion that the suggested direct and synthetic estimators are, respectively, more effective than the direct and synthetic estimators already in use. Further, the exemplifications of the proposed methods have been presented through the real data sets of the Swedish municipalities and the paddy crop acreage of Mohanlal Ganj tehsil, Uttar Pradesh, India. The results of the real data sets are presented from Table 7 to Table 10 which also shows the outperformance of proposed estimators. Consequently, we may advocate to use the suggested direct and synthetic estimators for the estimation of the domain means of small areas.

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