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A Transformed Class of Estimators of a Finite Population Mean using Two Auxiliary Variables in Two-Phase Sampling

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SUMMARY

In this paper, a transformed class of estimators has been developed for estimating the mean of a finite population under two-phase sampling using two auxiliary variables. The mathematical expressions for bias and mean square error (MSE) of the proposed class, as well as for the pre-existing estimators, have been derived to the first order of approximation. The proposed class of estimators has been compared with the other well-known estimators using the MSE criterion. Moreover, the optimum sample sizes of the first-phase and second-phase samples, along with the optimum MSEs of the concerned estimators, have been derived using the cost function analysis. The theoretical results have been empirically validated by considering real population datasets.

Keywords: Auxiliary variable, Bias, Mean square error, Percent absolute relative bias, Percent relative efficiency, Study variable, Two-phase sampling.

1. INTRODUCTION

It is well established that for estimating the population mean of a study variable, the transformation on auxiliary variable(s) is widely used for obtaining precise and efficient estimators, which outperform the usual unbiased estimator (i.e., the sample mean). Over the years, considerable developments have been made by several authors for the estimation of mean by utilizing transformed auxiliary variable(s) under various sampling designs, for instance, simple random sampling (SRS), two-phase sampling, and so on. Some noteworthy contributions under SRS design have been made by Sisodia and Dwivedi (1981), Upadhyaya and Singh (1999), Singh (2003), Singh and Tailor (2003), Kadilar and Cingi (2004), Gupta and Shabbir (2007), Vishwakarma *et al.* (2014), Vishwakarma and Kumar (2015), and Kumar and Vishwakarma (2017a).

The theory of estimation of mean has a significant role in various disciplines of research including agriculture, demographic studies, meteorology, and other diversified fields. For instance, the estimation of: average agricultural production, average life span of a species, average annual rainfall, and much more. The estimation theory is also widely applied in surveys dealing with small area estimation (SAE) problems.

To deal with the estimation of mean, the information on auxiliary variable(s) is obtained through various sources such as administrative records, census surveys, past experience, and so on. Sometimes, the prior information on auxiliary variable(s) is not available, and in that case the two-phase sampling design is utilized at the estimation stage. The concept of two-phase sampling was first introduced by Neyman (1938). The procedure of two-phase sampling design for estimating the population mean of a study variable involves two steps: (i) selecting a preliminary large sample (known as the first-phase sample) of size n' from a population consisting of N units for measuring the mean(s) of the auxiliary variable(s), and (ii) selecting a subsample (known as the second-phase sample) of size n from the preliminary sample of size n' for measuring the means of both the study variable and the auxiliary variable(s).

Some significant contributions under two-phase sampling design have been made by Sukhatme (1962),

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Srivastava (1970), Chand (1975), Sisodia and Dwivedi (1982), Mukerjee *et al.* (1987), Singh (2001), Singh and Ruiz Espejo (2007), Singh *et al.* (2007), Vishwakarma and Kumar (2016), Kumar and Vishwakarma (2017b), and Dubey *et al.* (2020).

The use of transformed auxiliary variable(s) for the estimation of mean has received considerable attention by survey statisticians and researchers in the past as well as in the recent times. Considering the given fact, an attempt is made in this paper to develop a transformed class of estimators for the population mean $\bar{Y}$ of the study variable Y using two auxiliary variables under two-phase sampling. The mean square error (MSE) criterion is applied for efficiency comparisons of the proposed class of estimators with the pre- existing estimators. Moreover, the validation of theoretical results has been made using an empirical analysis.

2. MATERIALS AND METHODS

2.1 Some Pre-Existing Estimators of Population Mean

Sukhatme (1962) developed the following ratio estimator under two-phase sampling for the estimation of population mean $\bar{Y}$ of the study variable Y:

$$\bar{y}_R^d = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right), \tag{1}$$

where $\bar{x}' = \sum_{i=1}^{n'} X_i / n'$ denotes the first-phase sample mean of the auxiliary variable X. Also, $\bar{y} = \sum_{i=1}^{n} Y_i / n$ and $\bar{x} = \sum_{i=1}^{n} X_i / n$ denote, respectively, the second-phase sample means of the variables Y and X.

Srivastava (1970) utilized a scalar quantity α and suggested the following ratio estimator for $\bar{Y}$ under two-phase sampling:

$$\bar{y}_{ds} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)^{\alpha}, \tag{2}$$

Chand (1975) utilized the information on two auxiliary variables X and Z, such that X is closely related to Y as compared to Z (i.e., $\rho_{YX} > \rho_{YZ} > 0$), and developed the following chain ratio type estimator for $\bar{Y}$:

$$\bar{y}_R^{dc} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}}{\bar{z}'}\right), \tag{3}$$

where $\bar{Z} = \sum_{i=1}^{N} Z_i / N$ and $\bar{z}' = \sum_{i=1}^{n'} Z_i / n'$ denote, respectively, the population mean and the first- phase sample mean of the auxiliary variable Z.

Mukerjee *et al.* (1987) defined the following regression estimator for $\bar{Y}$ under two-phase sampling:

$$\bar{y}_{MEA}^d = \bar{y} + b_{yx}\left(\bar{x}' - \bar{x}\right) + b_{yz}\left(\bar{z}' - \bar{z}\right), \tag{4}$$

where b_{yx} and b_{yz} denote, respectively, the sample regression coefficient of Y on X, and the sample regression coefficient of Y on Z. Also, $\bar{z} = \sum_{i=1}^{n} Z_i / n$ represents the second-phase sample mean of the auxiliary variable Z.

Singh and Upadhyaya (1995) suggested a modified chain ratio estimator for $\bar{Y}$ in two-phase sampling as follows:

$$\bar{y}_{SU} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z} + C_Z}{\bar{z}' + C_Z}\right), \tag{5}$$

where C_Z is the coefficient of variation of the auxiliary variable Z.

Singh (2001) developed a chain type estimator for $\bar{Y}$ in two-phase sampling as follows:

$$\bar{y}_S = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z} + \sigma_Z}{\bar{z}' + \sigma_Z}\right), \tag{6}$$

where σ_Z is the standard deviation of the auxiliary variable Z.

Upadhyaya and Singh (2001) developed a modified chain ratio estimator for $\bar{Y}$ in two-phase sampling as follows:

$$\bar{y}_{US} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\beta_{2(z)}\bar{Z} + C_Z}{\beta_{2(z)}\bar{z}' + C_Z}\right), \tag{7}$$

where $\beta_{2(z)}$ is the coefficient of kurtosis of the auxiliary variable Z.

Singh *et al.* (2007) utilized the information on correlation coefficient between the auxiliary variables X and Z, and suggested the following chain ratio-type estimator for $\bar{Y}$:

$$\bar{y}_{SEA} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z} + \rho_{XZ}}{\bar{z}' + \rho_{XZ}}\right), \tag{8}$$

where ρ_{XZ} is the correlation coefficient between the variables X and Z.

Singh and Choudhury (2012) suggested the following exponential chain ratio estimator for $\bar{Y}$:

$$\bar{y}_{\mathrm{Re}}^{dc} = \bar{y}\exp\left\{\frac{\left(\bar{x}'/\bar{z}'\right)\bar{Z}-\bar{x}}{\left(\bar{x}'/\bar{z}'\right)\bar{Z}+\bar{x}}\right\}, \tag{9}$$

Singh and Majhi (2014) suggested the following exponential type estimator for $\bar{Y}$:

$$\bar{y}_{SM} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\exp\left(\frac{\bar{Z}-\bar{z}'}{\bar{Z}+\bar{z}'}\right), \tag{10}$$

Mehta and Tailor (2020) developed the following chain ratio type estimator for $\bar{Y}$:

$$\bar{y}_{MT} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}+\beta_{2(z)}}{\bar{z}'+\beta_{2(z)}}\right). \tag{11}$$

To the first order of approximation, the mathematical expressions for Biases of various estimators described above are as follows:

$$B(\bar{y}_R^d) = \bar{Y}\left\{f_3(C_X^2-\rho_{YX}C_YC_X)\right\}, \tag{12}$$

$$B(\bar{y}_{ds}) = \bar{Y}\left[f_3\,\alpha\left\{\frac{(\alpha+1)}{2}C_X^2-\rho_{YX}C_YC_X\right\}\right], \tag{13}$$

$$B(\bar{y}_R^{dc}) = \bar{Y}\left\{f_2(C_Z^2-\rho_{YZ}C_YC_Z)+f_3(C_X^2-\rho_{YX}C_YC_X)\right\}, \tag{14}$$

$$B(\bar{y}_{MEA}^d) = Cov(\bar{x}',b_{yx})+Cov(\bar{z}',b_{yz})-Cov(\bar{x},b_{yx})-Cov(\bar{z},b_{yz}), \tag{15}$$

$$B(\bar{y}_{SU}) = \bar{Y}\left\{f_2(\psi_1^2C_Z^2-\psi_1\rho_{YZ}C_YC_Z)+f_3(C_X^2-\rho_{YX}C_YC_X)\right\}, \tag{16}$$

$$B(\bar{y}_S) = \bar{Y}\left\{f_2(\psi_2^2C_Z^2-\psi_2\rho_{YZ}C_YC_Z)+f_3(C_X^2-\rho_{YX}C_YC_X)\right\}, \tag{17}$$

$$B(\bar{y}_{US}) = \bar{Y}\left\{f_2(\psi_3^2C_Z^2-\psi_3\rho_{YZ}C_YC_Z)+f_3(C_X^2-\rho_{YX}C_YC_X)\right\}, \tag{18}$$

$$B(\bar{y}_{SEA}) = \bar{Y}\left\{f_2(\psi_4^2C_Z^2-\psi_4\rho_{YZ}C_YC_Z)+f_3(C_X^2-\rho_{YX}C_YC_X)\right\}, \tag{19}$$

$$B(\bar{y}_{\mathrm{Re}}^{dc}) = \bar{Y}\left\{f_2\left(\frac{3}{8}C_Z^2-\frac{1}{2}\rho_{YZ}C_YC_Z\right)+f_3\left(\frac{3}{8}C_X^2-\frac{1}{2}\rho_{YX}C_YC_X\right)\right\}, \tag{20}$$

$$B(\bar{y}_{SM}) = \bar{Y}\left\{f_2\left(\frac{3}{8}C_Z^2-\frac{1}{2}\rho_{YZ}C_YC_Z\right)+f_3\left(C_X^2-\rho_{YX}C_YC_X\right)\right\}, \tag{21}$$

$$B(\bar{y}_{MT}) = \bar{Y}\left\{f_2(\psi_5^2C_Z^2-\psi_5\rho_{YZ}C_YC_Z)+f_3(C_X^2-\rho_{YX}C_YC_X)\right\}, \tag{22}$$

Moreover, to the first order of approximation, the mathematical expressions for MSEs of various estimators described above are as follows:

$$MSE(\bar{y}_R^d) = \bar{Y}^2\left\{f_1C_Y^2+f_3\left(C_X^2-2\rho_{YX}C_YC_X\right)\right\}, \tag{23}$$

$$MSE(\bar{y}_{ds}) = \bar{Y}^2\left\{f_1C_Y^2+f_3\left(\alpha^2C_X^2-2\alpha\rho_{YX}C_YC_X\right)\right\}, \tag{24}$$

$$MSE(\bar{y}_R^{dc}) = \bar{Y}^2\left\{f_1C_Y^2+f_2(C_Z^2-2\rho_{YZ}C_YC_Z)+f_3(C_X^2-2\rho_{YX}C_YC_X)\right\}, \tag{25}$$

$$MSE(\bar{y}_{MEA}^d) = \bar{Y}^2C_Y^2\left\{f_1-f_3\left(\rho_{YX}^2+\rho_{YZ}^2-2\rho_{YX}\rho_{YZ}\rho_{XZ}\right)\right\}, \tag{26}$$

$$MSE(\bar{y}_{SU}) = \bar{Y}^2\left\{f_1C_Y^2+f_2(\psi_1^2C_Z^2-2\psi_1\rho_{YZ}C_YC_Z)+f_3(C_X^2-2\rho_{YX}C_YC_X)\right\}, \tag{27}$$

$$MSE(\bar{y}_S) = \bar{Y}^2\left\{f_1C_Y^2+f_2(\psi_2^2C_Z^2-2\psi_2\rho_{YZ}C_YC_Z)+f_3(C_X^2-2\rho_{YX}C_YC_X)\right\}, \tag{28}$$

$$MSE(\bar{y}_{US}) = \bar{Y}^2\left\{f_1C_Y^2+f_2(\psi_3^2C_Z^2-2\psi_3\rho_{YZ}C_YC_Z)+f_3(C_X^2-2\rho_{YX}C_YC_X)\right\}, \tag{29}$$

$$MSE(\bar{y}_{SEA}) = \bar{Y}^2\left\{f_1C_Y^2+f_2(\psi_4^2C_Z^2-2\psi_4\rho_{YZ}C_YC_Z)+f_3(C_X^2-2\rho_{YX}C_YC_X)\right\}, \tag{30}$$

$$MSE(\bar{y}_{\mathrm{Re}}^{dc}) = \bar{Y}^2\left\{f_1C_Y^2+f_2\left(\frac{1}{4}C_Z^2-\rho_{YZ}C_YC_Z\right)+f_3\left(\frac{1}{4}C_X^2-\rho_{YX}C_YC_X\right)\right\}, \tag{31}$$

$$MSE(\bar{y}_{SM}) = \bar{Y}^2\left\{f_1C_Y^2+f_2\left(\frac{1}{4}C_Z^2-\rho_{YZ}C_YC_Z\right)+f_3\left(C_X^2-2\rho_{YX}C_YC_X\right)\right\}, \tag{32}$$

$$MSE(\bar{y}_{MT}) = \bar{Y}^2\left\{f_1C_Y^2+f_2(\psi_5^2C_Z^2-2\psi_5\rho_{YZ}C_YC_Z)+f_3(C_X^2-2\rho_{YX}C_YC_X)\right\}, \tag{33}$$

Furthermore, the minimum attainable MSE of the estimator $\bar{y}_{ds}$ is given by

$$MSE(\bar{y}_{ds})_{\min} = \bar{Y}^2C_Y^2\left(f_1-f_3\rho_{YX}^2\right). \tag{34}$$

The notations used are as follows:

$$f_1 = \left(\frac{1}{n}-\frac{1}{N}\right),\ f_2 = \left(\frac{1}{n'}-\frac{1}{N}\right),\ f_3 = f_1-f_2 = \left(\frac{1}{n}-\frac{1}{n'}\right),$$

$$\psi_1 = \frac{\bar{Z}}{(\bar{Z}+C_Z)},\ \psi_2 = \frac{\bar{Z}}{(\bar{Z}+\sigma_Z)},\ \psi_3 = \frac{\beta_{2(z)}\bar{Z}}{(\beta_{2(z)}\bar{Z}+C_Z)},$$

$$\psi_4 = \frac{\bar{Z}}{(\bar{Z}+\rho_{XZ})},\ \psi_5 = \frac{\bar{Z}}{(\bar{Z}+\beta_{2(z)})},\ \beta_{2(z)} = \frac{\mu_{4(z)}}{\mu_{2(z)}^2},$$

$$\mu_{4(z)} = \frac{1}{N}\sum_{i=1}^{N}\left(Z_i-\bar{Z}\right)^4,\ \mu_{2(z)} = \frac{1}{N}\sum_{i=1}^{N}\left(Z_i-\bar{Z}\right)^2,$$

$$C_Y^2 = \frac{S_Y^2}{\bar{Y}^2},\ C_X^2 = \frac{S_X^2}{\bar{X}^2},\ C_Z^2 = \frac{S_Z^2}{\bar{Z}^2},\ \rho_{YX} = \frac{S_{YX}}{S_YS_X},$$

$$\rho_{YZ} = \frac{S_{YZ}}{S_YS_Z},\ \rho_{XZ} = \frac{S_{XZ}}{S_XS_Z},\ S_Y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^2,$$

$$S_X^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(X_i-\bar{X}\right)^2,\ S_Z^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(Z_i-\bar{Z}\right)^2,$$

$$S_{YX} = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)\left(X_i-\bar{X}\right),$$

$$S_{YZ} = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)\left(Z_i-\bar{Z}\right),$$

$$S_{XZ} = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(X_i-\bar{X}\right)\left(Z_i-\bar{Z}\right).$$

2.2 Proposed Class of Estimators

Extending the works of Singh *et al.* (2007), and Mehta and Tailor (2020), we propose the following transformed class of estimators for population mean $\bar{Y}$ under two-phase sampling:

$$T = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\alpha\bar{Z}+\gamma}{\alpha\bar{z}'+\gamma}\right), \tag{35}$$

where α and γ are the scalars, which may be either real numbers or functions of some known parameters of the auxiliary variable(s).

Some of the well-known existing estimators, as mentioned in Sub-section 2.1, are observed to be the members of the proposed class T on assigning suitable values to the scalars α and γ in (35), as demonstrated in Table 1.

Table 1. Members of the proposed class T

Sl. No.	Authors	Estimators	Assigned values of scalars in the proposed class T	
			α	γ
1	Sukhatme (1962)	$\bar{y}_R^d = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)$	0	1
2	Chand (1975)	$\bar{y}_R^{dc} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}}{\bar{z}'}\right)$	1	0
3	Singh and Upadhyaya (1995)	$\bar{y}_{SU} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}+C_Z}{\bar{z}'+C_Z}\right)$	1	C_Z
4	Singh (2001)	$\bar{y}_S = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}+\sigma_Z}{\bar{z}'+\sigma_Z}\right)$	1	σ_Z
5	Upadhyaya and Singh (2001)	$\bar{y}_{US} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\beta_{2(z)}\bar{Z}+C_Z}{\beta_{2(z)}\bar{z}'+C_Z}\right)$	$\beta_{2(z)}$	C_Z
6	Singh *et al.* (2007)	$\bar{y}_{SEA} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}+\rho_{XZ}}{\bar{z}'+\rho_{XZ}}\right)$	1	ρ_{XZ}
7	Mehta and Tailor (2020)	$\bar{y}_{MT} = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}+\beta_{2(z)}}{\bar{z}'+\beta_{2(z)}}\right)$	1	$\beta_{2(z)}$

2.3 Bias and MSE of the Proposed Class

To obtain the Bias and MSE of the proposed class T, we consider

$$\bar{y} = \bar{Y}(1+e_0), \quad \bar{x} = \bar{X}(1+e_1), \quad \bar{x}' = \bar{X}(1+e_1'), \quad \bar{z}' = \bar{Z}(1+e_2').$$

Also, we have

$$\left.\begin{aligned} &E(e_0)=E(e_1)=E(e_1')=E(e_2')=0,\\ &E(e_0^2)=f_1C_Y^2, E(e_1^2)=f_1C_X^2,\ E(e_1'^2)=f_2C_X^2,\ E(e_2'^2)=f_2C_Z^2,\\ &E(e_0e_1)=f_1\rho_{YX}C_YC_X, E(e_0e_1')=f_2\rho_{YX}C_YC_X, E(e_0e_2')=f_2\rho_{YZ}C_YC_Z,\\ &E(e_1e_1')=f_2C_X^2, E(e_1e_2')=f_2\rho_{XZ}C_XC_Z, E(e_1'e_2')=f_2\rho_{XZ}C_XC_Z. \end{aligned}\right\} \tag{36}$$

Now, expressing T in terms of e_0, e_1, e_1' and e_2', we have

$$T = \bar{Y}(1+e_0)(1+e_1')(1+e_1)^{-1}(1+\psi e_2')^{-1}, \tag{37}$$

where $\psi = \alpha\bar{Z}/(\alpha\bar{Z}+\gamma)$.

Multiplying out, simplifying, and retaining the error terms up to the second degree in (37), we have

$$T-\bar{Y} = \bar{Y}\big(e_0 - e_1 + e_1' - e_0e_1 + e_0e_1' - e_1e_1' + e_1^2 - \psi e_2' - \psi e_0e_2' + \psi e_1e_2' - \psi e_1'e_2' + \psi^2 e_2'^2\big), \tag{38}$$

Taking expectation on both sides of (38), and using results of (36), we obtain the Bias of the proposed class T up to the first order approximation as follows:

$$B(T) = \bar{Y}\left\{f_2(\psi^2C_Z^2 - \psi\rho_{YZ}C_YC_Z) + f_3(C_X^2 - \rho_{YX}C_YC_X)\right\}, \tag{39}$$

Again, on retaining the first order error terms in (38), we have

$$T-\bar{Y} = \bar{Y}(e_0 - e_1 + e_1' - \psi e_2'), \tag{40}$$

Squaring both sides of (40), taking the expectation and using results of (36), we obtain the MSE of the proposed class T to the first order of approximation as:

$$MSE\ (T) = \bar{Y}^2\big(f_1C_Y^2 + f_3C_X^2 + \psi^2 f_2C_Z^2 - 2f_3\rho_{YX}C_YC_X - 2\psi f_2\rho_{YZ}C_YC_Z\big)$$

or $$MSE\ (T) = \bar{Y}^2\left\{f_1C_Y^2 + f_2(\psi^2C_Z^2 - 2\psi\rho_{YZ}C_YC_Z) + f_3(C_X^2 - 2\rho_{YX}C_YC_X)\right\}. \tag{41}$$

The optimum value of ψ for which the MSE of the proposed class T in (41) is minimized, is given by:

$$\psi_{opt} = \frac{\rho_{YZ}C_Y}{C_Z}, \tag{42}$$

and hence the minimum attainable MSE of T is given by

$$MSE(T)_{\min} = \bar{Y}^2\left\{f_1C_Y^2 - f_2\rho_{YZ}^2C_Y^2 + f_3\left(C_X^2 - 2\rho_{YX}C_YC_X\right)\right\}. \tag{43}$$

Thus, we establish the following theorem.

Theorem 2.3.1 *To the first order of approximation,*

$$MSE(T) \geq \bar{Y}^2\left\{f_1C_Y^2 - f_2\rho_{YZ}^2C_Y^2 + f_3\left(C_X^2 - 2\rho_{YX}C_YC_X\right)\right\}, \tag{44}$$

with equality holding if $\psi = (\rho_{YZ}C_Y)/C_Z$.

2.4 Efficiency Comparisons

It is well known that the variance of sample mean $\bar{y}$ under simple random sampling without replacement (SRSWOR) scheme is given by

$$Var(\bar{y}) = f_1S_Y^2 = f_1\bar{Y}^2C_Y^2. \tag{45}$$

For making efficiency comparisons of the proposed estimator T with the well-known pre-existing estimators, we obtain the following conditions by utilizing equations (23) to (33), (41), and (45):

(i) $MSE(T) < Var(\bar{y})$ if

$$\psi^2 C_Z^2 - 2\psi\rho_{YZ}C_Y C_Z < \frac{f_3\left(2\rho_{YX}C_Y C_X - C_X^2\right)}{f_2}, \quad (46)$$

(ii) $MSE(T) < MSE\left(\bar{y}_R^d\right)$ if

$$\psi < \frac{2\rho_{YZ}C_Y}{C_Z}, \quad (47)$$

(iii) $MSE(T) < MSE\left(\bar{y}_{ds}\right)$ if

$$\psi^2 C_Z^2 - 2\psi\rho_{YZ}C_Y C_Z < \frac{f_3\left\{\left(\alpha^2-1\right)C_X^2 - 2(\alpha-1)\rho_{YX}C_Y C_X\right\}}{f_2}, \quad (48)$$

(iv) $MSE(T) < MSE\left(\bar{y}_R^{dc}\right)$ if

$$\left(\psi^2-1\right)C_Z^2 - 2(\psi-1)\rho_{YZ}C_Y C_Z < 0, \quad (49)$$

(v) $MSE(T) < MSE\left(\bar{y}_{MEA}^d\right)$ if

$$\psi^2 C_Z^2 - 2\psi\rho_{YZ}C_Y C_Z < \frac{f_3\left(2\rho_{YX}C_Y C_X + 2\rho_{XZ}\rho_{YX}\rho_{YZ}C_Y^2 - \rho_{YX}^2 C_Y^2 - \rho_{YZ}^2 C_Y^2 - C_X^2\right)}{f_2}, \quad (50)$$

(vi) $MSE(T) < MSE\left(\bar{y}_{SU}\right)$ if

$$\left(\psi+\psi_1\right)C_Z^2 - 2\rho_{YZ}C_Y C_Z < 0, \quad (51)$$

(vii) $MSE(T) < MSE\left(\bar{y}_S\right)$ if

$$\left(\psi+\psi_2\right)C_Z^2 - 2\rho_{YZ}C_Y C_Z < 0, \quad (52)$$

(viii) $MSE(T) < MSE\left(\bar{y}_{US}\right)$ if

$$\left(\psi+\psi_3\right)C_Z^2 - 2\rho_{YZ}C_Y C_Z < 0, \quad (53)$$

(ix) $MSE(T) < MSE\left(\bar{y}_{SEA}\right)$ if

$$\left(\psi+\psi_4\right)C_Z^2 - 2\rho_{YZ}C_Y C_Z < 0, \quad (54)$$

(x) $MSE(T) < MSE\left(\bar{y}_{\mathrm{Re}}^{dc}\right)$ if

$$\left(\psi^2-\frac{1}{4}\right)C_Z^2 - (2\psi-1)\rho_{YZ}C_Y C_Z < \frac{f_3}{f_2}\left(\rho_{YX}C_Y C_X - \frac{3}{4}C_X^2\right), \quad (55)$$

(xi) $MSE(T) < MSE\left(\bar{y}_{SM}\right)$ if

$$\left(\psi^2-\frac{1}{4}\right)C_Z^2 - (2\psi-1)\rho_{YZ}C_Y C_Z < 0. \quad (56)$$

(xii) $MSE(T) < MSE\left(\bar{y}_{MT}\right)$ if

$$\left(\psi+\psi_5\right)C_Z^2 - 2\rho_{YZ}C_Y C_Z < 0, \quad (57)$$

2.5 Cost Function Analysis

For obtaining the optimum sample sizes of the first-phase and second-phase samples, we consider a cost function of the form:

$$c = c_1 n' + c_2 n \quad (58)$$

where c is the total sampling cost. Also, c_1 is the cost per unit associated with the first-phase sample of size n', and c_2 is the cost per unit associated with the second-phase sample of size n.

The optimum values of n' and n which minimize the MSE of the proposed class T for a fixed cost $c \le c_0$, are obtained on using a Lagrangian function of the form:

$$L = MSE(T) + \lambda(c_1 n' + c_2 n - c_0) \quad (59)$$

where λ is the Lagrange's multiplier.

Now, from (41), we have

$$MSE(T) = f_1\bar{Y}^2 C_Y^2 + f_3\bar{Y}^2\left(C_X^2 - 2\rho_{YX}C_Y C_X\right) + f_2\bar{Y}^2\left(\psi^2 C_Z^2 - 2\psi\rho_{YZ}C_Y C_Z\right)$$

$$= f_1 S_Y^2 + f_3\bar{Y}^2\left(\frac{S_X^2}{\bar{X}^2} - 2\rho_{YX}\frac{S_Y}{\bar{Y}}\frac{S_X}{\bar{X}}\right) + f_2\bar{Y}^2\left(\psi^2\frac{S_Z^2}{\bar{Z}^2} - 2\psi\rho_{YZ}\frac{S_Y}{\bar{Y}}\frac{S_Z}{\bar{Z}}\right)$$

$$= f_1 S_Y^2 + f_3\left(R_1^2 S_X^2 - 2R_1 S_{YX}\right) + f_2\left(\psi^2 R_2^2 S_Z^2 - 2\psi R_2 S_{YZ}\right)$$

where the notations used are as follows:

$C_Y = S_Y/\bar{Y}$, $C_X = S_X/\bar{X}$, $C_Z = S_Z/\bar{Z}$, $R_1 = \bar{Y}/\bar{X}$, $R_2 = \bar{Y}/\bar{Z}$, $S_{YX} = \rho_{YX}S_Y S_X$, $S_{YZ} = \rho_{YZ}S_Y S_Z$.

Hence, on simplification, we get

$$MSE(T) = \left(\frac{1}{n}-\frac{1}{N}\right)S_Y^2 + \left(\frac{1}{n}-\frac{1}{n'}\right)\xi + \left(\frac{1}{n'}-\frac{1}{N}\right)\tau \quad (60)$$

where

$\xi = R_1^2 S_X^2 - 2R_1 S_{YX}$, and $\tau = \psi^2 R_2^2 S_Z^2 - 2\psi R_2 S_{YZ}$.

On substituting (60) in (59), we get

$$L = \left(\frac{1}{n}-\frac{1}{N}\right)S_Y^2 + \left(\frac{1}{n}-\frac{1}{n'}\right)\xi + \left(\frac{1}{n'}-\frac{1}{N}\right)\tau + \lambda(c_1 n' + c_2 n - c_0) \quad (61)$$

Now, differentiating L with respect to n' and n, equating the results to zero, and then using (58), we obtain the optimum values of n' and n as follows:

$$n' = \frac{c}{c_1 + c_2\sqrt{A}} \quad (62)$$

$$n = n'\sqrt{A} = \frac{c\sqrt{A}}{c_1 + c_2\sqrt{A}} \quad (63)$$

where $A = \dfrac{c_1\left(S_Y^2 + \xi\right)}{c_2\left(\tau - \xi\right)}$.

Hence, on substituting the values of n' and n from (62) and (63) in (60), we obtain the optimum MSE of the proposed class T as

$$MSE(T)_{opt} = \left\{\frac{(c_1 + c_2\sqrt{A})}{c\sqrt{A}} - \frac{1}{N}\right\}S_Y^2 + \left\{\frac{(c_1 + c_2\sqrt{A})}{c\sqrt{A}} - \frac{(c_1 + c_2\sqrt{A})}{c}\right\}\xi + \left\{\frac{(c_1 + c_2\sqrt{A})}{c} - \frac{1}{N}\right\}\tau$$

$$= \frac{(c_1 + c_2\sqrt{A})}{c}\left\{\frac{1}{\sqrt{A}}S_Y^2 + \left(\frac{1}{\sqrt{A}} - 1\right)\xi + \tau\right\} - \frac{1}{N}\left(S_Y^2 + \tau\right) \tag{64}$$

In a similar manner, the optimum values of n' and n for various pre-existing estimators are obtained, along with their optimum MSEs, and the findings are presented in Table 2.

The notations used in Table 2 are as follows:

$$A_1 = \frac{c_1\left(S_Y^2 - \xi_1\right)}{c_2\xi_1};\ \xi_1 = 2R_1S_{YX} - R_1^2S_X^2,$$

Table 2. Optimum n' and n along with the optimum MSEs of various estimators for fixed cost $c \le c_0$

Estimators	n'	n	Optimum MSEs
$\bar{y}$	---	$\frac{c}{c_2}$	$\left(\frac{c_2}{c} - \frac{1}{N}\right)S_Y^2$
$\bar{y}_R^d$	$\frac{c}{c_1 + c_2\sqrt{A_1}}$	$\frac{c\sqrt{A_1}}{c_1 + c_2\sqrt{A_1}}$	$\frac{(c_1 + c_2\sqrt{A_1})}{c}\left\{\frac{1}{\sqrt{A_1}}S_Y^2 - \left(\frac{1}{\sqrt{A_1}} - 1\right)\xi_1\right\} - \frac{1}{N}S_Y^2$
$\bar{y}_{ds}$	$\frac{c}{c_1 + c_2\sqrt{A_2}}$	$\frac{c\sqrt{A_2}}{c_1 + c_2\sqrt{A_2}}$	$\frac{(c_1 + c_2\sqrt{A_2})}{c}\left\{\frac{1}{\sqrt{A_2}}S_Y^2 - \left(\frac{1}{\sqrt{A_2}} - 1\right)\xi_2\right\} - \frac{1}{N}S_Y^2$
$\bar{y}_R^{dc}$	$\frac{c}{c_1 + c_2\sqrt{A_3}}$	$\frac{c\sqrt{A_3}}{c_1 + c_2\sqrt{A_3}}$	$\frac{(c_1 + c_2\sqrt{A_3})}{c}\left\{\frac{1}{\sqrt{A_3}}S_Y^2 + \left(\frac{1}{\sqrt{A_3}} - 1\right)\xi + \tau_1\right\} - \frac{1}{N}\left(S_Y^2 + \tau_1\right)$
$\bar{y}_{MEA}^d$	$\frac{c}{c_1 + c_2\sqrt{A_4}}$	$\frac{c\sqrt{A_4}}{c_1 + c_2\sqrt{A_4}}$	$\frac{(c_1 + c_2\sqrt{A_4})}{c}\left\{\frac{1}{\sqrt{A_4}}S_Y^2 - \left(\frac{1}{\sqrt{A_4}} - 1\right)\xi_3\right\} - \frac{1}{N}S_Y^2$
$\bar{y}_{SU}$	$\frac{c}{c_1 + c_2\sqrt{A_5}}$	$\frac{c\sqrt{A_5}}{c_1 + c_2\sqrt{A_5}}$	$\frac{(c_1 + c_2\sqrt{A_5})}{c}\left\{\frac{1}{\sqrt{A_5}}S_Y^2 + \left(\frac{1}{\sqrt{A_5}} - 1\right)\xi + \tau_2\right\} - \frac{1}{N}\left(S_Y^2 + \tau_2\right)$
$\bar{y}_S$	$\frac{c}{c_1 + c_2\sqrt{A_6}}$	$\frac{c\sqrt{A_6}}{c_1 + c_2\sqrt{A_6}}$	$\frac{(c_1 + c_2\sqrt{A_6})}{c}\left\{\frac{1}{\sqrt{A_6}}S_Y^2 + \left(\frac{1}{\sqrt{A_6}} - 1\right)\xi + \tau_3\right\} - \frac{1}{N}\left(S_Y^2 + \tau_3\right)$
$\bar{y}_{US}$	$\frac{c}{c_1 + c_2\sqrt{A_7}}$	$\frac{c\sqrt{A_7}}{c_1 + c_2\sqrt{A_7}}$	$\frac{(c_1 + c_2\sqrt{A_7})}{c}\left\{\frac{1}{\sqrt{A_7}}S_Y^2 + \left(\frac{1}{\sqrt{A_7}} - 1\right)\xi + \tau_4\right\} - \frac{1}{N}\left(S_Y^2 + \tau_4\right)$
$\bar{y}_{SEA}$	$\frac{c}{c_1 + c_2\sqrt{A_8}}$	$\frac{c\sqrt{A_8}}{c_1 + c_2\sqrt{A_8}}$	$\frac{(c_1 + c_2\sqrt{A_8})}{c}\left\{\frac{1}{\sqrt{A_8}}S_Y^2 + \left(\frac{1}{\sqrt{A_8}} - 1\right)\xi + \tau_5\right\} - \frac{1}{N}\left(S_Y^2 + \tau_5\right)$
$\bar{y}_{\mathrm{Re}}^{dc}$	$\frac{c}{c_1 + c_2\sqrt{A_9}}$	$\frac{c\sqrt{A_9}}{c_1 + c_2\sqrt{A_9}}$	$\frac{(c_1 + c_2\sqrt{A_9})}{c}\left\{\frac{1}{\sqrt{A_9}}S_Y^2 + \left(\frac{1}{\sqrt{A_9}} - 1\right)\xi_4 + \tau_6\right\} - \frac{1}{N}\left(S_Y^2 + \tau_6\right)$
$\bar{y}_{SM}$	$\frac{c}{c_1 + c_2\sqrt{A_{10}}}$	$\frac{c\sqrt{A_{10}}}{c_1 + c_2\sqrt{A_{10}}}$	$\frac{(c_1 + c_2\sqrt{A_{10}})}{c}\left\{\frac{1}{\sqrt{A_{10}}}S_Y^2 + \left(\frac{1}{\sqrt{A_{10}}} - 1\right)\xi + \tau_6\right\} - \frac{1}{N}\left(S_Y^2 + \tau_6\right)$
$\bar{y}_{MT}$	$\frac{c}{c_1 + c_2\sqrt{A_{11}}}$	$\frac{c\sqrt{A_{11}}}{c_1 + c_2\sqrt{A_{11}}}$	$\frac{(c_1 + c_2\sqrt{A_{11}})}{c}\left\{\frac{1}{\sqrt{A_{11}}}S_Y^2 + \left(\frac{1}{\sqrt{A_{11}}} - 1\right)\xi + \tau_7\right\} - \frac{1}{N}\left(S_Y^2 + \tau_7\right)$

$$A_2 = \frac{c_1\left(S_Y^2 - \xi_2\right)}{c_2\xi_2}\ ;\ \xi_2 = 2\alpha R_1 S_{YX} - \alpha^2 R_1^2 S_X^2\ ,$$

$$A_3 = \frac{c_1\left(S_Y^2 + \xi\right)}{c_2\left(\tau_1 - \xi\right)}\ ;\ \xi = R_1^2 S_X^2 - 2R_1 S_{YX}\ ,\ \tau_1 = R_2^2 S_Z^2 - 2R_2 S_{YZ}\ ,$$

$$A_4 = \frac{c_1\left(S_Y^2 - \xi_3\right)}{c_2\xi_3}\ ;\ \xi_3 = S_Y^2\left(\rho_{YX}^2 + \rho_{YZ}^2 - 2\rho_{YX}\rho_{YZ}\rho_{XZ}\right),$$

$$A_5 = \frac{c_1\left(S_Y^2 + \xi\right)}{c_2\left(\tau_2 - \xi\right)}\ ;\ \xi = R_1^2 S_X^2 - 2R_1 S_{YX}\ ,\ \tau_2 = \psi_1^2 R_2^2 S_Z^2 - 2\psi_1 R_2 S_{YZ}\ ,$$

$$A_6 = \frac{c_1\left(S_Y^2 + \xi\right)}{c_2\left(\tau_3 - \xi\right)}\ ;\ \xi = R_1^2 S_X^2 - 2R_1 S_{YX}\ ,\ \tau_3 = \psi_2^2 R_2^2 S_Z^2 - 2\psi_2 R_2 S_{YZ}\ ,$$

$$A_7 = \frac{c_1\left(S_Y^2 + \xi\right)}{c_2\left(\tau_4 - \xi\right)}\ ;\ \xi = R_1^2 S_X^2 - 2R_1 S_{YX}\ ,\ \tau_4 = \psi_3^2 R_2^2 S_Z^2 - 2\psi_3 R_2 S_{YZ}\ ,$$

$$A_8 = \frac{c_1\left(S_Y^2 + \xi\right)}{c_2\left(\tau_5 - \xi\right)}\ ;\ \xi = R_1^2 S_X^2 - 2R_1 S_{YX}\ ,\ \tau_5 = \psi_4^2 R_2^2 S_Z^2 - 2\psi_4 R_2 S_{YZ}\ ,$$

$$A_9 = \frac{c_1\left(S_Y^2 + \xi_4\right)}{c_2\left(\tau_6 - \xi_4\right)}\ ;\ \xi_4 = \frac{R_1^2 S_X^2}{4} - R_1 S_{YX}\ ,\ \tau_6 = \frac{R_2^2 S_Z^2}{4} - R_2 S_{YZ}\ ,$$

$$A_{10} = \frac{c_1\left(S_Y^2 + \xi\right)}{c_2\left(\tau_6 - \xi\right)}\ ;\ \xi = R_1^2 S_X^2 - 2R_1 S_{YX}\ ,\ \tau_6 = \frac{R_2^2 S_Z^2}{4} - R_2 S_{YZ}\ ,$$

$$A_{11} = \frac{c_1\left(S_Y^2 + \xi\right)}{c_2\left(\tau_7 - \xi\right)}\ ;\ \xi = R_1^2 S_X^2 - 2R_1 S_{YX}\ ,\ \tau_7 = \psi_5^2 R_2^2 S_Z^2 - 2\psi_5 R_2 S_{YZ}\ .$$

3. RESULTS AND DISCUSSION

3.1 Empirical Analysis

To examine the efficiency of the proposed class T as compared to the pre-existing estimators, we have considered three real population datasets. The analysis has been done using Wolfram Mathematica software. The descriptions of the populations, along with the values of respective parameters, are given below:

Population I- [Source: Anderson (1958)]

Y: Head length of second son,

X: Head length of first son,

Z: Head breadth of first son,

N=25, $n' = 10, n = 7, \bar{Y} = 183.84, \bar{X} = 185.72, \bar{Z} = 151.12$,

$\rho_{YX} = 0.7108, \rho_{YZ} = 0.6932, \rho_{XZ} = 0.7346, C_Y = 0.0546, C_X = 0.0526, C_Z = 0.0488, \beta_{2(z)} = 2.6519$.

Population II- [Source: Handique (2012)]

Y: Forest timber volume in cubic meter (Cum) in 0.1 ha sample plot,

X: Average tree height in the sample plot in meter (m),

Z: Average crown diameter in the sample plot in meter (m),

N=2500, $n' = 200, n = 25, \bar{Y} = 4.63, \bar{X} = 21.09, \bar{Z} = 13.55$,

$\rho_{YX} = 0.79, \rho_{YZ} = 0.72, \rho_{XZ} = 0.66, C_Y = 0.95, C_X = 0.98, C_Z = 0.64$.

Population III- [Source: Sukhatme and Chand (1977)]

Y: Apple trees of bearing age in 1964,

X: Bushels of apples harvested in 1964,

Z: Bushels of apples harvested in 1959,

N=200, $n' = 30, n = 20, \bar{Y} = 1031.82, \bar{X} = 2934.58$,

$\bar{Z} = 3651.49$, $\rho_{YX} = 0.93, \rho_{YZ} = 0.77, \rho_{XZ} = 0.84$,

$C_Y^2 = 2.55280, C_X^2 = 4.02504, C_Z^2 = 2.09379$.

The optimum values of the scalar ψ are computed for the above mentioned populations, and the findings are presented in Table 3. Moreover, the values of the scalars $\psi_1, \psi_2, \psi_3, \psi_4$, and ψ_5 are obtained for the above mentioned populations, and the findings are demonstrated in Table 4.

Table 3. Optimum values of the scalar ψ in various populations

Population	I	II	III
$\psi_{opt} = \frac{\rho_{YZ} C_Y}{C_Z}$	0.7756	1.0688	0.8502

Table 4. Values of the scalars in various populations

Scalars	Populations		
	I	II	III
$\psi_1 = \frac{\bar{Z}}{(\bar{Z} + C_Z)}$	0.9997	0.9549	0.9996
$\psi_2 = \frac{\bar{Z}}{(\bar{Z} + \sigma_Z)}$	0.9544	0.6098	0.4093
$\psi_3 = \frac{\beta_{2(z)}\bar{Z}}{(\beta_{2(z)}\bar{Z} + C_Z)}$	0.9999	*	*
$\psi_4 = \frac{\bar{Z}}{(\bar{Z} + \rho_{XZ})}$	0.9952	0.9536	0.9998
$\psi_5 = \frac{\bar{Z}}{(\bar{Z} + \beta_{2(z)})}$	0.9828	*	*

* Information on $\beta_{2(z)}$ is not available for the concerned populations.

The Biases, and percent absolute relative biases (PARBs) of various estimators of $\bar{Y}$ have been computed for the concerned populations, and the findings are demonstrated in Table 5. The PARBs are obtained on using the following formula:

$$PARB(\varphi) = \left|\frac{Bias\,(\varphi)}{\bar{Y}}\right| \times 100 = \left|\frac{E(\varphi) - \bar{Y}}{\bar{Y}}\right| \times 100,$$

where

$\varphi = \bar{y},\ \bar{y}_R^d,\ \bar{y}_{ds},\ \bar{y}_R^{dc},\ \bar{y}_{MEA}^d, \bar{y}_{SU}, \bar{y}_S, \bar{y}_{US},\ \bar{y}_{SEA}, \bar{y}_{\mathrm{Re}}^{dc},\ \bar{y}_{SM},\ \bar{y}_{MT},\ T$.

Moreover, the MSEs and percent relative efficiencies (PREs) of various estimators of $\bar{Y}$ have been obtained, and the findings are presented in Table 6. The PREs are computed with respect to the sample mean $\bar{y}$ on using the following formula:

$$PRE(\varphi, \bar{y}) = \frac{Var(\bar{y})}{MSE(\varphi)} \times 100,$$

where

$\varphi = \bar{y},\ \bar{y}_R^d,\ \bar{y}_{ds},\ \bar{y}_R^{dc},\ \bar{y}_{MEA}^d, \bar{y}_{SU}, \bar{y}_S, \bar{y}_{US},\ \bar{y}_{SEA},\ \bar{y}_{\mathrm{Re}}^{dc},\ \bar{y}_{SM},\ \bar{y}_{MT},\ T$.

Table 5. Biases and PARBs of various estimators of $\bar{Y}$

Estimator	Population I		Population II		Population III	
	Bias	PARB	Bias	PARB	Bias	PARB
$\bar{y}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\bar{y}_R^d$	0.0057	0.0031	0.0364	0.7872	17.9526	1.7399
$\bar{y}_{ds}$	-0.0006	0.0003	-0.0055	0.1188	-2.0041	0.1942
$\bar{y}_R^{dc}$	0.0116	0.0063	0.0358	0.7742	27.1208	2.6284
$\bar{y}_{MEA}^d$	**	**	**	**	**	**
$\bar{y}_{SU}$	0.0116	0.0063	0.0355	0.7667	27.0929	2.6257
$\bar{y}_S$	0.0102	0.0055	0.0340	0.7345	6.9059	0.6693
$\bar{y}_{US}$	0.0116	0.0063	*	*	*	*
$\bar{y}_{SEA}$	0.0115	0.0062	0.0355	0.7665	27.1046	2.6269
$\bar{y}_{\mathrm{Re}}^{dc}$	-0.0002	0.0001	-0.0026	0.0566	-2.7434	0.2659
$\bar{y}_{SM}$	0.0054	0.0029	0.0351	0.7572	14.8853	1.4426
$\bar{y}_{MT}$	0.0111	0.0060	*	*	*	*
T	0.0018	0.0010	0.0340	0.7334	6.8905	0.6678

*Information on $\beta_{2(z)}$ is not available for the concerned populations, **Data is not sufficient.

Table 6. MSEs and PREs of various estimators of $\bar{Y}$

Estimator	Population I		Population II		Population III	
	MSE	PRE	MSE	PRE	MSE	PRE
$\bar{y}$	10.36	100.00	0.7661	100.00	122303.00	100.00
$\bar{y}_R^d$	8.46	122.54	0.3831	200.01	87929.70	139.09
$\bar{y}_{ds}$	8.18	126.67	0.3435	223.02	83125.30	147.13
$\bar{y}_R^{dc}$	5.80	178.82	0.3371	227.27	43689.90	279.93
$\bar{y}_{MEA}^d$	9.23	112.25	0.5009	152.95	110763.00	110.42
$\bar{y}_{SU}$	7.59	136.60	0.7188	106.59	77081.90	158.67
$\bar{y}_S$	5.71	181.60	0.3454	221.80	54553.70	224.19
$\bar{y}_{US}$	5.80	178.83	*	*	*	*
$\bar{y}_{SEA}$	5.79	179.14	0.3375	227.04	43685.60	279.96
$\bar{y}_{\mathrm{Re}}^{dc}$	5.87	176.54	0.3614	212.00	49351.30	247.82
$\bar{y}_{SM}$	5.92	175.09	0.3500	218.91	50019.90	244.51
$\bar{y}_{MT}$	5.76	179.94	*	*	*	*
T	5.55	**186.65**	0.3369	**227.39**	42273.10	**289.32**

*Information on $\beta_{2(z)}$ is not available for the concerned populations.
Bold values indicate the maximum PREs.

3.2 Outcomes of the Analysis

From Table 5, it is revealed that the sample mean ($\bar{y}$) is an unbiased estimator of the population mean ($\bar{Y}$), whereas the proposed class T as well as the other existing estimators are biased estimators of $\bar{Y}$. Hence, the following results are obtained from Table 5:

(i) In population I, we have

$B(T) < B(\bar{y}_{SM}) < B(\bar{y}_R^d) < B(\bar{y}_S) < B(\bar{y}_{MT}) < B(\bar{y}_{SEA}) < B(\bar{y}_{SU}) < B(\bar{y}_{US}) < B(\bar{y}_R^{dc})$

(ii) In populations II, we have

$B(T) < B(\bar{y}_S) < B(\bar{y}_{SM}) < B(\bar{y}_{SEA}) < B(\bar{y}_{SU}) < B(\bar{y}_R^{dc}) < B(\bar{y}_R^d)$

(iii) In population III, we observe that:

$B(T) < B(\bar{y}_S) < B(\bar{y}_{SM}) < B(\bar{y}_R^d) < B(\bar{y}_{SU}) < B(\bar{y}_{SEA}) < B(\bar{y}_R^{dc})$

(iv) In all the three populations, the Biases of the estimators $\bar{y}_{ds}$ and $\bar{y}_{\mathrm{Re}}^{dc}$ are negative. Moreover, in populations I and II, we observe that: $B(\bar{y}_{ds}) < B(\bar{y}_{\mathrm{Re}}^{dc}) < B(\zeta)$, where $\zeta = \bar{y}_R^d,\ \bar{y}_R^{dc},\ \bar{y}_{SU}, \bar{y}_S, \bar{y}_{US},\ \bar{y}_{SEA},\ \bar{y}_{SM}, \bar{y}_{MT},\ T$.

However, in population III, it is observed that $B(\bar{y}_{\text{Re}}^{dc}) < B(\bar{y}_{ds}) < B(\zeta)$.

(v) In all the three populations, the patterns observed in Biases as mentioned above in (i), (ii), and (iii) also prevail for PARBs. Moreover, in populations I and II, it is observed that $PARB(\bar{y}_{\text{Re}}^{dc}) < PARB(\bar{y}_{ds}) < PARB(\zeta)$. Also, in population III, it is observed that $PARB(\bar{y}_{ds}) < PARB(\bar{y}_{\text{Re}}^{dc}) < PARB(\zeta)$.

Furthermore, the following results are obtained from Table 6:

(i) In all the three populations, the proposed class *T* has minimum MSEs, and hence consequently the maximum PREs, as compared to the sample mean ($\bar{y}$) and the other existing estimators.

(ii) Among the members of the proposed class *T*, the MSEs of estimators $\bar{y}_R^{dc}$ and $\bar{y}_{SEA}$ are nearly the same in the respective populations I and II. However, in population III, the MSEs of estimators $\bar{y}_R^{dc}$ and $\bar{y}_{SEA}$ vary considerably.

(iii) In population I, the MSE of Singh (2001) estimator $\bar{y}_S$ is minimum and nearly the same as that of some other members of the proposed class *T*, for instance, $\bar{y}_{MT}$, $\bar{y}_{SEA}$, $\bar{y}_{US}$, and $\bar{y}_R^{dc}$. Moreover, among the members of the proposed class *T*, the MSEs of Chand (1975) estimator $\bar{y}_R^{dc}$ is least as compared to the other members in the population II. Furthermore, in population III, Singh *et al.* (2007) estimator $\bar{y}_{SEA}$ has minimum MSE as compared to the other members of the class *T*.

(iv) In all the three populations, the sample mean ($\bar{y}$) has maximum MSEs and consequently the minimum PREs as compared to that of the proposed class *T*, and the other existing estimators.

4. CONCLUSION

On the basis of results of Table 5, we conclude that the proposed class *T* is positively biased in all the three populations, and has minimum bias as compared to the other positively biased estimators, in the concerned populations. Furthermore, on the basis of results of Table 6, we conclude that the proposed class *T* outperforms the sample mean ($\bar{y}$) and the other pre-existing estimators. Moreover, it has been established in Sub-section 2.2 that the proposed class *T* encompasses a wide range of members on assigning specific values to the scalars α and γ. The Biases and MSEs, along with the PARBs and PREs, of the members of proposed class *T* are compared among themselves as well as with the sample mean ($\bar{y}$) and the other existing estimators, and the findings are demonstrated in Tables 5 and 6, respectively.

Also, the optimum sample sizes of the first-phase and second-phase samples, under a specified cost function, have been obtained for the proposed class *T* as well as for the other pre-existing estimators of population mean $\bar{Y}$. In addition, the optimum MSEs have been computed for the specified cost, and the findings are elaborated in Sub-section 2.5.

In view of the theoretical and empirical results, we conclude that the proposed class *T* is superior to the other pre-existing estimators, and hence is more applicable for the estimation of population mean $\bar{Y}$ of the study variable *Y*.

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