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New Systematic Sampling –II

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SUMMARY

In this paper New Systematic Sampling Scheme-II has been suggested which besides maintaining simplicity of selection, provides the unbiased estimator of sampling variance. The efficiency of the suggested scheme has been compared with New Systematic Sampling-I, Usual Systematic Sampling and Simple Random Sampling. It has been observed that in situations where usual systematic sampling performs better than simple random sampling, similar to New Systematic Sampling Scheme-I the suggested sampling scheme also provides results better than simple random sampling. For some real life situations it may provide results better than even usual systematic sampling. Especially, in forest surveys, this could be used with advantage by taking strip samples horizontally and vertically with a random start. It is important to mention that the proposed scheme provides flexibility as the population size and sample size can be expressed in terms of rows and columns, resulting in different samples and hence efficiencies, though the uniqueness property is lost.

Keywords: Inclusion probability, Operational convenience, Systematic sampling, Unbiased variance estimation.

1. INTRODUCTION

Systematic sampling has got the nice feature of operational simplicity, in selecting the whole sample with just one random start. Apart from its operational simplicity, in many situations, systematic sampling provides more efficient estimators than simple random sampling. However, systematic sampling has a serious limitation that it is not possible to obtain an unbiased estimator of sampling variance which is one of the essential requirements for any sampling design. This is because while using systematic sampling all pairs of units in the population do not get non-zero chance of selection, which is a necessary condition for unbiased variance estimation. Murthy (1967) for the first time indicated that when n= (N+1)/2, it is possible to estimate unbiasedly the sampling variance using circular systematic sampling. D. Singh and Padam Singh (1977) suggested a new systematic sampling scheme (hereafter referred as New Systematic Sampling-I) using two sampling intervals with a random start, which enabled unbiased estimation of sampling variance.

In this paper, another new systematic sampling scheme (hereafter referred as New Systematic Sampling-II) has been suggested which also provides an unbiased estimator of variance. The efficiency of the suggested sampling scheme has been compared with the Conventional Systematic Sampling, Simple Random Sampling and New Systematic Sampling-I. For completeness and ready reference essentials of usual systematic sampling and New Systematic Sampling-I are briefly presented in Sections 2 & 3.

2. USUAL SYSTEMATIC SAMPLING

In systematic sampling, only the first unit is selected at random and the rest of units get automatically selected according to a predetermined pattern. The method of systematic sampling when N is a multiple of n, i.e. $N=nk$; where k is an integer is explained as under:

Let us assume that the $N=nk$ serial numbers of the population units in the frame are arranged in k columns and n rows as follows:

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1	2	3	…	*r*	…	*k*
k+1	*k*+2	*k*+3	…	*k*+*r*	…	2*k*
2*k*+1	2*k*+2	2*k*+3	…	2*k*+*r*	…	3*k*
…	…	…	…	...	…	…
…	…	…	…	...	…	…
(n-1)k+1	*(n-1)k*+2	*(n-1)k*+3	...	*(n-1)k*+*r*	...	*nk*

Then, for selecting a systematic sample of *n* units, we select a random number *r* from 1 to *k*. Then all units of the column to which '*r*' belongs are selected in the sample.

3. NEW SYSTEMATIC SAMPLING-I

Suppose a sample of size *n* is desired to be drawn from a population of size *N*. Let *u* (≤*n*) and *d* be two predetermined positive integers. The selection procedure under New Systematic Sampling-I involves of the following steps:

a) Select a random number *r* from 1 to *N*.

b) Starting with *r* select *u* continuous units and thereafter remaining *n-u*(=*v)* units with interval *d.*

In this, the choice of *u* and *d* is governed by the following conditions:

$$\frac{1}{2}N+1 \le u+vd \le N \text{ and } d \le u$$

4. SUGGESTED SAMPLING SCHEME - NEW SYSTEMATIC SAMPLING-II (NSS-II)

4.1 Situation 1: N=pq and n=p+q-1

The selection of a sample by New Systematic Sampling-II comprises of the following steps:

Step 1**:** Arrange the serial numbers of units of the population in *q* rows and *p* columns (*p*≥ *q*) as under:

1	2	3	…	*p*
p+1	*p*+2	*p*+3	…	2*p*
…	…	…	…	...
…	…	…	…	...
(q-1)p+1	…	…	…	*qp*

Step 2**:** Select a random start '*r*' from 1 to *N*.

Step 3**:** Take all units from the row and column corresponding to '*r*'.

The intuition for proposing this scheme is 'what happens when units of the row are also selected with the units of column corresponding to the random selection of *r'*. Surprisingly, this ensures non-zero chance of selection for all pairs of units in the sample.

4.1.1 Calculation of π_i 's and π_{ij} 's

For this sampling scheme, the π_i 's and π_{ij} 's are given by,

$\pi_i = n/N$; for all i, i=1,2…N

For units in the same column,

$$\pi_{ij} = \frac{NumberofUnitsintheColumn}{N} = \frac{q}{N}$$

For units in the same row,

$$\pi_{ij} = \frac{NumberofUnitsintheRow}{N} = \frac{p}{N}$$

For otherunits, $\pi_{ij} = \frac{2}{N}$

The sum of π_{ij} 's is given by;

For units in the same row;

$$\sum_{i=1}^{p}\sum_{j>1}^{p}\pi_{ij} = p/N * \begin{matrix}Number\ of\ pairs\ of\\ units\ in\ the\ row\end{matrix} * number\ of\ rows$$

$$= p/N \times \frac{p(p-1)}{2} \times q = \frac{pq\,p(p-1)}{2N} = \frac{p(p-1)}{2}$$

Similarly, for units in the same column;

$$\sum_{i=1}^{q}\sum_{j>1}^{q}\pi_{ij} = q/N * \begin{matrix}Number\ of\ pairs\ of\\ units\ in\ the\ column\end{matrix} * number\ of\ column$$

$$= q/N \times \frac{q(q-1)}{2} \times p = \frac{pq\,q(q-1)}{2N} = \frac{q(q-1)}{2}$$

For other units, not in the same row or same column,

$$\sum_{i=1}\sum_{j>1}\pi_{ij} = 2/N \times \begin{matrix}Number\ of\ Pairs\ of\\ Units\ from\ other\ than\\ Index\ row\ and\ column\end{matrix} * Number\ of\ Units$$

$$= 2N/N * ((p-1)*(q-1))/2 = (p-1)(q-1)$$

The sum of π_{ij} 's is given by;

$$\sum_{i=1}^{N}\sum_{j>1}^{N}\pi_{ij} = \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + (p-1)(q-1)$$

$$= \frac{(p+q-1)(p+q-2)}{2} = \frac{n(n-1)}{2}$$

Example 1 for Situation 4.1:

Consider a population of size 15 and let a sample of size 7 be drawn. Here, $N=p*q$ i.e., 15=5*3 and $n=p+q$-1=5+3-1=7

Arrange serial number of population units in a 2-way table with 3 rows and 5 columns as under:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

Select a random number 'r' from 1 to 15. Take all units from the row and column corresponding to 'r'. The possible samples of size 7 corresponding to different random starts 'r' are as under:

Random Start	Selected Units in the Sample						
1	1	2	3	4	5	6	11
2	1	2	3	4	5	7	12
3	1	2	3	4	5	8	13
4	1	2	3	4	5	9	14
5	1	2	3	4	5	10	15
6	6	7	8	9	10	1	11
7	6	7	8	9	10	2	12
8	6	7	8	9	10	3	13
9	6	7	8	9	10	4	14
10	6	7	8	9	10	5	15
11	11	12	13	14	15	1	6
12	11	12	13	14	15	2	7
13	11	12	13	14	15	3	8
14	11	12	13	14	15	4	7
15	11	12	13	14	15	5	10

Here, the inclusion probability for every units is 7/15 as every unit is appearing in 7 out of 15 samples. The inclusion probability of pairs of units π_{ij} is given by,

π_{ij} for units in the same row = Number of Units in the Row/N = 5/15

π_{ij} for units in the same column = Number of Units in the Column/N =3/15

π_{ij} for others = 2/N =2/15

4.2 Situation 2: N=pq and $n = q + t - 1$

All units from the row need not be selected always. Selection of more than half of the units from the row will be adequate to ensure non-zero chance of selection for every pair of units.

Let t be a number more than half of p, then

$$t \geq \frac{p+1}{2}\text{; if } p \text{ is odd and } \geq \frac{p}{2}+1\text{; if } p \text{ is even}$$

The New Systematic Sampling-II in this case is as under:

Step 1: Arrange the population units in a 2-way table with q rows and p columns.

Step 2: Select a random start 'r' from 1 to N.

Step 3: Take all units from the column and t units from the row (circularly) corresponding to 'r'.

It can be seen that under situation 2 all pairs of units have non-zero chance of inclusion. It may be mention that q+t-1 is the minimum possible sample size and that one can even select sample sizes greater than q+t-1.

Example A for Situation 4.2:

Consider sampling of 5 units from a population of size 15.

Here, $N=p*q$ i.e., 15=5*3; and t will be $\frac{p+1}{2} = \frac{5+1}{2} = 3$

For Population in Example 1, select a random number 'r' from 1 to 15. Take all the 3 units from the column and 3 units from the row circularly starting with 'r'.

All possible samples of size 5 corresponding to different random start 'r' are as under:

Random Start	Selected Units in the Sample				
1	1	2	3	6	11
2	2	3	4	7	12
3	3	4	5	8	13
4	4	5	1	9	14
5	5	1	2	10	15
6	6	7	8	1	11
7	7	8	9	2	12
8	8	9	10	3	13
9	9	10	6	4	14
10	10	6	7	5	15
11	11	12	13	1	6
12	12	13	14	2	7
13	13	14	15	3	8
14	14	15	11	4	9
15	15	11	12	5	10

Example B for Situation 4.2: For population in Example 1, selection of sample of size 6 is as under. Select a random number *r* from 1 to 15, then take all 3 units of column and 4 units circularly from the row, starting with *r*. All possible samples of size 6 are as under:

Random Start	Selected Units in the Sample					
1	1	2	3	4	6	11
2	2	3	4	5	7	12
3	3	4	5	1	8	13
4	4	5	1	2	9	14
5	5	1	2	3	10	15
6	6	7	8	9	1	11
7	7	8	9	10	2	12
8	8	9	10	6	3	13
9	9	10	6	7	4	14
10	10	6	7	8	5	15
11	11	12	13	14	1	6
12	12	13	14	15	2	7
13	13	14	15	11	3	8
14	14	15	11	12	4	9
15	15	11	12	13	5	10

4.2.1 Calculation of π_{ij} 's for Situation 4.2

In situation 4.2, N=*pq* and *n*=*q*+*t*-1, *t* being more than half of *p*.

Now, q+t-1 can be expressed as p+q-1-s, where s is the number of units not selected from the row, (*s* being less than the half of p).

So, t=p-s or s=p-t

For various cases under situation 4.2, π_{ij} 's are given by,

Case *a*: When number of units selected from the row is 1 less than *p,*(s=1)

i. For pairs of units in the same column = q/N
ii. For pairs of units in the same row = (p-s-1)/N,
iii. For other pairs of units, not in the same row or column =1/N, at a cross distance of 1 on either side and 2/N for remaining cases

Case *b*: When number of units selected from the row is 2 less than *p,* (s=2)

i. For pairs of units in the same column = q/N
ii. For pairs of units in the same row,

= (p-s-1)/N, when units are at a distance of 1 either side

= [p-(s-2)]/N, for units at a distance of 2 or more either side

i. For other pairs of units, not in the same row or column =1/N, at a cross distance of 1,2 on either side and 2/N for remaining cases

Case *c*: When number of units selected from the row is 3 less than *p,* (s=3)

i. For pairs of units in the same column = q/N
ii. For pairs of units in the same row
   = (p-s-1)/N, when units are at a distance of 1 either side
   = [p-(s-2)]/N, for units at a distance of 2 either side
   = [p-(s-3)]/N, for units at a distance of 3 or more
iii. For other pairs of units, not in the same row or column =1/N, at a cross distance of 1,2,3 on either side and 2/N for remaining cases

In general, when number of units selected from the row is *j* less than *p,* (s=*j*)

i. For pairs of units in the same column = q/N
ii. For pairs of units in the same row
   = (p-s-1)/N, when units are at a distance of 1 either side
   = [p-(s-2)]/N, for units at a distance of 2 either side
   = [p-(s-3)]/N, for units at a distance of 3 either side
   … … … … … … … … … … … …
   … … … … … … … … … … … …
   … … … … … … … … … … … …
   = [p-(s-j)]/N, for units at a distance of j or more
iii. For other pairs of units, not in the same row or column =1/N, at a cross distance of 1,2,…*j* on either side and 2/N for remaining cases

The empirical illustrations for N=30=15*2 and n=9 to 16 are given in Appendix.

4.3 Estimation Procedure

Having obtained the inclusion probabilities, the Horvitz-Thompson (1952) estimate of population mean under the proposed new systematic sampling II simplifies to

$$\hat{\hat{Y}}_N = \frac{1}{N}\sum_{i=1}^{n}\frac{y_i}{\pi_i} = \bar{y}_n \text{, the sample mean}$$

The Yates-Grundy (1953) form of the variance of the above estimate, and the variance estimator respectively reduces to,

$$V(\bar{y}_n) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j>1}^{N}\left(1-\frac{N^2}{n^2}\pi_{ij}\right)\left(Y_i - Y_j\right)^2$$

and

$$\hat{V}(\bar{y}_n) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j>1}^{N}\left(\frac{1}{\pi_{ij}}-\frac{N^2}{n^2}\right)\left(y_i - y_j\right)^2$$

4.4 Situation: When $N \neq pq$

In many situations, N may not be a product of two integers, p and q. In that case, N can be expressed as, $N = pq - w$. If w<p/2,then a sample of n=p+q-w-1 can be selected as under:

i. Select a random number r from 1 to N.

ii. Take all units of the columns and continuous (p-w) units from the row circularly starting with r

In this case, π_i 's for all units, though non-zero, will not be equal. This method will ensure non-zero chance of selection for all pairs of units. Horwitz-Thomson estimator can be used in this situation. The values of π_i 's and π_{ij} 's can be calculated for a specific situation. In the present era of computers, the inclusion probabilities for all pairs of units can be calculated using appropriate algorithm.

Alternatively, N can be expressed as N=pq+z and then a sample of size z can be selected from N and thereafter from remaining pq units, a sample of desired size can be selected using NSS-II.

4.5 Restriction on n for given N

Under NSS II, the minimum sample size n is $q + \left(\frac{p+1}{2} - 1\right)$, when p is odd

For real value solution of p we should have, $n \geq \sqrt{2N}$. This imposes a restriction on n for given N. The limitation on sample size is may not be very serious for sampling from a large population. However, a sample of required size n not satisfying the condition can be selected in two or more phase; the sampling at each phase being by new systematic sampling-II. Here, the term 'phase' is used in the sense that the ultimate sample of required size is a subsample of initial sample, as is the case in multiphase sampling.

Example of Two Phase Sampling

Suppose a sample of size 6 is required to be drawn from a population of size 60. This can be selected in two phase as under:

a) Phase 1: Select a sample of size 15 from population of 60.

N=60=10*6 and n = 10+6-1=15

b) Phase 2: Select a random sample of size 6 from above sample of 15.

N=15=5*3 and n =3+ (5+1)/2=6

4.6 Efficiency of new systematic sampling

As in the case of usual systematic sampling, the variance of sample mean under the proposed New Systematic Sampling-II can be rewritten as;

$$V(\bar{y}_{nss}) = \frac{N-1}{N}S^2 - \frac{n-1}{n}S^2_{nss}(w) \text{ where,}$$

$$S^2_{nss}(w) = \frac{1}{N(n-1)}\sum_{r=1}^{N}\sum_{i=1}^{n}\left(y_{ri} - \bar{y}_r\right)^2,$$ the mean sum of squares within new systematic samples.

As S^2, the population mean square is fixed for a given population, in order to obtain efficient estimator by adopting any of the systematic sampling schemes, the population units should be arranged in such a way that the within sample variation is as large as possible.

As the variance under usual systematic sampling and new systematic sampling schemes depends on the nature of the population, we discuss the relative efficiencies of different sampling schemes under consideration for various types of populations.

For population in random order, the sample mean for all sampling schemes will have same variance. For population with linear trend or autocorrelation, the usual systematic sampling will have the least variance, but new systematic sampling schemes (I & II) will still be more efficient than simple random sampling. For population with periodic trend, the New Systematic Sampling-II (as also New Systematic Sampling–I) which have part of the sample of continuous units and remaining with an interval, will be even better than the usual systematic sampling.

4.7 Empirical Comparison

Empirical populations considered by D. Singh and P. Singh (1977) have been taken for comparison of efficiency of the proposed scheme with other schemes. The data relates to operational holdings (in acres) and the number of fruit bearing trees of Venkatgir Taluk of Andhra Pradesh for 24 villages.

Table A. Holding Size and Number of Fruit bearing Trees for 24 Villages

Village	Population A Holding Size	Population B Number of Fruit bearing Trees
1	12	9085
2	10	4889
3	2	1002
4	2	499
5	30	4341
6	30	3299
7	20	1762
8	4	508
9	5	1905
10	38	5916
11	5	1011
12	2	3568
13	10	607
14	2	1515
15	17	3568
16	6	3619
17	6	269
18	14	3619
19	6	2480
20	7	1624
21	6	1640
22	2	2486
23	6	731
24	14	2480

The percentage standard error of sample mean for a sample of size 8 under different sampling schemes is presented in table below. The values of variance for simple random sampling, usual systematic sampling, New Systematic Sampling-I, have been taken from D. Singh and P. Singh (1977).

Table B. Comparison of Percentage Standard Errors of sample mean under different sampling schemes

Sampling Scheme		Percentage Standard Error of Estimate for Population A	Percentage Standard Error of Estimate for Population B
Simple Random Sampling		26.8	22.8
Usual Systematic Sampling		23.7	22.7
New Systematic Sampling I			
u	d		
2	2	20.8	18.1
3	3	22.0	22.6
3	2	21.8	18.3
4	4	27.8	23.1
4	3	23.0	22.1
4	2	24.2	19.6
5	5	31.0	22.4
5	4	27.1	22.0
5	3	23.7	20.2
6	6	24.1	20.1
6	5	30.7	22.1
6	4	29.2	21.7
7	3	28.0	19.2
7	7	28.3	17.8
7	6	27.9	17.6
Average		26.2	20.5
New Systematic Sampling II			
p	q		
6	4	22.0	22.4
8	3	21.1	23.1
Average		21.6	22.7

Empirically, it is observed that no general conclusion could be drawn about superiority of one scheme or the other in terms of the variance. It is very much dependent on the nature of population under study which is not completely known and the choice of scheme constants (like p, q, n, u, v, d etc.). It is noted that New Systematic Sampling-II performs even better than usual systematic sampling and New Systematic Sampling-I for the empirical population A and as good as other schemes for Population B. It is mentioned that above inference is based on the average standard error of the estimate for all situations considered in NSS-I and NSS-II. In view of this, the proposed New Systematic Sampling –II could be used in practice and it is expected that its performance will be as good as other competing sampling schemes.

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APPENDIX -COMBINATION MATRIX FOR PAIRS OF UNITS

Case under Situation 4.1:N=30=15*2, n=16=15+2-1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1		15	15	15	15	15	15	15	15	15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2			15	15	15	15	15	15	15	15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3				15	15	15	15	15	15	15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
4					15	15	15	15	15	15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
5						15	15	15	15	15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
6							15	15	15	15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
7								15	15	15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
8									15	15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
9										15	15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
10											15	15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
11												15	15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
12													15	15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
13														15	15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
14															15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
15																2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
16																	15	15	15	15	15	15	15	15	15	15	15	15	15	15
17																		15	15	15	15	15	15	15	15	15	15	15	15	15
18																			15	15	15	15	15	15	15	15	15	15	15	15
19																				15	15	15	15	15	15	15	15	15	15	15
20																					15	15	15	15	15	15	15	15	15	15
21																						15	15	15	15	15	15	15	15	15
22																							15	15	15	15	15	15	15	15
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24																									15	15	15	15	15	15
25																										15	15	15	15	15
26																											15	15	15	15
27																												15	15	15
28																													15	15
29																														15
30																														

Case a under situation 2:N=30=15*2, n=15=15+2-2

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1		13	13	13	13	13	13	13	13	13	13	13	13	13	13	2	1	2	2	2	2	2	2	2	2	2	2	2	2	1
2			13	13	13	13	13	13	13	13	13	13	13	13	13	1	2	1	2	2	2	2	2	2	2	2	2	2	2	2
3				13	13	13	13	13	13	13	13	13	13	13	13	2	1	2	1	2	2	2	2	2	2	2	2	2	2	2
4					13	13	13	13	13	13	13	13	13	13	13	2	2	1	2	1	2	2	2	2	2	2	2	2	2	2
5						13	13	13	13	13	13	13	13	13	13	2	2	2	1	2	1	2	2	2	2	2	2	2	2	2
6							13	13	13	13	13	13	13	13	13	2	2	2	2	1	2	1	2	2	2	2	2	2	2	2
7								13	13	13	13	13	13	13	13	2	2	2	2	2	1	2	1	2	2	2	2	2	2	2
8									13	13	13	13	13	13	13	2	2	2	2	2	2	1	2	1	2	2	2	2	2	2
9										13	13	13	13	13	13	2	2	2	2	2	2	2	1	2	1	2	2	2	2	2
10											13	13	13	13	13	2	2	2	2	2	2	2	2	1	2	1	2	2	2	2
11												13	13	13	13	2	2	2	2	2	2	2	2	2	1	2	1	2	2	2
12													13	13	13	2	2	2	2	2	2	2	2	2	2	1	2	1	2	2
13														13	13	2	2	2	2	2	2	2	2	2	2	2	1	2	1	2
14															13	2	2	2	2	2	2	2	2	2	2	2	2	1	2	1
15																1	2	2	2	2	2	2	2	2	2	2	2	2	1	2
16																	13	13	13	13	13	13	13	13	13	13	13	13	13	13
17																		13	13	13	13	13	13	13	13	13	13	13	13	13
18																			13	13	13	13	13	13	13	13	13	13	13	13
19																				13	13	13	13	13	13	13	13	13	13	13
20																					13	13	13	13	13	13	13	13	13	13
21																						13	13	13	13	13	13	13	13	13
22																							13	13	13	13	13	13	13	13
23																								13	13	13	13	13	13	13
24																									13	13	13	13	13	13
25																										13	13	13	13	13
26																											13	13	13	13
27																												13	13	13
28																													13	13
29																														13
30																														

Case b under situation 2 :N=30=15*2, n=14=15+2-3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1		12	11	11	11	11	11	11	11	11	11	11	11	11	12	2	1	1	2	2	2	2	2	2	2	2	2	2	1	1
2			12	11	11	11	11	11	11	11	11	11	11	11	11	1	2	1	1	2	2	2	2	2	2	2	2	2	2	1
3				12	11	11	11	11	11	11	11	11	11	11	11	1	1	2	1	1	2	2	2	2	2	2	2	2	2	2
4					12	11	11	11	11	11	11	11	11	11	11	2	1	1	2	1	1	2	2	2	2	2	2	2	2	2
5						12	11	11	11	11	11	11	11	11	11	2	2	1	1	2	1	1	2	2	2	2	2	2	2	2
6							12	11	11	11	11	11	11	11	11	2	2	2	1	1	2	1	1	2	2	2	2	2	2	2
7								12	11	11	11	11	11	11	11	2	2	2	2	1	1	2	1	1	2	2	2	2	2	2
8									12	11	11	11	11	11	11	2	2	2	2	2	1	1	2	1	1	2	2	2	2	2
9										12	11	11	11	11	11	2	2	2	2	2	2	1	1	2	1	1	2	2	2	2
10											12	11	11	11	11	2	2	2	2	2	2	2	1	1	2	1	1	2	2	2
11												12	11	11	11	2	2	2	2	2	2	2	2	1	1	2	1	1	2	2
12													12	11	11	2	2	2	2	2	2	2	2	2	1	1	2	1	1	2

13														12	11	2	2	2	2	2	2	2	2	2	2	1	1	2	1	1
14															12	1	2	2	2	2	2	2	2	2	2	2	1	1	2	1
15																1	1	2	2	2	2	2	2	2	2	2	2	1	1	2
16																	12	11	11	11	11	11	11	11	11	11	11	11	11	12
17																		12	11	11	11	11	11	11	11	11	11	11	11	11
18																			12	11	11	11	11	11	11	11	11	11	11	11
19																				12	11	11	11	11	11	11	11	11	11	11
20																					12	11	11	11	11	11	11	11	11	11
21																						12	11	11	11	11	11	11	11	11
22																							12	11	11	11	11	11	11	11
23																								12	11	11	11	11	11	11
24																									12	11	11	11	11	11
25																										12	11	11	11	11
26																											12	11	11	11
27																												12	11	11
28																													12	11
29																														12
30																														

Case c under situation 2:N=30=15*2, n=13=15+2-4

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1		11	10	9	9	9	9	9	9	9	9	9	9	10	11	2	1	1	1	2	2	2	2	2	2	2	2	1	1	1
2			11	10	9	9	9	9	9	9	9	9	9	9	10	1	2	1	1	1	2	2	2	2	2	2	2	2	1	1
3				11	10	9	9	9	9	9	9	9	9	9	9	1	1	2	1	1	1	2	2	2	2	2	2	2	2	1
4					11	10	9	9	9	9	9	9	9	9	9	1	1	1	2	1	1	1	2	2	2	2	2	2	2	2
5						11	10	9	9	9	9	9	9	9	9	2	1	1	1	2	1	1	1	2	2	2	2	2	2	2
6							11	10	9	9	9	9	9	9	9	2	2	1	1	1	2	1	1	1	2	2	2	2	2	2
7								11	10	9	9	9	9	9	9	2	2	2	1	1	1	2	1	1	1	2	2	2	2	2
8									11	10	9	9	9	9	9	2	2	2	2	1	1	1	2	1	1	1	2	2	2	2
9										11	10	9	9	9	9	2	2	2	2	2	1	1	1	2	1	1	1	2	2	2
10											11	10	9	9	9	2	2	2	2	2	2	1	1	1	2	1	1	1	2	2
11												11	10	9	9	2	2	2	2	2	2	2	1	1	1	2	1	1	1	2
12													11	10	9	2	2	2	2	2	2	2	2	1	1	1	2	1	1	1
13														11	10	1	2	2	2	2	2	2	2	2	1	1	1	2	1	1
14															11	1	1	2	2	2	2	2	2	2	2	1	1	1	2	1
15																1	1	1	2	2	2	2	2	2	2	2	1	1	1	2
16																	11	10	9	9	9	9	9	9	9	9	9	9	10	11
17																		11	10	9	9	9	9	9	9	9	9	9	9	10
18																			11	10	9	9	9	9	9	9	9	9	9	9
19																				11	10	9	9	9	9	9	9	9	9	9
20																					11	10	9	9	9	9	9	9	9	9
21																						11	10	9	9	9	9	9	9	9
22																							11	10	9	9	9	9	9	9
23																								11	10	9	9	9	9	9
24																									11	10	9	9	9	9
25																										11	10	9	9	9
26																											11	10	9	9

27																												11	10	9
28																													11	10
29																														11
30																														

Case d under situation 2:N=30=15*2, n=12=15+2-5

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1		10	9	8	7	7	7	7	7	7	7	7	8	9	10	2	1	1	1	1	2	2	2	2	2	2	1	1	1	1
2			10	9	8	7	7	7	7	7	7	7	7	8	9	1	2	1	1	1	1	2	2	2	2	2	2	1	1	1
3				10	9	8	7	7	7	7	7	7	7	7	8	1	1	2	1	1	1	1	2	2	2	2	2	2	1	1
4					10	9	8	7	7	7	7	7	7	7	7	1	1	1	2	1	1	1	1	2	2	2	2	2	2	1
5						10	9	8	7	7	7	7	7	7	7	1	1	1	1	2	1	1	1	1	2	2	2	2	2	2
6							10	9	8	7	7	7	7	7	7	2	1	1	1	1	2	1	1	1	1	2	2	2	2	2
7								10	9	8	7	7	7	7	7	2	2	1	1	1	1	2	1	1	1	1	2	2	2	2
8									10	9	8	7	7	7	7	2	2	2	1	1	1	1	2	1	1	1	1	2	2	2
9										10	9	8	7	7	7	2	2	2	2	1	1	1	1	2	1	1	1	1	2	2
10											10	9	8	7	7	2	2	2	2	2	1	1	1	1	2	1	1	1	1	2
11												10	9	8	7	2	2	2	2	2	2	1	1	1	1	2	1	1	1	1
12													10	9	8	1	2	2	2	2	2	2	1	1	1	1	2	1	1	1
13														10	9	1	1	2	2	2	2	2	2	1	1	1	1	2	1	1
14															10	1	1	1	2	2	2	2	2	2	1	1	1	1	2	1
15																1	1	1	1	2	2	2	2	2	2	1	1	1	1	2
16																	10	9	8	7	7	7	7	7	7	7	7	8	9	10
17																		10	9	8	7	7	7	7	7	7	7	7	8	9
18																			10	9	8	7	7	7	7	7	7	7	7	8
19																				10	9	8	7	7	7	7	7	7	7	7
20																					10	9	8	7	7	7	7	7	7	7
21																						10	9	8	7	7	7	7	7	7
22																							10	9	8	7	7	7	7	7
23																								10	9	8	7	7	7	7
24																									10	9	8	7	7	7
25																										10	9	8	7	7
26																											10	9	8	7
27																												10	9	8
28																													10	9
29																														10
30																														

Case e under situation 2: N=30=15*2, n=11=15+2-6

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1		9	8	7	6	5	5	5	5	5	5	6	7	8	9	2	1	1	1	1	1	2	2	2	2	1	1	1	1	1
2			9	8	7	6	5	5	5	5	5	5	6	7	8	1	2	1	1	1	1	1	2	2	2	2	1	1	1	1
3				9	8	7	6	5	5	5	5	5	5	6	7	1	1	2	1	1	1	1	1	2	2	2	2	1	1	1
4					9	8	7	6	5	5	5	5	5	5	6	1	1	1	2	1	1	1	1	1	2	2	2	2	1	1
5						9	8	7	6	5	5	5	5	5	5	1	1	1	1	2	1	1	1	1	1	2	2	2	2	1
6							9	8	7	6	5	5	5	5	5	1	1	1	1	1	2	1	1	1	1	1	2	2	2	2

7								9	8	7	6	5	5	5	5	2	1	1	1	1	1	2	1	1	1	1	1	2	2	2
8									9	8	7	6	5	5	5	2	2	1	1	1	1	1	2	1	1	1	1	1	2	2
9										9	8	7	6	5	5	2	2	2	1	1	1	1	1	2	1	1	1	1	1	2
10											9	8	7	6	5	2	2	2	2	1	1	1	1	1	2	1	1	1	1	1
11												9	8	7	6	1	2	2	2	2	1	1	1	1	1	2	1	1	1	1
12													9	8	7	1	1	2	2	2	2	1	1	1	1	1	2	1	1	1
13														9	8	1	1	1	2	2	2	2	1	1	1	1	1	2	1	1
14															9	1	1	1	1	2	2	2	2	1	1	1	1	1	2	1
15																1	1	1	1	1	2	2	2	2	1	1	1	1	1	2
16																	9	8	7	6	5	5	5	5	5	5	6	7	8	9
17																		9	8	7	6	5	5	5	5	5	5	6	7	8
18																			9	8	7	6	5	5	5	5	5	5	6	7
19																				9	8	7	6	5	5	5	5	5	5	6
20																					9	8	7	6	5	5	5	5	5	5
21																						9	8	7	6	5	5	5	5	5
22																							9	8	7	6	5	5	5	5
23																								9	8	7	6	5	5	5
24																									9	8	7	6	5	5
25																										9	8	7	6	5
26																											9	8	7	6
27																												9	8	7
28																													9	8
29																														9
30																														

Case f under situation 2: N=30=15*2, n=10=15+2-7

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1		8	7	6	5	4	3	3	3	3	4	5	6	7	8	2	1	1	1	1	1	1	2	2	1	1	1	1	1	1
2			8	7	6	5	4	3	3	3	3	4	5	6	7	1	2	1	1	1	1	1	1	2	2	1	1	1	1	1
3				8	7	6	5	4	3	3	3	3	4	5	6	1	1	2	1	1	1	1	1	1	2	2	1	1	1	1
4					8	7	6	5	4	3	3	3	3	4	5	1	1	1	2	1	1	1	1	1	1	2	2	1	1	1
5						8	7	6	5	4	3	3	3	3	4	1	1	1	1	2	1	1	1	1	1	1	2	2	1	1
6							8	7	6	5	4	3	3	3	3	1	1	1	1	1	2	1	1	1	1	1	1	2	2	1
7								8	7	6	5	4	3	3	3	1	1	1	1	1	1	2	1	1	1	1	1	1	2	2
8									8	7	6	5	4	3	3	2	1	1	1	1	1	1	2	1	1	1	1	1	1	2
9										8	7	6	5	4	3	2	2	1	1	1	1	1	1	2	1	1	1	1	1	1
10											8	7	6	5	4	1	2	2	1	1	1	1	1	1	2	1	1	1	1	1
11												8	7	6	5	1	1	2	2	1	1	1	1	1	1	2	1	1	1	1
12													8	7	6	1	1	1	2	2	1	1	1	1	1	1	2	1	1	1
13														8	7	1	1	1	1	2	2	1	1	1	1	1	1	2	1	1
14															8	1	1	1	1	1	2	2	1	1	1	1	1	1	2	1
15																1	1	1	1	1	1	2	2	1	1	1	1	1	1	2
16																	8	7	6	5	4	3	3	3	3	4	5	6	7	8
17																		8	7	6	5	4	3	3	3	3	4	5	6	7
18																			8	7	6	5	4	3	3	3	3	4	5	6
19																				8	7	6	5	4	3	3	3	3	4	5
20																					8	7	6	5	4	3	3	3	3	4

21																						8	7	6	5	4	3	3	3	3
22																							8	7	6	5	4	3	3	3
23																								8	7	6	5	4	3	3
24																									8	7	6	5	4	3
25																										8	7	6	5	4
26																											8	7	6	5
27																												8	7	6
28																													8	7
29																														8
30																														

Case i under situation 2:N=30=15*2, n=9=15+2-8

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1		7	6	5	4	3	2	1	1	2	3	4	5	6	7	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2			7	6	5	4	3	2	1	1	2	3	4	5	6	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1
3				7	6	5	4	3	2	1	1	2	3	4	5	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1
4					7	6	5	4	3	2	1	1	2	3	4	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1
5						7	6	5	4	3	2	1	1	2	3	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1
6							7	6	5	4	3	2	1	1	2	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1
7								7	6	5	4	3	2	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1
8									7	6	5	4	3	2	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1
9										7	6	5	4	3	2	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1
10											7	6	5	4	3	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1
11												7	6	5	4	1	1	1	1	1	1	1	1	1	1	2	1	1	1	1
12													7	6	5	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1
13														7	6	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1
14															7	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1
15																1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
16																	7	6	5	4	3	2	1	1	2	3	4	5	6	7
17																		7	6	5	4	3	2	1	1	2	3	4	5	6
18																			7	6	5	4	3	2	1	1	2	3	4	5
19																				7	6	5	4	3	2	1	1	2	3	4
20																					7	6	5	4	3	2	1	1	2	3
21																						7	6	5	4	3	2	1	1	2
22																							7	6	5	4	3	2	1	1
23																								7	6	5	4	3	2	1
24																									7	6	5	4	3	2
25																										7	6	5	4	3
26																											7	6	5	4
27																												7	6	5
28																													7	6
29																														7
30																														